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BY THE

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OF

MATHEMATICS,

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APPLICATION OF ALGEBRA TO THE THEORY OF CURVES AND SURFACES.

PART I.

THEORY OF CURVE LINES SITUATED IN A PLANE.

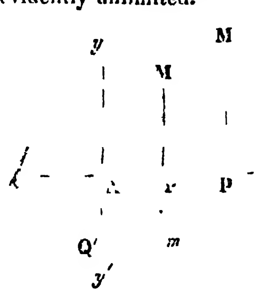
CHAP. I.—FIRST PRINCIPLES. LINES OF THE FIRST ORDER. TRANSFORMATION OF CO-ORDINATES.

1. When an equation contains two unknown quantities, x and y, the question will admit of innumerable solutions, and is therefore said to be indeterminate. By giving a number of arbitrary values to one of the unknown quantities, x, we may determine as many corresponding values of y. Thus, in the equation

$$y = 10x + 5,$$

if $x=0$, then $y=5$; if $x=1$, then $y=15$; if $x=2$, then $y=25$; and so on: the number of solutions being evidently unlimited.

2. If, in the indefinite straight line Ax, the parts AP, AP', &c., be taken to represent the values of x, and the lines PM, $P'M'$, &c., drawn parallel to the line Ay, be taken to represent the corresponding values of y; we may find as many points, M, M', &c., as we please, and a line passing through all these points is called the *locus* of the equation.



Conversely, this equation is called the *equation to the line*, which passes through all the points M, M', &c.

3. The line AP, which expresses any value of x, is called an

abscissa; and PM, the corresponding value of y, is called an *ordinate.* Any two corresponding values of x and y are called *co-ordinates.*

4. The lines Ax, Ay, are called the axes of x and y. They are generally drawn at right angles to each other, and are then denominated *rectangular axes;* when they are inclined to each other at a given angle, they are called *oblique axes.*

The point A is called the *origin* of the co-ordinates.

5. Since x is measured from a fixed point along the line Ax, given in position; if the values of x to the right of A be supposed *positive,* those to the left of A must be considered *negative* (vol. i. p. 417). In like manner, if the values of y, measured *upwards,* be positive, those measured *downwards* will be negative. This conventional rule being established, we shall have

x positive	and	y positive	in the angle	yAx
x negative	and	y positive	,,	yAx'
x negative	and	y negative	,,	$y'Ax'$
x positive	and	y negative	,,	$y'Ax$.

When, therefore, any particular values are given to x and y, we immediately perceive where the point is situated. Thus, if $x = +4$, and $y = -3$, the point will be somewhere within the angle $y'Ax$; and if we wish to determine its position, we take $AP = 4$, $AQ' = 3$; then the intersection of the two lines Pm, $Q'm$, drawn parallel to the two axes Ax, Ay, respectively, will be the point required.

When any point is situated in the axis xx', then $y = 0$; thus, for the point P, $x = 4$, $y = 0$.

When any point is situated in the axis yy', then $x = 0$; thus, for the point Q', $x = 0$, $y = -3$.

The point A is given by the values $x = 0$, $y = 0$.

The two equations $x = a$, $y = b$, by which any point is determined, are called the *equations to this point.*

6. Curve lines are usually divided into two classes, *algebraic* and *transcendental.* When the equation to a line can be expressed in a finite number of terms, containing only integral powers of x and y, and constant quantities, it is called an *algebraic* curve: all others are called *transcendental* curves. Thus, the lines represented by the equations

$$x^2 + y^2 - r^2 = 0; \quad ay = x^2; \quad ay^m = x^n;$$

are algebraic curves. And those represented by the equations

$$y = \sin x; \quad y = \log x; \quad y = a^x;$$

are transcendental curves.

7. Algebraic curves are divided into orders, according to the degree of the equation which expresses the relation between x and y. Thus,

$Ay + Bx + C = 0$ is the equation to a line of the 1st order

$Ay^2 + (Bx + C)y + Dx^2 + Ex + F = 0$,, ,, 2d order

$Ay^n + (Bx + C)y^{n-1} + (Dx^2 + Ex + F)y \quad + \&c. = 0$, is the general equation of the nth order, in which the highest sum of the indices in any term is equal to n.

8. The general equation of any degree includes not only all lines of

that order, but likewise all those of an inferior order. Thus, the equation of the second degree,

$$(y - ax + b)(y - a'x + b') = 0,$$

represents two lines of the first order, whose equations are $(y - ax + b) = 0$, and $y - a'x + b' = 0$; since either of these equations will satisfy the original equation. And if $a' = a$, $b' = b$, these two lines will become one line only of the first order.

In like manner, the equation of the third degree,

$$(y - ax + b)(cy^2 - dx) = 0,$$

represents one line of the first order, whose equation is $y - ax + b = 0$; and a line of the second order, whose equation is $cy^2 - dx = 0$.

LINES OF THE FIRST ORDER.

9. All equations of the first degree are comprehended in the general form

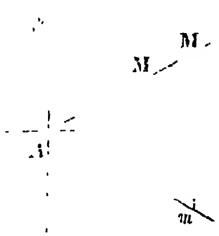
$$Ay + Bx + C = 0; \quad \text{or,} \quad y = -\frac{B}{A}x - \frac{C}{A};$$

and putting $-\frac{B}{A} = a$, $-\frac{C}{A} = b$, this equation becomes

$$y = ax + b,$$

which is the general equation of the first degree. We shall first consider the case when $b = 0$.

10. Prop. I.—*To determine the locus of the line whose equation is* $y = ax$.



Let Ax, Ay, be the two rectangular axes; and let AP, AP', be two given values of x, and PM, $P'M'$, the corresponding values of y. Then, since the equation $y = ax$ is always true, we have

$$PM = a \times AP; \quad P'M' = a \times AP';$$

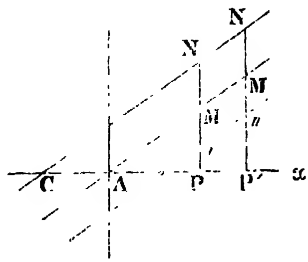
$$\therefore PM : P'M' :: a \times AP : a \times AP'$$
$$:: AP : AP'.$$

And because the angles at P and P' are equal, the triangles APM, $AP'M'$, are similar (Geom. prop. 71); therefore the angle $P'AM' = PAM$, and M' is in the straight line AM. Hence it follows that all the points M, M', &c., are in a straight line, drawn through A, the origin of the co-ordinates.

When a is negative, or the equation is $y = -ax$, then positive values of x correspond to negative values of y, and negative values of x correspond to positive values of y. Hence the line AM will take the position Am; and it may be proved, in the same manner as in the first case, that the locus of the point m is a straight line Am, which passes through the origin A.

11. *Cor.*—Since $a = \frac{y}{x} = \frac{PM}{AP}$, it follows that a is the trigonometrical tangent of the angle PAM.

12. PROP. II.—*To determine the locus of the line whose equation is* $y = ax + b$.



In this case, the ordinate y always exceeds the ordinate in the last proposition by the same quantity, b. If, therefore, any ordinate, PM, be produced, and MN be taken equal to b; then, if a straight line, NN', be drawn through the point N parallel to AM, every straight line $M'N'$, parallel to MN, will be equal to b. Hence, putting $AP = x$, $PN = y$, we have

$$PN = y = ax + b.$$

The straight line NN', therefore, is the locus of the equation $y = ax + b$.

If b be negative, the distance Mn must be measured in the opposite direction from the point M. Hence the line NN' will, in this case, occupy the position nn'.

13. *Cor.* 1.—The constant quantity b is the distance, AB, at which the straight line NN' cuts the axis Ay. And the constant quantity a is the trigonometrical tangent of the angle PAM or ACB.

14. *Cor.* 2.—If in the equation $Ay + Bx + C = 0$, $B = 0$, then $y = -\frac{C}{A}$ for all values of x; therefore the locus of N is a straight line, parallel to Ax. If $B = 0$, $C = 0$, the line CN will coincide with Ax.

If $A = 0$, then $x = -\frac{C}{B}$ for all values of y; therefore the locus of N is a straight line, parallel to Ay. If $A = 0$, $C = 0$, the line CN will coincide with Ay.

15. *Scholium.*—Hence it appears that an equation of the first degree always represents a straight line. The easiest way of constructing this line will be to find the points where it cuts the two axes, by putting successively $x = 0$, $y = 0$. Thus, if the equation be $3y - 2x - 6 = 0$, then

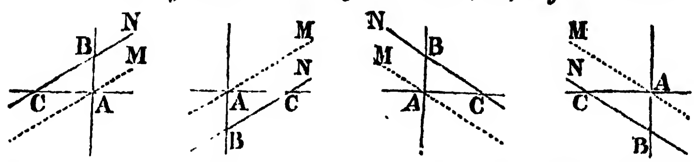
$$\text{when } x = 0, \quad 3y - 6 = 0, \quad y = 2;$$
$$\text{when } y = 0, \quad -2x - 6 = 0, \quad x = -3.$$

Hence, if we take $AB = 2$, and $AC = -3$, in the last figure, and through the points B, C, draw the line BC, it will be the line required.

If the equation be $y = ax$, we can only determine one point by this method; for when $x = 0$, y is also $= 0$, and therefore the line passes through the origin of co-ordinates. A second point, however, may easily be found, by giving to x any determinate value, such as 1, and then finding the corresponding value of y.

If, therefore, we give to a and b their proper signs, we shall have the following equations, represented by the straight lines CN, CN, &c., given in the figures below.

$y = ax + b$; $\quad y + ax - b$; $\quad y = -ax + b$; $\quad y = -ax - b$.



If the equation be $x = a$, the line is parallel to the axis of y, at the distance a.

If the equation be $y = b$, the line is parallel to the axis of x, at the [illegible] b.

16. PROP. III.—*The equation to a straight line is one of the first degree.* (See fig. to prop. 2.)

Let CN be the given straight line, cutting the two axes Ax, Ay, in C and B. Take any point, N, in the line BN, and draw NP perpendicular to Ax. Put $AP = x$, $PN = y$, $AB = b$, $AC = c$; we have then, from similar triangles,

$$AC : AB :: CP : PN; \quad \text{or,} \quad c : b :: c + x : y;$$

$$\therefore \quad cy = bx + bc; \quad \text{or,} \quad y = \frac{b}{c}x + b.$$

Examples.

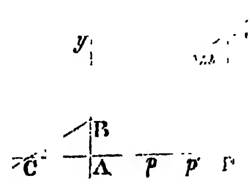
Construct the straight lines which belong to the following equations:

1.	$y - x = 0$	5.	$3y + 12x - 10 = 0$
2.	$5y + 2x = 0$	6.	$4y + 6x + 9 = 0$
3.	$10y - 2x - 7 = 0$	7.	$5x - 10 = 0$
4.	$6y - 4x + 18 = 0$	8.	$4y + \ \ 2 = 0$

PROBLEMS ON THE STRAIGHT LINE.

17. PROB. I.—*To find the equation to a straight line,* BM, *which passes through two given points,* m, m′.

Let $y = Ax + B$ be the equation to the straight line BM, in which x and y denote any values whatever, and A and B, constant quantities, to be determined from the conditions of the problem. Let α, β, and α', β', be the co-ordinates of these points, which are supposed to be given. Then, since the equation $y = Ax + B$ is true for every point in the straight line, BM, it is true for the points m, m'. We have, therefore,



$$y = Ax + B; \qquad \beta = A\alpha + B; \qquad \beta' = A\alpha' + B;$$

and the values of A and B are to be determined from the two last of these equations, and substituted in the first. Eliminate B by subtracting the second of these equations from the first and third; we then get

$$y - \beta = A(x - \alpha); \qquad \beta' - \beta = A(\alpha' - \alpha).$$

From the last equation we have $A = \frac{\beta' - \beta}{\alpha' - \alpha}$; and substituting this value in the preceding one, we finally obtain

$$y - \beta = \frac{\beta' - \beta}{\alpha' - \alpha}(x - \alpha)$$

for the equation to the straight line mm'.

18. *Cor.* 1.—The equation $y - \beta = A(x - \alpha)$, which was obtained by eliminating B from the two equations $y = Ax + B$, and $\beta = A\alpha + B$, is the equation to a straight line passing through the point m; for if $x = \alpha$, then will $y - \beta = 0$, and $y = \beta$. The coefficient A remains still indeterminate, since an infinite number of straight lines may be made to pass through this point.

19. *Cor.* 2.—The distance mm' is manifestly equal to

$$\sqrt{[(\alpha' - \alpha)^2 + (\beta' - \beta)^2]}.$$

20. PROB. II.—*To find the equation to a straight line which passes through a given point, and is parallel to a given straight line.*

Let $y = ax + b$ be the equation to the given straight line,
and $y = Ax + B$ the equation to the required straight line.
Also, let α, β, be the co-ordinates of the given point through which the line is to pass. Because this is a point in the straight line whose equation is $y = Ax + B$, we have (art. 17),

$$\beta = A\alpha + B, \quad \text{and} \quad y - \beta = A(x - \alpha).$$

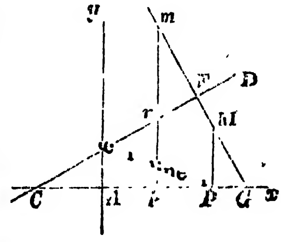
And because the two lines are parallel, they will cut the axis of x at the same angle; therefore the tangents of these angles are equal, or $A = a$. Hence the equation to the straight line required is

$$y - \beta = a(x - \alpha).$$

21. *Cor.*—The equation to a straight line which passes through a given point (α, β), and makes an angle θ, with the axis of x, is

$$y - \beta = \tan\theta\,(x - \alpha).$$

22. PROB. III.—*To find the equation to a straight line which passes through a given point,* m, *and is perpendicular to a given straight line,* CD.



Let $y = ax + b$ be the equation to the given straight line CD, and $y = Ax + B$ the equation to the line EM, which passes through the given point m, and is perpendicular to CD. Let the angle $DCx = \theta$, and $mGx = \phi$; also, let α, β, be the co-ordinates of the point m. Then, because the straight line mE passes through m, we have (art. 18),

$$y - \beta = A(x - \alpha).$$

And since mE is perpendicular to CD, the angle $\phi = 90° + \theta$; consequently

$$A = \tan \phi = \tan (90^\circ + \theta) = -\tan (90^\circ - \theta) \text{ (Trig. art. 67)};$$

$$\therefore \quad A = -\cot\theta = -\frac{1}{\tan\theta} = -\frac{1}{a} \ldots\ldots\ldots (1).$$

$$\text{Hence} \quad y - \beta = -\frac{1}{a}(x - \alpha)$$

is the equation required.

23. *Cor.*—If $y = Ax + B$ and $y = ax + b$ be the equations to two straight lines which are perpendicular to each other, we have, from equation (1),

$$A = -\frac{1}{a}; \quad \text{or,} \quad Aa + 1 = 0.$$

24. Prob. IV.—*To find the length of the perpendicular* mE, *from the given point* m *upon a given straight line* CD.

As before, let $y = ax + b$ be the equation to CD, and α, β, the co-ordinates of the point m. Also, put $mE = p$, and the angle $DCx = \theta$. We have then,

$$p = mn \times \sin mnE = mn\cos\theta = (pm - pn)\cos\theta \ldots\ldots (2).$$

But $pm = \beta$. Also, since n is a point in CD, whose equation is $y = ax + b$, therefore,

$$pn = a \times Ap + b = a\alpha + b.$$

$$\text{Likewise} \quad \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{(1 + \tan^2\theta)}} = \frac{1}{\sqrt{(1 + a^2)}}.$$

Substituting these values in equation (2), we have

$$p = \frac{\beta - a\alpha - b}{\sqrt{(1 + a^2)}}.$$

Problems for Practice.

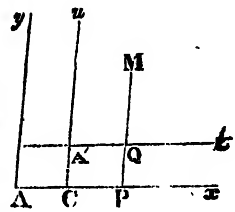
1. To find the equation to a straight line which passes through a given point, and makes a given angle with a given straight line.
2. To determine the point of intersection of two given straight lines.
3. To find the angle contained between two given straight lines.
4. To find the equation to a straight line which bisects the angle contained by two given straight lines.

Note.—A point is said to be given, when the co-ordinates of that point are known; and a straight line is said to be given, when its equation is given.

THE TRANSFORMATION OF CO-ORDINATES.

25. Before we proceed to the discussion of equations of the second and higher orders, it is necessary to explain the method of changing the origin and direction of the co-ordinate axes; by which means the equation may often be considerably simplified, and the properties of the curve more easily investigated.

26. PROP. I.—*To change the origin of the co-ordinates, the new axes being parallel to the former.*



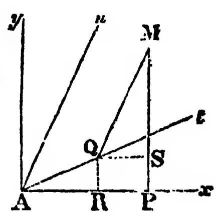
Let Ax, Ay, be the original axes; $A't$, $A'u$, the new axes, respectively parallel to the former. Take any point, M, in the curve, and draw the ordinate MQP parallel to Ay or $A'u$. Let $AC = a$, $A'C = b$, $AP = x$, $PM = y$, $A'Q = t$, $QM = u$. We have then

$$\left.\begin{aligned} x &= AP = AC + A'Q = a + t \\ y &= PM = A'C + QM = b + u \end{aligned}\right\}\ldots\ldots(A).$$

Substituting these values of x and y in the equation to the curve, we obtain a new equation in terms of t and u, the new co-ordinates.

When C is to the left of A, a will be negative; and if A' be below the axis Ax, b will be negative.

27. PROP. II.—*To change the directions of the co-ordinates, the new axes having the same origin as the former.*



Let Ax, Ay, be the original axes, At, Au, the new axes, and M any point in the curve. Let $AP = x$, $PM = y$, $AQ = t$, $QM = u$. Draw QR parallel to Ay and QS parallel to Ax; and let the angle tAx be represented by (tx), the angle xAy by (xy); and so on. Now,

$$x = AP = AR + RP = AR + QS$$
$$y = PM = PS + SM = RQ + SM.$$

We have also, by Trigonometry,

$$AR : AQ :: \sin AQR : \sin ARQ :: \sin(ty) : \sin(xy);$$
$$\therefore\ AR \sin(xy) = t \sin(ty).$$
$$QS : QM :: \sin QMS : \sin QSM :: \sin(uy) : \sin(xy);$$
$$\therefore\ QS \sin(xy) = u \sin(uy).$$

Hence, adding these two equations together,

$$x \sin(xy) = t \sin(ty) + u \sin(uy)\ldots\ldots\ldots(B).$$

Again, we have

$$RQ \quad AQ \quad \sin QAR : \sin ARQ :: \sin(tx) : \sin(yx);$$
$$\therefore\ RQ \sin(yx) = t \sin(tx).$$
$$SM \quad QM \quad \sin SMQS : \sin QSM :: \sin(ux) : \sin(yx);$$
$$\therefore\ SM \sin(yx) = u \sin(ux).$$

And adding these two equations together,

$$y \sin(yx) = t \sin(tx) + u \sin(ux)\ldots\ldots\ldots(B).$$

These equations being symmetrical, may easily be remembered in this form.

28. *Cor.* 1.—When the axis At is situated below Ax, the angle tAx or (tx) must be considered negative, and therefore also its sine will be negative (Trig. art. 51). In like manner, when the axis Au is to the left of Ay, the angle uAy must be considered negative.

29. *Cor.* 2.—If the angle $tAx = \alpha$, the angle $uAx = \beta$, and $yAx = 90°$; we have

$$\sin (tx) = \sin \alpha; \quad \sin (ty) = \cos \alpha; \quad \sin (xy) = 1;$$
$$\sin (ux) = \sin \beta; \quad \sin (uy) = \cos \beta.$$
$$\therefore \; x = t \cos \alpha + u \cos \beta; \quad y = t \sin \alpha + u \sin \beta \ldots\ldots (C).$$

30. *Cor.* 3.—If the axes of x and y be at right angles to each other, and also the axes of t and u, then will $\beta = 90° + \alpha$. Hence

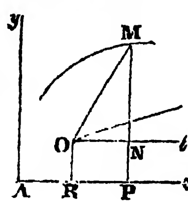
$$\sin \beta = \sin (90°+\alpha) = \sin (90°-\alpha) = \cos \alpha \text{ (Trig. art. 65)};$$
$$\cos \beta = \cos (90°+\alpha) = -\cos (90°-\alpha) = -\sin \alpha \text{ (Trig. art. 66)};$$
$$\therefore \; x = t \cos \alpha - u \sin \alpha; \quad y = t \sin \alpha + u \cos \alpha \ldots\ldots (D).$$

31. *Scholium.*—Since the values of x and y are in all cases expressed by simple equations, the degree of an equation can never be changed by the transformation of co-ordinates.

32. The position of a point upon a plane may also be determined by means of its distance from a given point, and the angle which that distance makes with a line given in position. The given point is called the *pole*, and the variable distance, the *radius vector*.

This angle and the radius vector are sometimes called *polar co-ordinates;* and the equation which expresses the relation subsisting between them, at any point in a curve, is called the *polar equation* to the curve.

33. Prop. III.—*To transform an equation between rectangular co-ordinates into another between polar co-ordinates; and conversely.*



(1). Let Ax, Ay, be the axes of x and y at right angles to each other; M any point in the curve; O the pole, and OM the radius vector. Draw Ot parallel to Ax; and put $AP = x$, $PM = y$, $AR = a$, $RO = b$, and the angle $MOt = \varphi$. We have then,

$$x = AP = a + r \cos \varphi$$
$$y = PM = b + r \sin \varphi.$$

Substituting these values in the equation between rectangular co-ordinates we obtain an equation between r and φ, or between polar co-ordinates.

(2.) *Conversely.—To find* r *and* φ *in terms of* x *and* y. We have, in the triangle OMN,

$$r^2 = (x - a)^2 + (y - b)^2$$

$$\tan \varphi = \frac{\;}{x - a} \qquad \sin \varphi = \frac{y - b}{r}; \qquad \cos \varphi = \frac{x - a}{\;}$$

34. *Cor.* 1.—If the axis Ot, from whence the angle φ is measured, be not parallel to Ax, but makes an angle α with it, when they are produced: we must substitute $\varphi \pm \alpha$ for φ in the preceding equations.

35. *Cor.* 2.—If the axes of x and y be oblique to each other, the investigation is nearly the same; but as the expressions are seldom required, and are more complicated, we shall not introduce them here.

CHAP. II.—LINES OF THE SECOND ORDER.

36. The general equation of the second degree between x and y is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots (1);$$

in which we may suppose the axes of the co-ordinates to be at right angles to each other, and not diminish the generality of the equation; for if they were not rectangular, they might easily be transformed by equations (B) into others which were so.

By resolving this equation as a quadratic, we obtain the value of y equal to

$$-\frac{bx}{2a} - \frac{d}{2a} \pm \frac{1}{2a}\sqrt{\left[(b^2 - 4ac)x^2 + (2bd - 4ae)x + d^2 - 4af\right]}$$

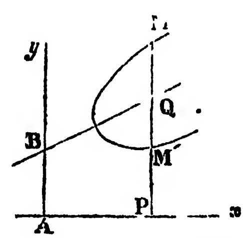
And if, for the sake of simplicity, we put

$$-\frac{b}{2a} = k, \quad -\frac{d}{2a} = l, \quad \frac{1}{2a} = m, \quad b^2 - 4ac = n, \text{ \&c.,}$$

we get

$$y = kx + l \pm m\sqrt{(nx^2 + px + q)} \ldots\ldots (2).$$

37. If we suppose BQ to be the straight line whose equation is $y = kx + l$, and from any point, Q, in this line take QM, QM', each equal to the irrational part $m\sqrt{(nx^2+px+q)}$; then will M, M', be two points in the curve; for $PQ = kx + l$, and therefore



$$PM = kx + l + m\sqrt{(nx^2 + px + q)}$$
$$PM' = kx + l - m\sqrt{(nx^2 + px + q)}.$$

38. Def.—The straight line BQ, which bisects all the lines MM', drawn parallel to the axis of y, is called a *diameter* of the curve.

If BQ bisects these lines at right angles, it is called an *axis of the curve*. This must be distinguished from the axes of x and y, which are called the axes of *co-ordinates*.

39. Prop. I.—*To determine the general form of the curve from the last equation.*

Let α, β, be the two roots of the equation $nx^2 + px + q = 0$; then we have, from the principles of Algebra (arts. 107 & 270),

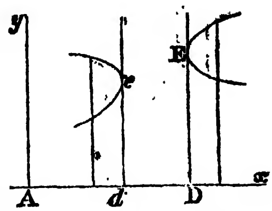
$$nx^2 + px + q = n(x - \alpha)(x - \beta),$$

for all values of x; and therefore

$$y = kx + l \pm m\sqrt{[n(x - \alpha)(x - \beta)]} \ldots\ldots (3).$$

This proposition, therefore, will divide itself into three different cases, according as n is positive, negative, or zero.

40. Prop. II.—*To determine the curve when* n *or* $b^2 - 4ac$ *is positive.*



(1.) Let the roots α, β, be real and unequal, and $\alpha > \beta$; also, let $AD = \alpha$, $Ad = \beta$. When $x > \alpha$, each of the factors under the radical, n, $x - \alpha$, $x - \beta$, is positive, consequently the values of y are real. However great, therefore, the values of x be taken, there are always two real corresponding values of y; hence there are two infinite arcs to the right of the ordinate DE. When $x < \alpha$ and $> \beta$, n and $x - \beta$ are positive, and $x - \alpha$ is negative; therefore $n(x-\alpha)(x-\beta)$ is negative, and the roots of y are imaginary. Hence no part of the curve lies between the ordinates DE and de. In this case, the curve between DE and de is said to be *imaginary*.

When $x < \beta$, $x - \alpha$ and $x - \beta$ are both negative, therefore $n(x-\alpha)(x-\beta)$ is again positive, and the values of y are again real, however great the negative values of x be taken. Hence there are two infinite branches to the left of de. This curve is called a *hyperbola*.

If $x = \alpha$, or $x = \beta$, the two values of y become equal to each other, and therefore the ordinates DE, de, are tangents to the curve, at the points E and e.

(2.) Let the roots α, β, be equal. We have in this case

$$y = kx + l \pm m\sqrt{[n(x-\alpha)(x-\alpha)]}$$
$$= kx + l \pm m(x-\alpha)\sqrt{n};$$

$$\therefore \quad y = (k + m\sqrt{n})x + l - m\alpha\sqrt{n}, \quad \text{or} \quad y = (k - m\sqrt{n})x + l + m\alpha\sqrt{n}.$$

These are the equations to two straight lines; and since the coefficients of x are unequal, these lines will cut the axis of x at different angles, and intersect each other.

(3.) Let the roots α, β, be imaginary.

These will be of the form $\gamma + \delta\sqrt{-1}$ and $\gamma - \delta\sqrt{-1}$. Hence

$$nx^2 + px + q = n(x - \gamma - \delta\sqrt{-1})(x - \gamma + \delta\sqrt{-1})$$
$$= n[(x-\gamma)^2 + \delta^2].$$

And since the expression within the brackets is the sum of two squares, it is always positive, whatever be the value of x; therefore the values of y are always real. Hence the curve consists of two infinite branches to the right of A, and two infinite branches to the left of A.

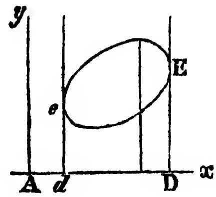
This curve also is a *hyperbola*; and it will afterwards be proved that this is the same as the first curve, with the axes in a different position.

41. Prop. III.—*To determine the curve when* n *or* $b^2 - 4ac$ *is negative.*

(1.) We have in this case

$$y = kx + l \pm m\sqrt{[-n(x-\alpha)(x-\beta)]}.$$

It may be proved, as in the last proposition, that when x is greater than either α or β, the expression under the radical, $-n(x-\alpha)(x-\beta)$, is negative, and the roots of y are imaginary. When $x < \alpha$ and $> \beta$, this expression is positive, and the values of y are real. When x is less



than either α or β, the roots of y again become imaginary. Hence the curve is limited by the lines DE, de; and no part of it lies to the right of DE, or to the left of de. This curve is of an oval form, and is called an ellipse.

(2.) When the roots α, β, are equal. In this case,

$$y = kx + l \pm m(x - \alpha)\sqrt{-n}.$$

Hence the values of y are always imaginary, except when $x = \alpha$; therefore the ellipse is reduced to a single point.

(3.) When the roots α, β, are imaginary, and of the form $\gamma \pm \delta\sqrt{-1}$. In this case,

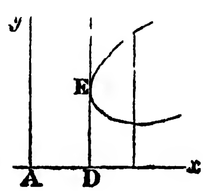
$$-nx^2 + px + q = -n[(x - \gamma)^2 + \delta^2],$$

which being always negative, the values of y are impossible; therefore the curve is imaginary.

42. PROP. IV.—*To determine the curve when* n *or* $b^2 - 4ac = 0$. We have in this case, $y = kx + l \pm m\sqrt{(px + q)}$;

or, putting $: -\frac{q}{p}$,

$$y = kx + l \pm m\sqrt{[p(x - \alpha)]}.$$



(1.) If p be positive, and $x > \alpha$, the values of y are real, and the curve has two infinite arcs to the right of DE. If $x < \alpha$, the values of y are imaginary, and therefore no part of the curve lies to the left of DE.

(2.) If p be negative, and $x > \alpha$, y is imaginary, and therefore no part of the curve lies to the right of DE. If $x < \alpha$, both p and $x - \alpha$ are negative; therefore the values of y are real, and the curve has two infinite arcs to the left of DE. This curve is called *a parabola.*

43. *Cor.*—If $n = 0$, $p = 0$, then $y = kx + l \pm m\sqrt{q}$, which are the equations to two straight lines when q is positive. These straight lines also are parallel, because the coefficients of x are equal (art. 11). If q be negative, y is impossible, and there is no curve at all.

44. PROP. V.—*To determine the form of the curve when* a $= 0$, *in equation* (1).

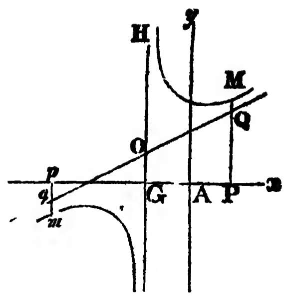
In this case, $(bx + d)y + cx^2 + ex + f = 0$;

$$\therefore \quad y = -\frac{cx^2 + ex + f}{bx + d}$$

nd by division, we obtain an equation of this form,

$$y = gx + h + \frac{r}{bx + d} \ldots\ldots\ldots\ldots\ (4).$$

Let OQ be the straight line whose equation is $y = gx + h$. At any point, P, draw PQ perpendicular to AP, and take $QM =$



$\frac{r}{bx + d}$, then M will be a point in the curve. It is evident that $\frac{r}{bx + d}$ diminishes as $bx + d$ increases, and that when $bx + d$ becomes indefinitely great, $\frac{r}{bx + d}$ becomes indefinitely small, and consequently the curve approaches indefinitely near to the line OQ, but yet never actually meets it.

If $bx + d = 0$, or $x = -\frac{d}{b}$, y is infinite; if, therefore, we take $AG = -\frac{d}{b}$, and draw GH parallel to Ay, the line GH will never meet the curve.

If $bx + d$ become negative, qm must be measured on the other side of the line Oq; and when $-(bx + d)$ is indefinitely great, $-\frac{r}{bx + d}$ will be indefinitely small; therefore the curve again approaches indefinitely near to the line Oq, but never meets it.

45. Def.—Straight lines which continually approach a curve, and come nearer to it than by any given distance, but being produced ever so far, never meet it, are called ***asymptotes.****

Examples.

To determine the limits and general form of the curves to which the following equations belong:—

1. $y^2 - 2xy + 2x^2 - 2y - 4x + 9 = 0$.
2. $y^2 + 2xy + 3x^2 - 4x = 0$.
3. $y^2 - 4xy + 2x^2 - 2y + 16x - 17 = 0$.
4. $y^2 - 4xy + 5x^2 + 2y - 4x + 2 = 0$.
5. $y^2 - 3xy + x^2 + 1 = 0$.
6. $y^2 - 6xy + 8x^2 + 2x - 1 = 0$.
7. $y^2 + 3xy + x^2 + y + x = 0$.
8. $y^2 - 2xy + x^2 - y - 1 = 0$.
9. $y^2 + 2xy + x^2 - 1 = 0$.
10. $2xy - 4x^2 + 6x - 3 = 0$.

REDUCTION OF THE GENERAL EQUATION OF THE SECOND DEGREE TO ITS MOST SIMPLE FORM.

46. Prop. VI.—*The second term in the general equation,*

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots\ldots (1),$$

* From ἀσύμπτωτος, which signifies *never coinciding.*

may be taken away by passing from one system of rectangular co-ordinates to another, the origin remaining the same.

Assume (formula D)

$$x = t\cos\alpha - u\sin\alpha, \qquad y = t\sin\alpha + u\cos\alpha,$$

and substitute these values in equation (1); we then obtain an equation of this form,

$$Au^2 + Btu + Ct^2 + Du + Et + f = 0,$$

in which B is equal to the expression

$$2(a-c)\sin\alpha\cos\alpha + b(\cos^2\alpha - \sin^2\alpha);$$

and in order that the term containing tu may disappear, we must put this expression $= 0$. And because (Trig. art. 74),

$$2\sin\alpha\cos\alpha = \sin 2\alpha; \quad \cos^2\alpha - \sin^2\alpha = \cos 2\alpha; \quad \text{we have}$$

$$0 = (a-c)\sin 2\alpha + b\cos 2\alpha \dots\dots\dots\dots (\beta.)$$

$$\therefore \quad \frac{\sin 2\alpha}{\cos 2\alpha} = \tan 2\alpha = \frac{b}{c-a}.$$

And since the tangent has all degrees of magnitude, from 0 to $+\infty$, or from 0 to $-\infty$, the arc 2α is always real, and therefore B may be always made to disappear.

Hence equation (1) can always be reduced to an equation of the form

$$Au^2 + Ct^2 + Du + Et + f = 0 \dots\dots\dots (5).$$

47. Prop. VII.—*To find the values of* A *and* C, *in terms of the coefficients of equation* (1).

We have from the transformation in the last proposition,

$$A = a\cos^2\alpha - b\sin\alpha\cos\alpha + c\sin^2\alpha$$
$$C = a\sin^2\alpha + b\sin\alpha\cos\alpha + c\cos^2\alpha.$$

Hence, adding and subtracting the last equation, we get

$$A + C = a + c \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (\gamma).$$
$$A - C = a(\cos^2\alpha - \sin^2\alpha) - 2b\sin\alpha\cos\alpha - c(\cos^2\alpha - \sin^2\alpha);$$
$$\therefore \quad A - C = (a-c)\cos 2\alpha - b\sin 2\alpha.$$

To eliminate α, add the square of the last equation to the square of equation (β), and we obtain

$$(A-C)^2 = (a-c)^2 + b^2;$$
$$\therefore \quad A - C = \sqrt{[(a-c)^2 + b^2]} \dots\dots\dots (\gamma).$$

Taking half the sum and half the difference of equations (β) and (γ), we get

$$A = \tfrac{1}{2}(a+c) \pm \tfrac{1}{2}\sqrt{[(a-c)^2 + b^2]}$$
$$C = \tfrac{1}{2}(a+c) \mp \tfrac{1}{2}\sqrt{[(a-c)^2 + b^2]};$$

the sign of the radical in C being contrary to that in A.

48. *Cor.* 1.—When $b^2 - 4ac = 0$, or the curve is a parabola,

$$(a-c)^2 + b^2 = (a-c)^2 + 4ac = (a+c)^2;$$
$$\therefore \quad A = a + c, \text{ or } 0; \qquad C = 0, \text{ or } a + c.$$

Hence, in the case of the parabola, when B is made to vanish, either

A or C vanishes at the same time. If we suppose $C = 0$, equation (5) is reduced to

$$Au^2 + Du + Et + f = 0 \ldots\ldots\ldots\ldots\ldots (6).$$

49. *Cor.* 2.—If E also vanishes, the equation becomes

$$Au^2 + Du + f = 0 \ldots\ldots\ldots\ldots\ldots (7).$$

50. *Cor.* 3.—When $b^2 - 4ac$ is positive, or $b^2 > 4ac$, then $\sqrt{[(a-c)^2 + b^2]} > a + c$. Hence it is evident that A and C will have different signs. When $b^2 - 4ac$ is negative, A and C will have the same sign.

51. PROP. VIII.—*The simple powers of* t *and* u, *in equation* (5), *may be taken away, by changing the origin of co-ordinates; the new axes being parallel to the last.*

$$\text{Since} \quad Au^2 + Du = A\left(u^2 + \frac{D}{A}u\right) = A\left(u + \frac{D}{2A}\right)^2 - \frac{D^2}{4A}$$

$$Ct^2 + Et = C\left(t^2 + \frac{E}{C}t\right) = C\left(t + \frac{E}{2C}\right)^2 - \frac{E^2}{4C}$$

If we put

$$u + \frac{D}{2A} = y, \quad t + \frac{E}{2C} = x, \quad f - \frac{D^2}{4A} - \frac{E^2}{4C} = -F,$$

we shall obtain the equation

$$Ay^2 + Cx^2 = F \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

52. *Cor.*—It is manifest that the first power of t cannot be taken away in equation (6), but the equation may be simplified by putting, as in this proposition, $u + \dfrac{D}{2A} = y$, and also

$$Et + F - \frac{D^2}{4A} = E\left[t + \frac{F}{E} - \frac{D^2}{4AE}\right] = Ex.$$

Equation (6) will then become

$$Ay^2 + Ex = 0 \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

By the same substitution, equation (7) will become $Ay^2 = F$ equation is included in equation (8), C in this case being = '

53. *Scholium.*—From the two last propositions it appea general equation (1) can always be reduced to one of the tw

$$My^2 + Nx^2 = P; \qquad My^2 + Nx = 0.$$

And we shall now consider the nature of the lines represente equations.

54. PROP. IX.—*To determine the form of the curve lines to the two last equations belong,*

If we divide every term of the first equation by P, we get

$$\frac{M}{P}y^2 + \frac{N}{P}x^2 = 1 \ldots$$

Now the coefficients $\frac{M}{P}$, $\frac{N}{P}$, may be either both positive, or one positive and the other negative, or both negative. In the first case, put $\frac{P}{M} = b^2$, $\frac{P}{N} = a^2$, and the equation becomes

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1; \quad \text{or,} \quad a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots\ldots (E).$$

The curve represented by this equation will be shown in article 65 to be of an oval form, and is called an ellipse.

If the first of these coefficients be positive and the other negative, put $\frac{P}{M} = b^2$, and $\frac{P}{N} = -a^2$. the equation then becomes

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \quad \text{or,} \quad a^2y^2 - b^2x^2 = a^2b^2 \ldots\ldots\ldots (H).$$

This curve is called a hyperbola, and its form will be considered in article 130. If the first of these coefficients be negative and the other positive, the curve is still a hyperbola, and has the same form, the two axes having interchanged places.

If both these coefficients be negative, the equation is manifestly impossible, and the curve is imaginary.

If any of the coefficients M, N, or $P = 0$, that equation will either represent two straight lines, or the lines will be imaginary. Thus, if $N = 0$, $My^2 = P$ will represent two parallel straight lines, when M and P have the same sign. If $P = 0$, the equation $My^2 + Nx^2 = 0$ will represent two straight lines, which intersect each other when M and N have different signs.

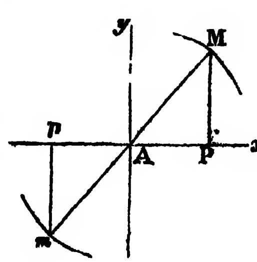
(2.) If we divide every term of the second equation $My^2 + Nx = 0$ by M, and put $\frac{N}{M} = -2p$, we get

$$y^2 = 2px \ldots\ldots\ldots\ldots\ldots\ldots\ldots (P).$$

The locus of this equation is called a parabola, and it will be discussed in article 191. The curve has the same form, whether p be positive or negative.

55. Def.—When every chord passing through a point is bisected by it, this point is called the centre of the curve.

56. Prop. X.—*The origin of co-ordinates, in the curve whose equation is* $My^2 + Nx^2 = P$, *is a centre of the curve.*



For if we substitute $-x$ and $-y$ for $+x$ and $+y$, we shall have the same equation, $My^2 + Nx^2 = P$. Consequently the point m, whose co-ordinates are $-x$, $-y$, is a point in the curve. We have also, in the triangles APM, Apm, $AP = Ap$, $PM = pm$, and the angle $APM = Apm$, therefore the angle $MAP = mAp$; and MAm is a straight line, bisected in A.

57. *Scholium.*—Whenever the equation

remains the same on the substitution of $-x$ and $-y$ for $+x$ and $+y$, the curve has a centre, in which is placed the origin of co-ordinates. And this will be the case,

(1.) When the sum of the indices of x and y is even, whether there be a constant term or not; as

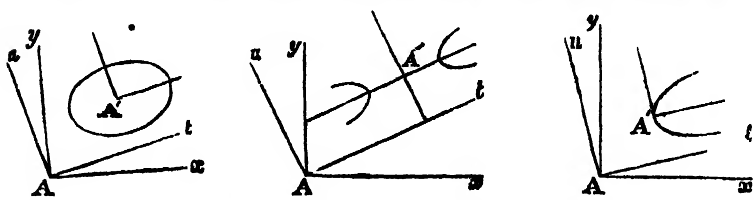
$$Ay^4 + Bxy + Cx^4 + D = 0.$$

(2.) When the sum of the indices of x and y is odd, and there is no constant term; as

$$Ay^3 + Bxy^2 + Cy + Dx = 0.$$

58. Prop. XI.—*To explain the changes which have taken place in the position of the axes, in consequence of the alteration in the form of the original equation.*

These will be sufficiently understood from the following diagrams:—



(1.) In each of these figures the lines Ax, Ay, represent the position of the axes to the curve, in the original equation (1).

(2.) In the first transformation (prop. 6), the origin remains at A, but the curve is referred to the new rectangular axes, At, Au; and the corresponding equation to the curve is equation (5), in the ellipse and hyperbola; and equation (6), in the parabola.

In the second transformation (prop. 8), the origin is transferred from A to A', the centre of the curve in the ellipse and hyperbola, and to A', the vertex in the parabola. The new axes are parallel to the former, At, Au.

59. Prop. XII.—*A straight line cannot cut a line of the second order in more than two points.*

Let $y = gx + h$ be the equation to a straight line, and $ay^2 + bxy + cx^2 + dy + ex + f = 0$ the equation to a line of the second order. Now, at the points of intersection, the values of x and y in the straight line are the same as those in the curve. If, therefore, we substitute $gx + h$ for y in the equation of the second degree, we shall evidently have an equation of the form

$$Mx^2 + Nx + P = 0.$$

From this quadratic are obtained two values of x, which, substituted in the equation to the straight line, give two corresponding values of y; and these will give the two points in which the line cuts the curve.

60. *Cor.* 1.—When the two roots of the quadratic are equal, the points of intersection coincide, and the straight line is a tangent to the curve. When the roots are imaginary, the straight line falls entirely without the curve.

61. *Cor.* 2.—In the same manner it may be proved that a straight line cannot intersect a line of the nth order in more than n points.

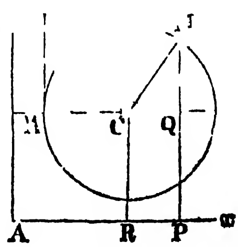
CHAP. III.—THE CIRCLE AND THE ELLIPSE.

62. The equation to the ellipse, in art. 51, is

$$a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots\ldots\ldots (E);$$

and if $a = b$, the ellipse becomes a circle. Before we proceed, therefore, to deduce the various properties of the ellipse, we shall briefly consider the different forms of the equation to the circle.

THE CIRCLE.



63. If we put $b = a$ in the preceding equation, it becomes

$$y^2 + x^2 = a^2;\ \text{or},\ CQ^2 + QM^2 = a^2 \ldots (1).$$

This equation shows that every point M, in the curve, is at the same distance from the origin of co-ordinates, C; and therefore the curve is a circle whose centre is C, and radius $= CM = a$.

64. If we produce QC to the circumference A', and put $A'Q = x' = x + a$, we have $x = x' - a$, and the preceding equation becomes $y^2 + (x' - a)^2 = a^2$; or,

$$y^2 = 2ax' - x'^2 \ldots\ldots\ldots\ldots (2).$$

65. Prop.—*To find the equation to the circle, when any point, within or without the circle, is assumed as the origin of co-ordinates.*

Let the point A be the origin of co-ordinates, and Ax, Ay, the rectangular axes. Let M be any point in the circumference; draw MP, CR, perpendicular to Ax. Put $AP = x$, $PM = y$, $AR = \alpha$, $RC = \beta$; we have then, $CQ^2 + QM^2 = CM^2$, or

$$(x - \alpha)^2 + (y - \beta)^2 = a^2 \ldots\ldots\ldots (3),$$

which is the most general equation to the circle when the axes are rectangular.

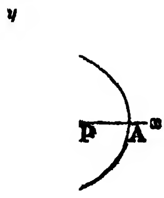
66. *Scholium.*—If, in the general equation of the second degree (art. 36), $b = 0$, and $a = c$, this may evidently be reduced to the same form as equation (3), and consequently the curve will be a circle. And if this equation be transformed to another where the origin remains the same and the axes are rectangular, we shall also find $B = 0$ and $A = C$ (arts. 46, 47). This result might have been anticipated, since it appears from the last article that the rectangle xy cannot enter into the equation of the circle. The general equation to the circle, therefore, when the axes are rectangular, is of the form

$$y^2 + x^2 + dy + ex + f = 0 \ldots (4).$$

Since the circle may be considered as a *species of ellipse*, where a is equal to b in equation (E); we shall not at present dwell on the properties of the circle, but proceed to the more general form of the equation, where the values of a and b are unequal.

THE ELLIPSE.

67. Prop. I.—*To determine the figure of the ellipse from its equation.*



The origin of the co-ordinates being at the centre, the equation to the ellipse is

$$a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots (E);$$

$$\therefore \quad y = \pm \frac{b}{a}\sqrt{(a^2 - x^2)}.$$

When $x = 0$, $y = \pm b$. When x is positive, and $< a$, y has two equal values, with contrary signs. When $x = a$, $y = 0$. When $x > a$, y is impossible, and therefore no part of the curve lies to the right of A. Hence the two opposite arcs BMA, bmA, are equal and similar. It is also evident that y continually diminishes from $x = 0$ to $x = a$.

If x be negative, y has the same values as when x is positive; the same values of y must therefore recur, from $x = 0$ to $x = -a$; and there are two equal and opposite arcs, Ba, ba. Hence the whole curve is divided into two equal parts by the axis of x, and also into two equal parts by the axis of y; and it is therefore symmetrical with respect to these axes.

The same properties might have been deduced by finding the value of x in terms of y.

68. *Cor.*—From the equation to the curve, we have.

$$CM = \sqrt{(x^2+y^2)} = \sqrt{\left[x^2 + \frac{b^2}{a^2}(a^2 - x^2)\right]} = \sqrt{\left[b^2 + \frac{a^2-b^2}{a^2}x^2\right]}.$$

Therefore CM is least when x is least in magnitude, that is, when $x = 0$; in which case, CM becomes equal to b or CB. Also, CM is greatest when x is greatest in magnitude, that is, when $x = Ca$ or CA; in which case, CM becomes also equal to a. Hence it follows that CM continually increases from CB or Cb, its least value, to CA or Ca, its greatest value.

69. The lines Aa, Bb, are evidently *axes of the curve* (art. 38). The greater axis, Aa, is called the *transverse* or *major axis*, and the lesser axis, Bb, the *conjugate* or *minor axis*.

The points A, a, B, b, are called the *vertices* of the curve.

70. Prop. II.—*To find the equation to the ellipse when the co-ordinates are measured from the vertex.*

Let PM, as before, $= y$ and $aP = x'$, then $x = aP - aC = x' - a$, therefore

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) = \frac{b^2}{a^2}[a^2 - (x' - a)^2];$$

$$\therefore \quad y^2 = \frac{b^2}{a^2}(2ax' - x'^2) \ldots\ldots\ldots\ldots (E').$$

This equation, expressed in the form of a proportion, becomes

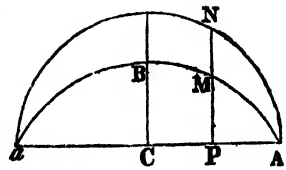
$$a^2 : b^2 :: x'(2a - x') : y^2 \qquad \text{that is,}$$

$$CA^2 : CB^2 :: aP \times PA : PM^2.$$

71. *Cor.* 1.—If $P'M'$ be any other ordinate, then (see the last figure)

$$PM^2 : P'M'^2 :: aP \times PA : aP' \times P'A.$$

72. *Cor.* 2.—If a circle be described upon Aa as diameter, and the ordinate PM be produced to meet the circumference in N, then $PN^2 = aP \times PA$; therefore



$$CA^2 : CB^2 :: PN^2 : PM^2;$$

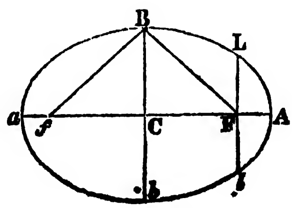
or, $CA : CB :: PN : PM.$

Hence the ordinate in the ellipse has to the corresponding ordinate of the circle, the constant ratio of the axis minor to the axis major.

73. *Cor.* 3.—In like manner, if a circle be described upon Bb as diameter, and corresponding ordinates in the ellipse and circle be drawn to this axis, the ordinate in the ellipse will have to the corresponding ordinate of the circle, the constant ratio of the axis major to the axis minor.

THE FOCI.

74. If in equation (E') we put $\frac{b^2}{a} = p$, and suppress the accents of x, this equation becomes



$$y^2 = 2px - \frac{p}{a}x^2.$$

The line $2p$ is called the *principal parameter*, or *latus rectum;* and by the definition it is a third proportional to the transverse and conjugate axes.

The *foci*, F, f, are two points in the transverse axis, where the ordinates are equal to half the latus rectum. The reason of this name will be afterwards explained.

The distance, CF, between the focus and centre is called the *excentricity* of the ellipse. In some treatises on Conic Sections, the ratio $\frac{CF}{CA}$ is called the excentricity.

75. Prop. III.—*To find the distance of the focus from the centre of the ellipse.*

Let the origin of the co-ordinates be at the centre. Also let $CF = c$; then, by the definition, $FL = p = \frac{b^2}{a}$. Substitute these values for x and y in equation (E), we then obtain

$$a^2\frac{b^4}{a^2} + b^2c^2 = a^2b^2, \text{ or, } b^2 + c^2 = a^2.$$

$$\therefore \quad c = \sqrt{(a^2 - b^2)}.$$

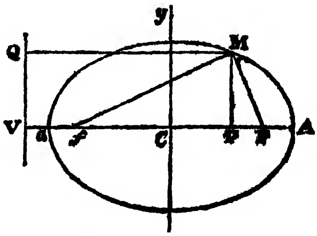
76. *Cor.* 1.—Since $a^2 = b^2 + c^2$, by this proposition; and in the right-angled triangle BCF, $BF^2 = b^2 + c^2$,

$$\therefore \quad BF = a; \text{ and in like manner } Bf = a.$$

77. *Cor.* 2.—Since $b^2 = a^2 - c^2 = (a + c)(a - c)$,

$$\therefore \quad aF \times FA = BC^2.$$

78. Prop. IV.—*To determine the distance of the focus from any point in the curve.*



Let $CF = Cf = c$. We have then

$$FM^2 = (c - x)^2 + y^2$$

$$= c^2 - 2cx + x^2 + \frac{b^2}{a^2}(a^2 - x^2)$$

$$= c^2 + b^2 - 2cx + x^2 - \frac{b^2}{a^2}$$

But $c^2 + b^2 = a^2$; and

$$x^2 - \quad a^2 - b^2 \quad \frac{c^2x^2}{a^2}.$$

Hence $FM^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$; and putting $\frac{c}{a} = e$,

$$FM = a - \frac{cx}{a} = a - ex.$$

In like manner, we shall find

$$fM = a + \frac{cx}{a} = a + ex.$$

79. *Cor.* 1.—If CV be taken a third proportional to Cf and CA, and VQ be drawn perpendicular, and MQ parallel to CV; we have

$$MQ = VP = \frac{a^2}{c} + x = \frac{a^2 + cx}{c}$$

Also $$fM = a + \frac{cx}{a} = \frac{a^2 + cx}{a}.$$

$$\therefore \ fM : MQ :: c : a.$$

The line VQ is called the *directrix* of the curve, and by some writers this constant ratio is taken for the definition of the ellipse.

80. *Cor.* 2.—Adding the two focal distances together, we get this remarkable expression,

$$FM + fM = 2a;$$

that is, the sum of the two lines drawn from the foci to any point in the curve is equal to the major axis. In many geometrical treatises, this equation is taken for the definition of the ellipse; and as it is one of its most distinguishing properties, we shall take the converse proposition, and from this definition prove that the curve is the same as that which we have called an ellipse.

81. Prop. V.—*To find the locus of a point whose distances from two fixed points are together always equal to a given line, 2a.*

Let F, f, be the two given points, M any point in the curve. Join Ff, and bisect it in C. Produce Ff indefinitely, and draw Cy, MP,

perpendicular to Ff. Assume Cx, Cy, for the axes of co-ordinates, and put $CP = x$, $PM = y$, $CF = Cf = c$. Also, let $FM = a - z$; and because $FM + fM = 2a$, therefore $fM = a + z$. Now in the right-angled triangles FMP, fMP, we have

$$y^2 + (c - x)^2 = (a - z)^2$$
$$y^2 + (c + x)^2 = (a + z)^2.$$

From which equations we must eliminate z. Subtracting the first equation from the second, we have

$$4cx = 4az; \text{ or, } \quad z = \frac{cx}{a}.$$

Substituting this value of z in either of the preceding equations, we get

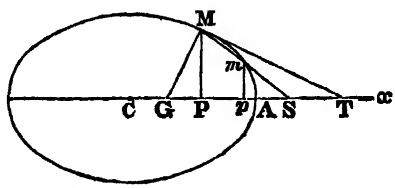
$$y^2 + c^2 + x^2 = a^2 + \frac{c^2x^2}{a^2};$$

$$\therefore\ a^2y^2 + (a^2 - c^2)\,x^2 = a^2(a^2 - c^2). \text{ And putting } a^2 - c^2 = b^2,$$
$$a^2y^2 + b^2x^2 = a^2b^2,$$

which is the equation to an ellipse whose axes are $2a$ and $2b$.

THE TANGENT AND NORMAL.

82. In Geometry, a straight line is said to *touch* a circle, when, being produced, it meets the circumference in one point only; but in order to have a definition which is equally true for all curves, and of easy application in Analytical Geometry, it is usual to define the tangent as follows:—



83. Def.—If a secant, MmS, be drawn, cutting any curve in two points, M, m; and if it turn about the point M, until m moves up to M, and at length coincides with it, the secant, in its ultimate position MT, will be a *tangent* to the curve.

The part of the axis PT, intercepted between the ordinate and the tangent, is called the *subtangent*.

A straight line, MG, perpendicular to the tangent at the point of contact, is called a *normal*.

The part of the axis PG, intercepted between the ordinate and the normal, is called the *subnormal*.

84. Prop. VI.—*To find the equation to a tangent to the ellipse.*

Let x', y', be the co-ordinates of the given point, and x'', y'', those of any other point in the ellipse. The equation to a straight line drawn through these two points is (art. 17)

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \ldots\ldots\ldots\ldots (1).$$

And because these two points are in the curve, we have

$$a^2y'^2 + b^2x'^2 = a^2b^2; \qquad a^2y''^2 + b^2x''^2 = a^2b^2;$$
$$\therefore\ a^2(y''^2 - y'^2) + b^2(x''^2 - x'^2) = 0;$$
$$\text{or,} \quad a^2(y'' + y')(y'' - y') + b^2(x'' + x')(x'' - x') = 0;$$

$$\therefore \quad \frac{y''-y'}{x''-x'} = -\frac{b^2}{a^2}\frac{x''+x'}{y''+y'}.$$

Hence equation (1) becomes, by substitution,

$$y - y' = -\frac{b^2}{a^2}\frac{x''+x'}{y''+y'}(x - x') \ldots\ldots (2).$$

This is the equation to the secant, passing through the two points (x', y'), (x'', y''). Let now the point (x'', y''), be supposed to move up to the point (x', y'), and ultimately to coincide with it, the secant then becomes a tangent (art. 80), and $x'' = x'$, $y'' = y'$; hence the equation to the tangent is

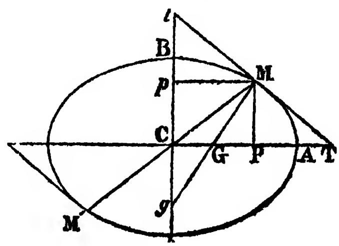
$$y - y' = -\frac{b^2}{a^2}\frac{x'}{y'}(x - x').$$

Multiplying by a^2y', and transposing, and also substituting a^2b^2 for $a^2y'^2 + b^2x'^2$, we obtain

$$a^2yy' + b^2xx' = a^2b^2 \ldots\ldots\ldots\ldots (T),$$

the equation to the tangent most frequently used.

85. Prop. VII.—*To find the intersection of the tangent with the two axes.*



(1.) The equation to the tangent being

$$a^2yy' + b^2xx' = a^2b^2,$$

let it cut the axis of x at T; then at this point $y = 0$; therefore

$$b^2xx' = a^2b^2, \text{ and } CT = x = \frac{a^2}{x'};$$

$$\therefore \quad CP \times CT = CA^2.$$

(2.) Again, let the tangent cut the axis of y at t. In this case $y = Ct$, and $x = 0$, therefore

$$a^2yy' = a^2b^2, \quad \text{and} \quad Ct = y = \frac{b^2}{y'};$$

$$\therefore \quad Cp \times Ct = CB^2.$$

Hence it follows, that each of the semiaxes is a mean proportional between the abscissa and the part of the axis intercepted between the centre and the tangent.

86. *Cor.*—Since the value of CT is independent of the minor axis b and the ordinate y', it will be constant for the same abscissa, in all ellipses described upon the same major axis Aa. Also, since the circle may be considered as an ellipse whose axes are equal to each other; if a circle be described with the centre C and radius CA, and the ordinate PM be produced to meet the circumference of this circle in N, the tangents to the ellipse and the circle, at the points M and N, will meet the axis in the same point, T.

87. Prop. VIII.—*To find the trigonometrical tangent and sine of the angle, which a tangent to the ellipse makes with the major axis.*

(1.) By transposition and division, equation (T) becomes

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}.$$

If we compare this with the general equation to the straight line

$y = Ax + B = x\tan\theta + B$, we have

$$A = \tan\theta = -\frac{b^2x'}{a^2y'},$$

the trigonometrical tangent of the angle which the tangent to the ellipse makes with the axis.

(2.) We have likewise

$$\sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta} = \frac{b^4x'^2}{a^4y'^2 + b^4x'^2};$$

also, $\quad a^4y'^2 + b^4x'^2 = a^2(a^2b^2 - b^2x'^2) + b^4x'^2$

$$= a^4b^2 - (a^2 - b^2)b^2x'^2 = a^4b^2 - b^2c^2x'^2;$$

$$\therefore \quad a^4y'^2 + b^4x'^2 = a^2b^2(a^2 - e^2x'^2).$$

Hence $\quad \sin\theta = \dfrac{bx'}{a\sqrt{(a^2 - e^2x'^2)}}.$

88. *Cor.* 1.—If α be the angle which CM makes with the axis CA then $\tan\alpha = \dfrac{y'}{x'}$; therefore

$$\tan\theta = -\frac{b^2}{a^2\tan\alpha}; \quad \text{or,} \quad \tan\alpha\tan\theta = -\frac{b^2}{a^2}.$$

89. *Cor.* 2.—If we substitute $-x'$ and $-y'$ for $+x'$ and $+y'$, we shall still find $\tan\theta$ equal to $-\dfrac{b^2x'}{a^2y'}$. Hence, if MC be produced (see the preceding figure) to meet the curve again in M', the tangents at M and M' will be parallel.

90. Prop. IX.—*To find the equation to the normal, at any point,* (x', y'), *in the ellipse.*

Since the normal passes through the point (x', y'), its equation will be of the form (art. 18)

$$y - y' = A'(x - x').$$

Also, if $y = Ax + B$ be the equation to the tangent at the point (x', y'), we have $A' = -\dfrac{1}{A}$ (art. 23), because the normal is perpendicular to the tangent. But (art. 87)

$$A = -\frac{b^2x'}{a^2y'}; \quad \therefore \quad A' = \frac{a^2y'}{b^2x'}.$$

Hence the equation to the normal is

$$y - y' = \frac{a^2y'}{b^2x'}(x - x')\ldots\ldots(N).$$

91. Prop. X.—*To find the intersection of the normal with the axes.*

(1.) Let the normal MG cut the axis aA in G, and the axis Bb in g. If, then, in the last equation we put $y = 0$, we have $x = CG$, and the equation becomes

$$-y' = \frac{a^2y'}{b^2x'}(x - x'); \quad \therefore \quad x - x' = -\frac{b^2x'}{a^2}.$$

Hence $$CG = \frac{a^2 - b^2}{a^2}x' = e^2x'.$$

(2.) Again, if in equation (*N*) we put $x = 0$, we have $y = Cg$, and the equation becomes $y - y' = -\frac{a^2y'}{b^2}$; therefore

$$Cg = -\frac{a^2 - b^2}{b^2}y' = -\frac{e^2}{1 - e^2}y'.$$

92. *Cor.* 1.—Since $GP = x' - e^2x' = \frac{b^2}{a^2}x'$, we have

$$MG^2 = y'^2 + \frac{b^4x'^2}{a^4} = \frac{a^4y'^2 + b^4x'^2}{a^4}$$

$$= \frac{a^2b^2(a^2 - e^2x'^2)}{a^4} \text{ (art. 87).}$$

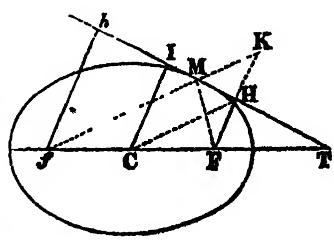
$$\therefore \quad MG = \frac{b}{a}\sqrt{(a^2 - e^2x'^2)}.$$

93. *Cor.* 2.—In like manner, since $pg = y' + \frac{e^2y'}{1 - e^2} = \frac{a^2}{b^2}y'$, we have

$$Mg = x'^2 + \frac{a^4y'^2}{b^4} = \frac{b^4x'^2 + a^4y'^2}{b^4};$$

$$\therefore \quad Mg = \frac{a}{b}\sqrt{(a^2 - e^2x'^2)}.$$

94. Prop. XI.—*To find the lengths of the perpendiculars from the centre, and the foci upon the tangent.*



(1.) Let $CI = P$, $FH = p$, $fh = p'$, $FM = r$, $fM = r'$; we have then

$$P = CT \sin\theta = \frac{a^2}{x'}\frac{bx'}{a\sqrt{(a^2 - e^2x'^2)}}$$

$$= \frac{ab}{\sqrt{(a^2 - e^2x'^2)}}.$$

(2.) Next we have

$$p = FT\sin\theta = (CT - CF)\sin\theta$$

$$= \left(\frac{a^2}{x'} - ae\right)\frac{bx'}{a\sqrt{(a^2 - e^2x'^2)}}$$

$$= \frac{b(a - ex')}{\sqrt{(a^2 - e^2x'^2)}} = b\sqrt{\left(\frac{a - ex'}{a + ex'}\right)} = b\sqrt{\frac{r}{r'}}.$$

And in like manner we find

$$p' = b\sqrt{\left(\frac{a + ex'}{a - ex'}\right)} = b\sqrt{\frac{r'}{r}}.$$

95. *Cor.*—Hence $pp' = b^2$.

96. **Prop. XII.**—*The tangent* Hh *makes equal angles with the focal distances* FM, fM.

Let the angle $FMH = \varphi$, and the angle $fMh = \varphi'$; then

$$\sin\varphi = \frac{p}{r} = b\sqrt{\frac{r}{r'}} \times \frac{1}{r} = \frac{b}{\sqrt{(rr')}};$$

$$\text{also,} \qquad \sin\varphi' = \frac{r}{r'} = \frac{b}{\sqrt{(rr')}}.$$

Hence $\sin\varphi = \sin\varphi'$, therefore φ is either equal to φ' or $180 - \varphi'$. But φ is evidently not equal to fMH or $180 - \varphi'$, therefore $\varphi = \varphi'$.

97. *Cor.*—$\sin\varphi = \dfrac{b}{\sqrt{(rr')}} = \dfrac{b}{\sqrt{(a^2 - e^2x'^2)}}.$

Scholium.

98. From this property of the ellipse, the two points F, f, are called *foci*, or burning points. For if the concavity of the ellipse be a polished surface, rays of light and heat issuing from one focus, will by the laws of optics be reflected to the other. In the parabola, rays of light parallel to the axis will be reflected to the focus; and in the hyperbola, rays of light proceeding towards one focus will be reflected to the other.

99. **Prop. XIII.**—*The locus of the point* H *is the circumference of a circle, described upon the major axis as diameter.*

Draw CI perpendicular to the tangent, and join CH. We have then

$$CH^2 = CI^2 + IH^2 = CT^2 \sin\theta + CF^2\cos^2\theta$$

$$= \frac{a^2}{x'^2}\sin^2\theta + a^2e^2(1 - \sin^2\theta)$$

$$= a^2e^2 + \left(\frac{a^4}{x'^2} - a^2e^2\right)\sin^2\theta = a^2e^2 + \frac{a^2(a^2 - e^2x'^2)}{x'^2}\sin^2\theta.$$

$$\text{But} \qquad \sin^2\theta = \frac{b^2x'^2}{a^2(a^2 - e^2x^2)} \text{ (art. 83)},$$

$$\therefore \quad CH^2 = a^2e^2 + b^2 = a^2.$$

Hence $CH = a$; and the locus of the point H is the circumference of a circle, described with the centre C, and radius CA.

100. *Scholium.*—This proposition admits of a very simple and elegant geometrical demonstration. For if FH, fM, be produced to meet in K, the angles FMH, HMK, are equal (art. 96); and it may easily be shown that

$$fK = fM + MK = fM + MF = 2a,$$

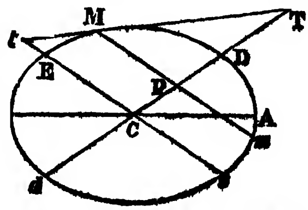
and that CH is parallel to fK. Hence

$$CH = \tfrac{1}{2}fK = a.$$

THE DIAMETERS AND SUPPLEMENTARY CHORDS.

101. **Def.**—Two diameters are called *conjugate*, when each bisects the chords parallel to the other.

102. PROP. XIV.—*The equation to the curve between the rectangular co-ordinates may be changed into another, in which the oblique axes are diameters conjugate to each other.*



Let CD, CE, be two new axes, drawn through the centre C; also, let the angle $DCA = \alpha$, $ECA = \beta$. Draw any chord, MPm, parallel to Ee, and let $CP = t$, $PM = u$. We have then, from art. 29,

$$x = t \cos \alpha + u \cos \beta;$$
$$y = t \sin \alpha + u \sin \beta;$$

and substituting these values in equation (E), $a^2y^2 + b^2x^2 = a^2b^2$, we obtain an equation of the form

$$Au^2 + Btu + Ct^2 = a^2b^2 \quad \ldots\ldots\ldots\ldots (1),$$

in which

$$A = a^2 \sin^2 \beta + b^2 \cos^2 \beta; \qquad C = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha;$$
$$B = 2a^2 \sin \alpha \sin \beta + 2b^2 \cos \alpha \cos \beta \ldots. (2).$$

If now we put $B = 0$, the equation will be reduced to

$$Au^2 + Ct^2 = a^2b^2 \ldots\ldots\ldots\ldots (3);$$

and since each value of t gives two equal values of u with contrary signs, the axis of t is a diameter of the curve. In like manner, each value of u gives two equal values of t with contrary signs, therefore the axis of u also is a diameter. Hence, since each of these axes bisects all the chords drawn parallel to the other, they are conjugate diameters.

103. *Cor.* 1.—In dividing the equation

$$B = 2a^2 \sin \alpha \sin \beta + 2b^2 \cos \alpha \cos \beta = 0$$

by $2a^2 \cos \alpha \cos \beta$, it becomes

$$\tan \alpha \tan \beta = -\frac{b^2}{a^2}.$$

Whatever value, therefore, be given to α, β will always have a real value. Hence it follows that every line drawn through the centre is a diameter, and that every diameter has its conjugate.

104. *Cor.* 2.—If a tangent be drawn at the extremity of the diameter Dd, we have, from art. 88,

$$\tan \alpha \tan \theta = -\frac{b^2}{a^2} = \tan \alpha \tan \beta;$$

hence $\tan \theta = \tan \beta$, and $\theta = \beta$. The diameter Ee, therefore, is parallel to the tangent at D.

105. *Cor.* 3.—If we divide equation (3) by a^2b^2, and put

$$b'^2 = \frac{a^2b^2}{A} = \frac{a^2b^2}{a^2 \sin^2 \beta + b^2 \cos^2 \beta};$$
$$a'^2 = \frac{a^2b^2}{C} = \frac{a^2b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha},$$

we get, in substituting x and y for t and u,

$$\frac{y^2}{b'^2} + \frac{x^2}{a'^2} = 1 \quad \text{or,} \quad a'^2y^2 + b'^2x^2 = a'^2b'^2 \ldots\ldots (e).$$

And when $x = 0$, $y = \pm b'$; when $y = 0$, $x = \pm a'$. Hence the lengths of the conjugate diameters are $2a'$, $2b'$.

106. *Scholium.*—Since equation (e) has exactly the same form as the equation between rectangular co-ordinates, it is evident that all the properties which are independent of the inclination of the ordinates will be equally true for the conjugate diameters as for the two axes. It follows, therefore, that the squares of the ordinates to any diameter are proportional to the rectangles of the abscissæ; or

$$CD^2 : CE^2 :: dP \times PD : PM^2.$$

Also, the equation to the tangent, at any point, (x', y'), is

$$a'^2yy' + b'^2xx' = a'^2b'^2.$$

And lastly, if T, t, be the intersections of the tangent with the two conjugate diameters dD, eE; then

$$CP \times CT = CD^2; \qquad Cp \times Ct = CE^2.$$

107. Prop. XV.—*The axes of the ellipse are the only pair of conjugate diameters which are at right angles to each other.*

If we put $\beta = 90° + \alpha$, we have

$$\sin \beta = \cos \alpha; \qquad \cos \beta = -\sin \alpha.$$

Substituting these values in equation (2), it becomes

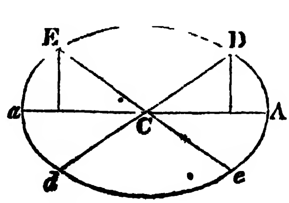
$$(a^2 - b^2) \sin \alpha \cos \alpha = 0.$$

And since a and b are unequal, this equation can only be satisfied by making $\sin \alpha = 0$, or $\cos \alpha = 0$; in which cases the conjugate diameters will coincide with the major and minor axes.

The same result might also be obtained from art. 23. For when two lines, which make the angles θ, ϕ, with the axis of x, are perpendicular to each other, then $\tan \theta \tan \phi = -1$; but in this case, $\tan \alpha \tan \beta = -\frac{b^2}{a^2}$, which is not equal to -1.

108. *Cor.*—When the ellipse becomes a circle, we have $a = b$; in which case $\tan \alpha \tan \beta = -1$, whatever be the value of α. All the conjugate diameters in the circle, therefore, are perpendicular to each other.

109. Prop. XVI.—*The co-ordinates of D, the extremity of any diameter, being given; to find those of* E, *the extremity of the diameter conjugate to it.*



Let $y = x \tan \alpha$, and $y = x \tan \beta$, be the equations to the diameters CD, CE. Also, let x', y', be the co-ordinates of the point D, and x'', y'', those of the point E. Because the point (x'', y''), is in the curve

$$a^2y''^2 + b^2x''^2 = a^2b^2.$$

Also

$$y'' = x'' \tan\beta = -\frac{b^2}{a^2}\,\frac{1}{\tan\alpha}\,x'' = -\frac{b^2x'}{a^2y'}\,x'';$$

from which two equations we are required to find the values of x'' and y''.

From the last equation, $a^2y''^2 = a^2\,\dfrac{b^4x'^2}{a^4y'^2}\,x''^2 = \dfrac{b^4x'^2}{a^2y'^2}\,x''^2$.

Substituting this value in the first equation,

$$\frac{b^4x'^2}{a^2y'^2}\,x''^2 + b^2x''^2 = a^2b^2,$$

reducing, $\qquad (b^2x'^2 + a^2y'^2)\,x''^2 = a^4y'^2.$

Dividing the last equation by $b^2x'^2 + a^2y'^2 = a^2b^2$, we get

$$x''^2 = \frac{a^2y'^2}{b^2}; \quad \text{and} \quad x'' = \pm\frac{ay'}{b}.$$

$$\text{Also} \qquad y'' = -\frac{b^2x'}{a^2y'}\,x'' = \mp\frac{bx'}{a},$$

the sign of y'' being contrary to that of x''.

110. Prop. XVII.—*The sum of the squares of any two conjugate diameters is equal to the sum of the squares of the two axes.*

Retaining the same notation as in the last article, we have

$$x''^2 + x'^2 = \frac{a^2y'^2}{b^2} + x'^2 = \frac{a^2y'^2 + b^2x'^2}{b^2} = a^2;$$

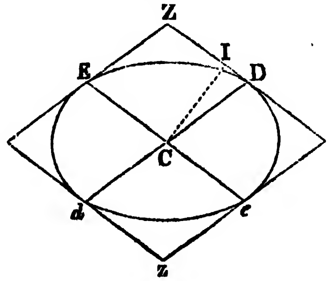
$$y''^2 + y'^2 = \frac{b^2x'^2}{a^2} + y'^2 = \frac{b^2x'^2 + a^2y'^2}{a^2} = b^2;$$

$$\text{Hence} \qquad x''^2 + y''^2 + x'^2 + y'^2 = a^2 + b^2;$$

$$\therefore \quad a'^2 + b'^2 = a^2 + b^2.$$

111. Prop. XVIII.—*All parallelograms formed by drawing tangents at the extremities of any two conjugate diameters are equal in area.*

The parallelogram Zz is evidently divided into four equal parallelograms by the diameters Dd, Ee. Also, the parallelogram $CZ = CE \times CI$. Now



$$CE^2 = x''^2 + y''^2 = \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2}$$

$$= a^2 - x'^2 + \frac{b^2x'^2}{a^2} = a^2 - \frac{a^2 - b^2}{a^2}\,x'^2$$

$$= a^2 - e^2x'^2;$$

$$\therefore \quad \text{parallelogram } CZ = \sqrt{(a^2 - e^2x'^2)} \times \frac{ab}{\sqrt{(a^2 - e^2x'^2)}} \quad \text{(art. 94)}$$

$$= ab.$$

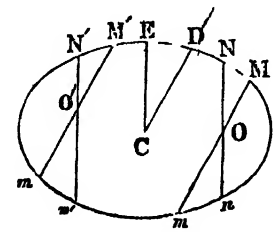
Hence parallelogram Zz = rectangle of the two axes.

112. *Cor.*—Hence, also, we have

$$CE^2 = (a - ex')(a + ex') = FD \times fD;$$

that is, the rectangle contained by the focal distances of any point in the curve is equal to the square of the corresponding semi-conjugate diameter.

113. Prop. XIX.—*If through any point,* O, *within or without an ellipse, two straight lines,* Mm, Nn, *be drawn to meet the curve, parallel to two other straight lines given in position, the rectangle contained by the segments of the one will have a constant ratio to the rectangle contained by the segments of the other; that is, the ratio* OM × Om : ON × On *is constant.*



Let the axes of co-ordinates be transferred to the point O, the new axes being OM, ON. Then the equation to the curve will be of the form (1),

$$ay^2 + bxy + cx^2 + dy + ex + f = 0;$$

and if $x = 0$, it becomes

$$ay^2 + dy + f = 0,$$

in which equation the values of y are ON, $-On$; and therefore, from the theory of equations (Alg. art. 106),

$$ON \times On = -\frac{f}{a}.$$

In the same manner, by making $y = 0$, we find $OM \times Om = -\dfrac{f}{c}$;

$$\therefore \quad OM \times Om : ON \times On :: \frac{f}{c} : \frac{f}{a} :: a : c.$$

Now, let the origin be transferred to O', the new axes being parallel to the former. This will be effected (art. 26) by putting

$$x = t + \alpha, \qquad y = u + \beta;$$

and it is evident that in substituting these values in equation (1), the form of the new equation will be

$$au^2 + btu + ct^2 + Du + Et + F = 0,$$

in which the coefficients a, b, c, are the same as before. We have, therefore, as above,

$$O'M' \times O'm' : O'N' \times O'n' :: a : c;$$

and consequently

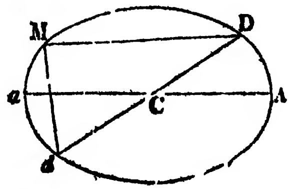
$$OM \times Om : ON \times On :: O'M' \times O'm' : O'N' \times O'n'.$$

114. *Cor.*—If the point O' coincide with the centre of the ellipse, and CD, CE, be parallel to Mm, Nn; we shall have

$$OM \times Om : ON \times On :: CD^2 : CE^2.$$

115. Def.—Two straight lines drawn from any point in the curve to the extremities of a diameter are called *supplemental chords*. If that diameter be the axis major, they are called the *principal* supplemental chords.

116. PROP. XX.—*Having given the equation to the chord* DM, *to find the equation to the supplemental chord* dM.



Let x', y', be the co-ordinates to the point D, referred to the axes of the ellipse; then will $-x'$, $-y'$, be those of the point d. Also, let α, α', be the angles which the chords DM, dM, make with the major axis. Then, since these chords pass through the points D, d, the equations to these lines will be (art. 21)

$$y - y' = \tan\alpha\,(x - x'); \qquad y + y' = \tan\alpha'\,(x + x').$$

Let x'', y'', be the co-ordinates of the point M. Because this point is common to both the lines DM, dM, we have

$$\tan\alpha = \frac{y'' - y'}{x'' - x'}; \qquad \tan\alpha = \frac{y'' + y'}{x'' + x'};$$

$$\therefore\ \tan\alpha\tan\alpha' = \frac{y''^2 - y'^2}{x''^2 - x'^2}.$$

But because D, M are points in the ellipse,

$$a^2y'^2 + b^2x'^2 = a^2b^2; \qquad a^2y''^2 + b^2x''^2 = a^2b^2;$$

$$\therefore\ a^2(y''^2 - y'^2) + b^2(x''^2 - x'^2) = 0.$$

$$\text{Hence} \quad \frac{y''^2 - y'^2}{x''^2 - x'^2} = -\frac{b^2}{a^2} = \tan\alpha\tan\alpha';$$

and the equation to the chord dM is

$$y + y' = -\frac{b^2}{a^2\tan\alpha}(x + x').$$

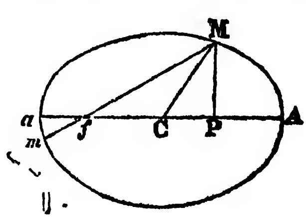
117. *Cor.*—If α, β, be the angles which two conjugate diameters make with the major axis, the first of which is drawn parallel to DM, then (art. 101)

$$\tan\alpha\tan\beta = -\frac{b^2}{a^2} = \tan\alpha\tan\alpha';$$

consequently $\tan\beta = \tan\alpha'$, and $\beta = \alpha'$. Hence it follows, that if a diameter be drawn parallel to DM, its conjugate will be parallel to the supplementary chord dM.

POLAR EQUATIONS.

118. PROP. XXI.—*To find the polar equation to the ellipse, the centre being the pole.*



Put $CM = r$, $MCA = \varphi$, we have then

$$x = r\cos\varphi; \qquad y = r\sin\varphi.$$

Substituting these values in the equation to the ellipse, $a^2y^2 + b^2x^2 = a^2b^2$, we get

$$r^2(a^2\sin^2\varphi + b^2\cos^2\varphi) = a^2b^2;$$

or, $$r^2[a^2(1-\cos^2\varphi)+a^2(1-e^2)\cos^2\varphi]=a^4(1-e^2);$$

$$\therefore\quad r^2=\frac{a^2(1-e^2)}{1-e^2\cos^2\varphi}.$$

119. PROP. XXII.—*To find the polar equation to the ellipse, the focus being the pole.*

This equation may be found in the same way as the last, by substituting in the equation to the curve, and then reducing the terms; but we may deduce it more easily from prop. 4.

Let $fM=r$, and the angle $AfM=\varphi$; then

$$r=a+ex=a+e(r\cos\varphi-ae),$$

from whence we obtain

$$r=\frac{a(1-e^2)}{1-e\cos\varphi};$$

an equation much used in Astronomy.

120. *Cor.*—If Mf be produced to meet the curve again in m, we shall have

$$r'=fm=\frac{a(1-e^2)}{1-e\cos(\pi+\varphi)}=\frac{a(1-e^2)}{1+e\cos\varphi};$$

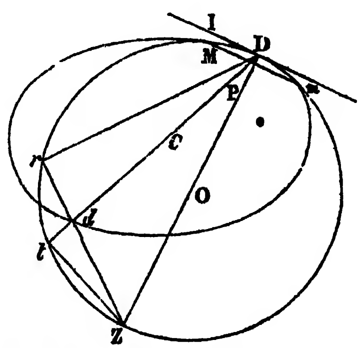
$$\therefore\quad\frac{1}{r}+\frac{1}{r'}=\frac{2}{a(1-e^2)}=\frac{2}{p}.$$

Hence, $$rr'=\tfrac{1}{2}p(r+r').$$

THE CURVATURE AND QUADRATURE OF THE ELLIPSE.

121. DEF.—If a circle be described through three points of a curve, and two of them continually approach the third point, and at length coincide with it; the circle in its ultimate position is called the *circle of curvature* at that point: and the radius of this circle is called the *radius of curvature.*

122. PROP. XXIII.—*To find the radius of curvature at any point of an ellipse.*



Let D be any point in the ellipse; draw the diameter Dd, and take any point, M, in the curve near D. Draw the chord Mm parallel to the tangent DI, and through the points M, D, m, let the circle Drz be drawn; then when the points M, m, move up to D, this will ultimately be the circle of curvature. Also, draw Dz perpendicular to the tangent at D; this will ultimately be the diameter of curvature.

Let Mm cut Dd in P, and put $DP=t$, the chord $Dt=C$, the chord of curvature passing through the centre of the ellipse, or the value of Dt when M and m move up to

$D = \sigma$; also, put the semi-diameter $CD = a'$, the semi-conjugate diameter $= b'$, and the radius of curvature $= \varrho$. Conceive CI to be drawn perpendicular to the tangent DI; we have then (art. 106)

$$PM^2 = \frac{b'^2}{a'^2}(2a't - t^2).$$

And because the chord Mm is bisected by the diameter Dd,

$$\therefore\ PM^2 = PM \times Pm = DP \times Pt = t\,(C - t);$$

$$\therefore\ C - t = \frac{b'^2}{a'^2}(2a' - t).$$

Let M and m move up to D, then $t = 0$, and C becomes c, the chord of curvature; therefore

$$c = \frac{2b'^2}{a'}.$$

Hence $\quad 2\varrho = \dfrac{c}{\sin CDI} = \dfrac{2b'^2}{a'} \times \dfrac{CD}{CI}$; or, $\varrho = \dfrac{b'^2}{CI}$.

But $b'^2 = a^2 - e^2x'^2$ (art. 112); and $CI = \dfrac{ab}{\sqrt{(a^2 - e^2x'^2)}}$ (art. 94);

$$\therefore\ \varrho = \frac{(a^2 - e^2x'^2)^{\frac{3}{2}}}{ab}.$$

123. *Cor.* 1.—If the normal at the point $D = n$, we have (art. 92)

$$n = \frac{b}{a}\sqrt{\left(a^2 - e^2x'^2\right)}; \quad \text{therefore}$$

$$\varrho = \frac{a^2}{b^4}n^3 = \frac{n^3}{p^2}.$$

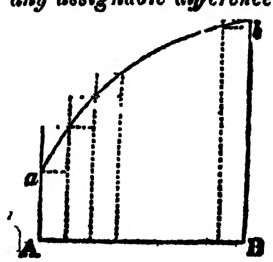
124. *Cor.* 2.—If Dr be the chord of curvature, passing through the focus, and the angle $rDI = \varphi$; then

$$Dr = 2\ \sin\varphi = \frac{2\,(a^2 - e^2x'^2)^{\frac{3}{2}}}{ab}\,\frac{b}{\sqrt{(a^2 - e^2x'^2)}};$$

$$\therefore\ Dr = \frac{2\,(a^2 - e^2x'^2)}{a} = \frac{2b'^2}{a}.$$

125. LEMMA.—*If in any curvilineal space,* AabB, *a number of equidistant ordinates,* Aa, y′, y″ . . . Bb, *be drawn, and the inscribed parallelograms be completed; then if the number of these parallelograms be increased indefinitely, the sum of the inscribed parallelograms will approach nearer to the curvilineal space* AabB *than by any assignable difference.*

Let $Aa = a$, $Bb = b$, and the base $AB = c$; also, let AB be divided into n parts, each equal to h, so that $c = nh$. Then if the inscribed and circumscribed parallelograms be completed as in the figure, and the sum of the inscribed parallelograms be put $= s$, the sum of the circumscribed parallelograms $= S$, and the curvilineal area $AabB = P$; we have



$$s = ah + y'h + y''h \ldots\ldots + y^{(n-1)}h$$
$$S = \quad y'h + y''h \ldots\ldots\ldots\ldots + bh;$$
$$\therefore \quad S - s = bh - ah = (b-a)h = \frac{(b-a)c}{n}.$$

And when n becomes indefinitely great, the difference $S - s$ continually diminishes, and may become less than any assignable quantity. But the curvilineal space P being always contained between S and s,

$$P - s < S - s < \frac{(b-a)c}{n};$$

therefore the difference $P - s$ may become less than any assignable quantity.

We have here supposed the ordinates to be perpendicular to AB; but the proof is nearly the same when they are oblique.

126. *Cor.* 1.—If the distances between the ordinates be not equal to one another, but they are all diminished indefinitely, s will still approach nearer to P than by any assignable difference. For if h be the greatest of these distances, it may easily be shown that

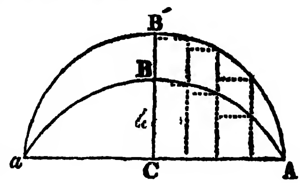
$$S - s < (b-a)h; \quad \therefore \quad P - s < S - s < (b-a)h.$$

And when h is diminished indefinitely, $P - s$ becomes less than any assignable quantity.

127. *Cor.* 2.—In the same manner it may be proved that the sum of the circumscribed parallelograms, when the distances of the ordinates are diminished indefinitely, will approach nearer to P than by any difference that can be assigned.

128. PROP. XXIV.—*To find the area of the ellipse.*

Let CA, CB, be the two semiaxes of the ellipse, and $AB'a$ a semicircle described upon the major axis Aa. Let CA be divided into n parts, each equal to h, so that $CA = a = nh$. Also, let the inscribed rectangles in the ellipse be completed, and the corresponding rectangles in the circle.



Let $b, y', y'' \ldots\ldots 0$ be the ordinates of the ellipse,

and $a, Y', Y'' \ldots\ldots 0$ the corresponding ordinates of the circle.

We have then

$b : a :: y' : Y' :: y'' : Y'' ::$ &c.

$\quad :: y' + y'' + y''' +$ &c. $: Y' + Y'' + Y''' +$ &c. (Alg. art. 184),

$\quad ::$ sum of inscribed rectangles in the ellipse $:$ do. in the circle.

And when n is increased indefinitely, these sums, by the lemma, will approach nearer to the areas CBA, $CB'A$, than by any assignable difference; therefore this proportion must be true also for these areas.

Hence $\quad b : a ::$ area of the ellipse $:$ area of the circle.

$$\therefore \quad \text{area of the ellipse} = \frac{b}{a}\pi a^2 = \pi ab.$$

CHAP. IV. — THE HYPERBOLA.

129. When M and N have different signs, in art. 50, we obtained two equations,

$$a^2y^2 - b^2x^2 = -a^2b^2 \ldots\ldots\ldots (H)$$
$$a^2y^2 - b^2x^2 = \quad a^2b^2 \ldots\ldots\ldots (H_2).$$

From the first of these we get $\quad y = \frac{b}{a}\sqrt{(x^2 - a^2)}$;

and from the second, $\quad x = \frac{a}{b}\sqrt{(y^2 - b^2)}$.

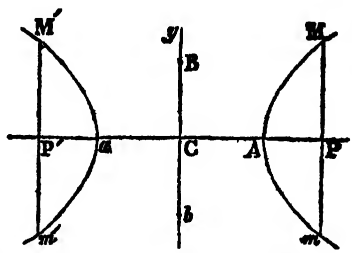
Now the last equation has exactly the same form as the preceding one, a and x having interchanged places with b and y. We shall therefore confine our attention to equation (H), the one most frequently used.

130. If we compare equation (H) with the equation (E) to the ellipse,

$$a^2y^2 + b^2x^2 = a^2b^2,$$

we perceive that the only difference between them is in the sign of b^2; for if in the equation to the ellipse we substitute $-b^2$ for $+b^2$, we get the equation to the hyperbola. Hence it is evident that many of the properties of the hyperbola will be analogous to those of the ellipse, and that their investigations will be exactly alike. We shall, therefore, in the greater number of cases, merely enunciate the propositions, and leave their demonstrations as exercises for the student. Where the properties are altogether different, we shall distinguish the propositions by the letters of the alphabet.

131. Prop. I.—*To determine the figure of the hyperbola from it equation.*



From equation (H), we have

$$y = \pm\frac{b}{a}\sqrt{\left(x^2 - a^2\right)}.$$

When $x = 0$, $y = b\sqrt{-1}$; therefore the hyperbola does not intersect the axis Cy. When x is positive and $< a$, the values of y are imaginary; therefore no part of the curve lies between C and A. When $x = a$, $y = 0$; therefore the hyperbola cuts the axis of x, at the point A. When $x > a$, then for each value of x there are two equal values of y, with contrary signs. As x increases, the values of y increase; and when x becomes indefinitely great, the values of y become so likewise. Hence the two opposite branches, AM, Am, are infinite, and also equal and similar.

If x be negative, y has the same values as when x is positive; no part of the curve, therefore, will lie between C and a; and to the left of a there will be two infinite branches, aM', am', equal and similar to the two branches AM, Am.

Hence the hyperbola consists of two equal and similar branches, extending indefinitely to the right of A; and also of two equal and

similar branches, extending indefinitely to the left of a; and it is symmetrically placed with respect to the axes of x and y.

132. *Cor.*—From the equation to the curve, we have

$$CM = \sqrt{(x^2+y^2)} = \sqrt{\left[x^2 + \frac{b^2}{a^2}\left(x^2 - a^2\right)\right]} = \sqrt{\left[\frac{a^2+b^2}{a^2}x^2 - b^2\right]}.$$

Therefore CM is least when x is least in magnitude, that is, when $x = a$; in which case CM becomes also equal to a. Hence CA or Ca is the least line that can be drawn from the centre to the curve.

133. If CB and Cb be taken on the axis of y, each equal to b, the line Bb is called the *conjugate axis*, although it does not meet the *curve*. The real axis Aa is called the *transverse* axis; and this may be either greater than, equal to, or less than, the conjugate axis. When the two axes are equal, the curve is called the *equilateral* hyperbola; and it has to the hyperbola with unequal axes the same relation that the circle has to the ellipse.

The points A, a, are called vertices of the curve.

134. PROP. II.—*To find the equation to the hyperbola, when the co-ordinates are measured from the vertex*, A.

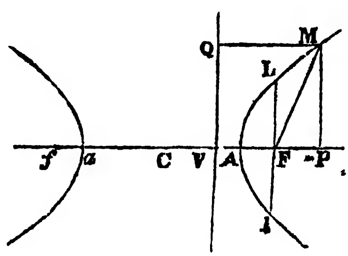
The equation is $\quad y'^2 = \frac{b^2}{a^2}\left(2ax' + x'^2\right) \ldots\ldots. (H');$

$$\therefore \quad CA^2 : CB^2 :: aP \times PA : PM^2.$$

135. *Cor.*—If $P'M'$ be any other ordinate, then

$$PM^2 : P'M'^2 :: aP \times PA : aP' \times P'A.$$

THE FOCI.



136. If in equation (H') we put $\frac{b^2}{a} = p$, it becomes

$$y^2 = 2px + \frac{p}{a}x^2.$$

The line $2p$ is called the *principal parameter* or *latus rectum*, and by the definition it is a third proportional to the transverse and conjugate axes.

The foci F, f, are two points in the transverse axis, where the ordinates are equal to half the latus rectum. The reason of this name we have already explained, in art. 98, in the ellipse.

The distance CF between the focus and centre is called the *excentricity*. In some treatises the ratio $\frac{CF}{CA}$ is called the excentricity.

137. PROP. III.—*To find the distance of the focus from the centre of the hyperbola.*

$$CF = Cf = \sqrt{(a^2 + b^2)}.$$

138. *Cor.* $\quad AF \times Fa = BC^2.$

139. PROP. IV.—*To determine the distance of the focus from any point in the curve.*

If the ratio $\frac{CF}{CA} = \frac{c}{a}$ be put $= e$, then

$$FM = ex - a; \qquad fM = ex + a.$$

140. *Cor.* 1.—If CV be taken a third proportional to CF and CA and VQ be drawn perpendicular, and MQ parallel to CA, then

$$FM : MQ :: c : a.$$

The line VQ is called the directrix of the curve; and this property is sometimes taken for the definition of the hyperbola; c being greater than a in the hyperbola, and less than a in the ellipse.

141. *Cor.* 2. $$fM - FM = 2a;$$
that is, the difference of the two lines drawn from the foci to any point in the curve is equal to the transverse axis. In many geometrical treatises this equation is taken for the definition of the hyperbola; and as it is one of its most distinguishing properties, we will take the converse problem, and from this definition prove that the curve is the same as that which we have called a hyperbola.

142. PROP. V.—*To find the locus of a point, the difference of whose distances from two given points,* F, f, *is always equal to a given line,* 2a.

If $Ff = 2c$, the locus is a hyperbola, whose axes are $2a$, and $2\sqrt{(c^2 - a^2)}$.

THE TANGENT AND NORMAL.

143. PROP. VI.—*To find the equation to a tangent of the hyperbola.*

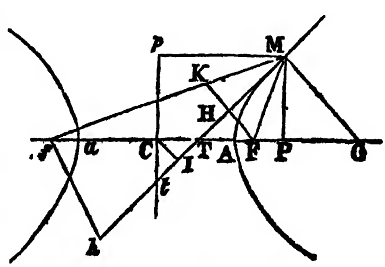
If x', y', be the co-ordinates of the point M, the equation to the tangent is

$$a^2yy' - b^2xx' = -a^2b^2 \ldots\ldots\ldots\ldots (T).$$

144. PROP. VII.—*To find the intersection of the tangent with the axes.*

If T and t be the intersections of the tangent with the transverse and conjugate axes, then

$$CP \times CT = CA^2;$$
$$Cp \times Ct = CB^2.$$



145. *Cor.*—The value of CT is constant for the same abscissa, in all hyperbolas described upon the same transverse axis, Aa.

146. PROP. VIII.—*To find the trigonometrical tangent and sine of the angle which a tangent to the hyperbola makes with the transverse axis.*

If θ be the angle which the tangent makes with the transverse axis,

$$\tan\theta = \frac{b^2x'}{a^2y'}; \qquad \sin\theta = \frac{bx'}{a\sqrt{(e^2x'^2 - a^2)}}.$$

147. *Cor.* 1.—Also, if α be the angle which CM makes with the transverse axis,

$$\tan\theta\tan\alpha = \frac{b^2}{a^2}.$$

148. *Cor.* 2.—If MCM' be a diameter, the tangents at M and M' are parallel.

149. **Prop. IX.**—*To find the equation to the normal at any point* (x', y') *in the hyperbola.*

The required equation is

$$y - y' = -\frac{a^2y'}{b^2x'}(x - x') \ldots\ldots\ldots (N).$$

150. **Prop. X.**—*To find the intersection of the normal with the axes.*

$$CG = \frac{a^2 + b^2}{a^2}x' = e^2x'$$

$$Cg = \frac{a^2 + b^2}{b^2}y' = \frac{e^2}{e^2 - 1}y'.$$

151. *Cor.*—The length of the normal $MG = \frac{b}{a}\sqrt{\left(e^2x'^2 - a^2\right)}$,

and $$Mg = \frac{a}{b}\sqrt{\left(e^2x'^2 - a^2\right)}.$$

152. **Prop. XI.**—*To find the lengths of the perpendiculars from the centre, and from the foci upon the tangent.*

The perpendicular $CI = \dfrac{ab}{\sqrt{(e^2x'^2 - a^2)}}$

" $$FH = b\sqrt{\left(\frac{ex' - a}{ex' + a}\right)} = b\sqrt{\frac{r}{r'}}.$$

" $$fH = b\sqrt{\left(\frac{ex' + a}{ex' - a}\right)} = b\sqrt{\frac{r'}{r}}.$$

153. *Cor.* $$FH \times fh = b^2.$$

154. **Prop. XII.**—*The tangent* MT *makes equal angles with the focal distances* FM, fM.

For $$\sin\phi = \sin\phi' = \frac{b}{\sqrt{(rr')}} = \frac{b}{\sqrt{(e^2x'^2 - a^2)}};$$

therefore $\phi = \phi'$.

155. **Prop. XIII.**—*The locus of the point* H *is a circle, described upon the transverse axis as diameter.*

Scholium.—This proposition admits of a simple geometrical demonstration, in the same manner as the similar proposition in the ellipse

(art. 100). For if FH be produced to meet fM in K, it may easily be shown that

$$fK = fM - MK = fM - MF = 2a,$$

and that CH is parallel to fK. Hence

$$CH = \tfrac{1}{2} fK = a.$$

THE DIAMETERS AND SUPPLEMENTARY CHORDS.

156. Prop. XIV.—*The equation to the hyperbola, between the rectangular co-ordinates, may be changed into another, in which the oblique axes shall be diameters conjugate to each other.*

By substituting, in equation (H),

$$x = t \cos \alpha + u \cos \beta; \qquad y = t \sin \alpha + u \sin \beta,$$

we obtain an equation of the form

$$Au^2 + Btu + Ct^2 = -a^2b^2 \ldots\ldots\ldots (1),$$

in which we have

$$A = a^2 \sin^2 \beta - b^2 \cos^2 \beta; \qquad C = a^2 \sin^2 \alpha - b^2 \cos^2 \alpha;$$
$$B = 2a^2 \sin \alpha \sin \beta - 2b^2 \cos \alpha \cos \beta \ldots\ldots (2).$$

And putting $B = 0$, the equation becomes

$$Au^2 + Ct^2 = -a^2b^2 \ldots\ldots\ldots\ldots (3);$$

in which each of the axes bisects all the chords drawn parallel to the other, and therefore they are conjugate diameters.

157. *Cor.* 1.—Since $B = 0$, therefore $\tan \alpha \tan \beta = \dfrac{b^2}{a^2}$. Hence β has a real value for every value of α; therefore every line drawn through the centre is a diameter, and every diameter has its conjugate.

158. *Cor.* 2.—If a tangent be drawn at the extremity of the diameter Dd, it is parallel to the diameter Ee.

159. *Cor.* 3.—If we divide equation (3) by a^2b^2, and put

$$a'^2 = \frac{-a^2b^2}{C} = \frac{-a^2b^2}{a^2 \sin^2 \alpha - b^2 \cos^2 \alpha};$$
$$b'^2 = \frac{a^2b^2}{A} = \frac{a^2b^2}{a^2 \sin^2 \beta - b^2 \cos^2 \beta};$$

we get the equation

$$\frac{y^2}{b'^2} - \frac{x^2}{a'^2} = -1; \quad \text{or,} \quad a'^2y^2 - b'^2x^2 = -a'^2b'^2 \ldots\ldots\ldots (h).$$

When $y = 0$, $x = \pm a'$; but when $x = 0$, $y = b'\sqrt{-1}$: the axis of y, therefore, never meets the curve. The lines CE, Ce, however, are taken equal to b', and Ee is called the conjugate diameter. When x abstracted from its sign is less than a', the values of y are imaginary; therefore no part of the curve lies between D and d.

160. *Cor.* 4.—The length of any semidiameter which meets the curve, and makes an angle α with the transverse axis, is.

$$\frac{ab}{\sqrt{(-a^2\sin^2\alpha + b^2\cos^2\alpha)}}.$$

And the length of any semiconjugate diameter which does not meet the curve, and which makes an angle β with the transverse axis, is

$$\frac{ab}{\sqrt{(a^2\sin^2\beta - b^2\cos^2\beta)}}.$$

161. *Scholium.*—Since equation (h) has exactly the same form as the equation between the rectangular co-ordinates, it is evident that all the properties which are independent of the inclination of the ordinates will be equally true for the conjugate diameters as for the two axes. Hence

$$CD^2 : CE^2 :: dP \times PD : PM^2.$$

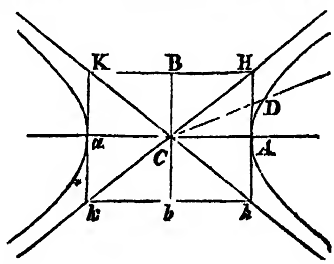
Also the equation to the tangent, at any point, (x', y'), is

$$a'^2yy' - b'^2xx' = -a'^2b'^2.$$

And if T, t, be the intersections of the tangent with the two conjugate diameters dD, eE;

$$CP \times CT = CD^2; \qquad Cp \times Ct = -CE^2.$$

162. Prop. *A.*—*To determine those diameters which meet the curve.*



Let $y = x\tan\alpha$ be the equation to any diameter, and x', y', the co-ordinates of the point of intersection with the curve. We have then

$$y' = x'\tan\alpha;$$

$$a^2y'^2 - b^2x'^2 = -a^2b^2;$$

from which two equations we find

$$x' = \pm\frac{ab}{\sqrt{(b^2 - a^2\tan^2\alpha)}}.$$

Now these values are real, if $b^2 > a^2\tan^2\alpha$; or, abstracted from its sign, if $\tan\alpha < \frac{b}{a}$. If, therefore, the line HAh be drawn perpendicular to CA, and AH, Ah, be taken each equal to b, and also the lines HCk, hCK, be drawn; any diameter drawn within the angles HCh, KCk, will meet the hyperbola, and any line drawn within the angles HCK, hCk, will not meet it. If $b = a\tan\alpha$, x' is infinite; the lines CH, Ch, therefore, will never meet the curve, but they will approach indefinitely near to it.

163. *Cor.*—Since $\tan\alpha\tan\beta = \frac{b^2}{a^2}$ (art. 157); if $\tan\alpha < \frac{b}{a}$, $\tan\beta$ will be $> \frac{b}{a}$. Hence, if the diameter CD meets the curve, the conjugate diameter CE will not meet it.

164. Prop. XV.—*The axes of the hyperbola are the only pair of conjugate diameters which are at right angles to each other.*

165. Prop. XVI.—*The co-ordinates,* x', y', *of the extremity of any diameter being given, to find the co-ordinates* x'', y'', *of the extremity of the diameter conjugate to it.*

This proposition may be investigated by finding the value of $\tan\beta$, in the same manner as in the corresponding proposition in the ellipse; and then finding x'', from the expression $x'' = b'\cos\beta$. The value of b' is given in art. 159. In this way we get

$$x'' = \pm\frac{ay'}{b}; \qquad y'' = \mp\frac{bx'}{a}.$$

166. Prop. XVII.—*The difference of the squares of any two conjugate diameters is equal to the difference of the squares of the two axes.*

$$\text{For} \quad x'^2 - x''^2 = x'^2 - \frac{a^2y'^2}{b^2} = \frac{b^2x'^2 - a^2y'^2}{b^2} = a^2;$$

$$y'^2 - y''^2 = y'^2 - \frac{b^2x'^2}{a^2} = \frac{a^2y'^2 - b^2x'^2}{a^2} = -b^2.$$

$$\text{Hence} \quad x'^2 + y'^2 - x''^2 - y''^2 = a^2 - b^2;$$

$$\therefore \quad a'^2 - b'^2 = a^2 - b^2.$$

167. Prop. XVIII.—*All parallelograms formed by lines drawn parallel to two conjugate diameters, through their extremities, are equal in area.*

$$\text{For} \quad CE^2 = x''^2 + y''^2 = e^2x'^2 - a^2,$$

$$\text{and} \quad CI = \frac{ab}{\sqrt{(e^2x'^2 - a^2)}};$$

$\therefore$ parallelogram $CZ = CE \times CI = ab$;

and parallelogram Zz = rectangle of the two axes.

168. *Cor.* $\quad CE^2 = (ex' - a)(ex' + a) = FD \times fD$.

169. Prop. XIX.—*If through any point,* O, *within or without a hyperbola, the straight lines* Mm, Nn, *be drawn to meet the curve, parallel to two other straight lines given in position; the rectangle contained by the segments of the one will have a constant ratio to the rectangle contained by the segments of the other; that is, the ratio* OM × Om : ON × On *is constant.*

If the axes of co-ordinates be transferred to the point O, and the new axes be OM, ON; the equation to the curve will be of the form

$$a'y^2 + b'xy + cx^2 + dy + ex + f = 0,$$

a', b' being accented to distinguish them from the semiaxes of the curve; and the proof will be precisely the same as for the ellipse.

170. *Cor.*—If the point O' coincide with C, the centre of the hyperbola, and CD, CE, be drawn parallel to Mm, Nn, the equation to the curve will become (art. 52)

$$a'y^2 + b'xy + cx^2 + f' = 0, \quad \therefore \frac{a'}{f'}y^2 + \frac{b'}{f'}xy + \frac{c}{f'}x^2 + 1 = 0;$$

where a', b', c, are the same as before; but in this case the axes of co-ordinates will never meet the curve. Now it appears from art. 156, that the equation to the curve, for any axes passing through the centre, and inclined to the transverse axis at the angles α, β, is

$$\frac{A}{a^2b^2}y^2 + \frac{B}{a^2b^2}xy + \frac{C}{a^2b^2}x^2 + 1 = 0,$$

where $A = a^2 \sin^2 \beta - b^2 \cos^2 \beta$, $C = a^2 \sin^2 \alpha - b^2 \cos^2 \alpha$. And, if these axes be the same as those above, it is evident that the equations must be identical. Hence (art. 159)

$$\frac{a}{f'} = \frac{A}{a^2b^2} = \frac{1}{CE^2}; \qquad \frac{c}{f'} = \frac{C}{a^2b^2} = \frac{1}{CD^2};$$

$$\therefore\ OM \times Om : ON \times On :: a' : c :: CD^2 : CE^2.$$

171. Def.—Two straight lines drawn from any point in the curve to the extremities of a diameter are called *supplemental chords.* If that diameter be the transverse axis, they are called *principal* supplemental chords.

172. Prop. XX.—*Having given the equation to the chord* DM, *to find the equation to the supplemental chord* dM.

If the equation to DM be $y - y' = \tan \alpha\, (x - x')$, the equation to dM will be $y + y' = \dfrac{b^2}{a^2 \tan \alpha}(x + x')$.

173. *Cor.*—The conjugate diameter CE is parallel to the supplementary chord dM.

THE ASYMPTOTES.

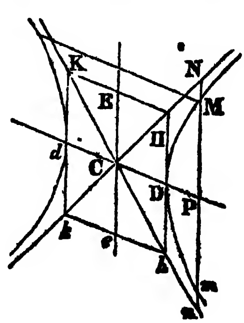
174. Prop. *B.*—*When the equation to a curve can be reduced to the form*

$$y = Ax + B + \frac{C}{x} + \frac{D}{x^2} + \&c.,$$

the straight line whose equation is $y' = Ax + B$ *is an asymptote to the curve.*

For as x increases, the terms $\dfrac{C}{x}$, $\dfrac{D}{x^2}$, &c., continually diminish; and when x is indefinitely great, these terms become indefinitely small. Hence, since $y - y'$ becomes indefinitely small, the straight line whose equation is $y' = Ax + B$ approaches nearer to the curve than by any assignable difference, and yet never meets it; and therefore the straight line is an asymptote to the curve (art. 41).

175. Prop. *C.*—*The asymptotes of the hyperbola coincide with the diagonals of the parallelogram constructed upon any pair of conjugate diameters.*



The equation to the hyperbola is

$$y = \pm \frac{b'}{a'}\sqrt{(x^2 - a'^2)}.$$

Expanding this expression by the binomial theorem (Alg. art. 389), we get

$$\pm y = \frac{b'x}{a'}\left(1 - \frac{a'^2}{2x^2} - \frac{a'^4}{8x^4} + \&c.\right)$$

$$= \frac{b'x}{a'} - \frac{a'b'}{2x} - \frac{a'^3b'}{8x^3} + \&c.$$

Hence the lines represented by the two equations

$$y = \frac{b'x}{a'}, \qquad y = -\frac{b'x}{a'},$$

will be asymptotes to the curve; and these two lines are evidently the diagonals of the parallelogram *HKkh*.

176. *Cor.* 1.—Because *HDh* is parallel to the conjugate diameter *Ee*, *Hh* touches the hyperbola at the point *D*. Also $DH = CE = Dh$.

177. *Cor.* 2.—Since the equation to the tangent, at any point, (x', y'), is (art. 162)

$$y = \frac{b'^2 x'}{a'^2 y'} x - \frac{b'^2}{y'},$$

and the equation to the curve is $a'^2 y'^2 - b'^2 x'^2 = -a'^2 b'^2$, or

$$\frac{x'}{y'} = \frac{a'}{b'} \sqrt{\left(1 + \frac{b'^2}{y'^2}\right)}.$$

By the substitution of this value in the equation to the tangent, we get

$$y = \frac{b'x}{a'} \sqrt{\left(1 + \frac{b'^2}{y'^2}\right)} - \frac{b'^2}{y'}.$$

And when y' is indefinitely great, $\frac{b'^2}{y'}$ and $\frac{b'^2}{y'^2}$ are indefinitely small, therefore the equation to the tangent ultimately becomes $y = \frac{b'x}{a'}$, which is the equation to the asymptote. Hence the asymptote may be considered as a tangent to the curve, at an infinite distance.

178. Prop. *D.—If any chord of the hyperbola be produced to meet the asymptotes, the parts of it intercepted between the curve and the asymptotes will be equal.*

Let *Mm* be any chord of a hyperbola, and let it be produced to meet the asymptotes in *N*, *n*. Bisect *Mm* in *P*, and draw the diameter *CP*; then the equations to the asymptotes *CN*, *Cn*, are

$$y = \frac{b'}{a'} x, \qquad y = -\frac{b'}{a'} x;$$

consequently $PN = Pn$, and therefore $MN = mn$.

179. *Cor.*—Hence also we have

$$MN \times Mn = (PN - PM)(PN + PM) = PN^2 - PM^2.$$

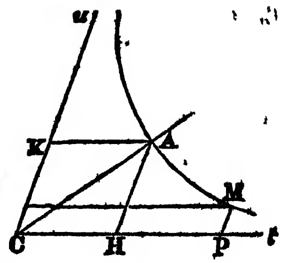
$$\text{But} \quad PN^2 = \frac{b'^2}{a'^2} x^2; \qquad PM^2 = \frac{b'^2}{a'^2} (x^2 - a'^2);$$

$$\therefore \; MN \times Mn = b'^2.$$

180. Prop. *E.—To find the equation to the hyperbola when the two asymptotes are the axes of co-ordinates.*

Let the inferior asymptote be the axis of t, and let 2ϕ be the angle contained between the asymptotes. Then, in the equations (art. 28)

$$x = t \cos \alpha + u \cos \beta,$$
$$y = t \sin \alpha + u \sin \beta,$$



we have $\alpha = -\phi$, $\beta = \phi$; therefore

$$x = t\cos\phi + u\cos\phi = (t+u)\cos\phi,$$
$$y = u\sin\phi - t\sin\phi = (u-t)\sin\phi.$$

Substituting these values in equation (H), we get

$$a^2\sin^2\phi\,(u-t)^2 - b^2\cos^2\phi\,(u+t)^2 = -a^2b^2.$$

And because $\tan\phi = \frac{b}{a}$, therefore

$a^2\sin^2\phi = b^2\cos^2\phi$; hence reducing,

$$4a^2\sin^2\phi\,.\,tu = 4ab\sin\phi\cos\phi\,.\,tu = a^2b^2;$$

$$\therefore\quad tu = \frac{ab}{2\sin 2\phi}.$$

181. *Cor.*—The area of the parallelogram $CM = tu \sin 2\phi = \frac{1}{4}ab$. Hence the area of the parallelogram CM is invariable, and equal to $\frac{1}{8}$th of the rectangle on the two axes.

POLAR EQUATIONS.

182. Prop. XXI.—*The polar equation to the hyperbola, the centre being the pole, is*

$$r^2 = \frac{a^2(e^2-1)}{1-e^2\cos^2\phi}.$$

183. Prop. XXII.—*The polar equation to the hyperbola, the focus* F *being the pole, and the angle* MFx $= \phi$, *is*

$$r = \frac{a(e^2-1)}{1-e\cos\phi}.$$

184. *Cor.*—If MF be produced to meet the curve again in m, and Fm be put $= r'$, then

$$\frac{1}{r} + \frac{1}{r'} = \frac{2}{a(e^2-1)} = \frac{2}{p}.$$

THE CURVATURE AND QUADRATURE OF THE HYPERBOLA.

185. Prop. XXIII.—*The radius of curvature, at any point of a hyperbola, is equal to*

$$\frac{(e^2x'^2-a^2)^{\frac{3}{2}}}{ab} = \frac{(\text{normal})^3}{p^2}.$$

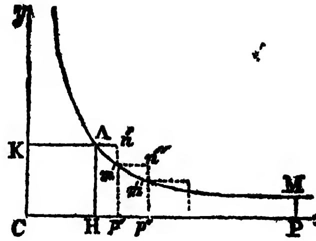
186. *Cor.*—The chord of curvature passing through the focus is equal to

$$\frac{2(e^2x'^2-a^2)}{a}.$$

187. Prop. XXIV.—*The area of a hyperbola, between its asymptotes, is proportional to the logarithm of the abscissa.*

Suppose AM, for the sake of simplicity, to be an equilateral hyperbola; Cx, Cy, the asymptotes, and A the vertex. Draw AH, AK, pa-

rallel to the asymptotes, and also any number of ordinates, $p'm'$, $p''m''$, &c., parallel to Cy; complete the rectangles Hn', $p'n''$, &c.



Let $CH = CK = 1$, $CP = x$, $PM = y$. Also, let the abscissæ Cp', Cp'', &c., be denoted by x', x'', &c.; and the ordinates $p'm'$, $p''m''$, &c., by y', y'', &c. Then the equation to the curve gives

$$x'y' = x''y'' = x'''y''' = \text{\&c.} = CH \times HA = 1.$$

Also the rectangle $Hn' = HA \times Hp' = 1 \times (x' - 1) = x' - 1$

" $p'n'' = p'm' \times p'p'' = y' \left(x'' - x'\right) = \dfrac{x''}{x'} - 1$

" $p''n''' = p''n'' \times p''p''' = y'' \left(x''' - x''\right) = \dfrac{x'''}{x''} - 1$

and so on.

Now, as the disposition of the points p', p'', &c., is altogether arbitrary, we may suppose the abscissæ $1, x', x''$, &c.... x, to be in geometrical progression. Putting, then, $x' = 1 + v$, we have

$$x' = 1+v\,;\; x'' = (1+v)^2\,;\; x''' = (1+v)^3 \ldots\ldots x = (1+v)^n.$$

$$\text{Also} \qquad \frac{x'}{1} = \frac{x''}{x'} = \frac{x'''}{x''} = \ldots\ldots = 1 + v\,;$$

consequently each of the rectangles Hn', $p'n''$, &c., is $= x' - 1 = v$.

If, therefore, we take the abscissæ

$$1, \quad 1 + v, \quad (1 + v)^2, \quad (1 + v)^3 \ldots\ldots\ldots\ldots (1 + v)^n,$$

the sum of the rectangles contained between AH and these abscissæ will be respectively

$$0, \qquad v, \qquad 2v, \qquad 3v \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots nv.$$

Hence, while the abscissæ increase in geometrical progression, the areas increase in arithmetical progression, and consequently the sum of the rectangles contained between HA and any ordinate, PM, is proportional to the logarithm of CP.

As this conclusion is entirely independent of n, the number of divisions, it will be equally true if n be indefinitely great. But in this case the sum of the rectangles will approach indefinitely near to the curvilineal area $HPMA$; consequently the hyperbolic area $HPMA$ is proportional to the logarithm of CP.

188. *Scholium.*—If the area $HPMA$ be supposed *equal* to the logarithm of CP, we may easily determine the *base* of this system of logarithms. For since

$$\log (1 + v)^n = nv, \text{ when } n \text{ is infinite};$$

and in any system of logarithms, the base is equal to the number whose logarithm is 1 (Alg., art. 391); putting, therefore, $nv = 1$, or $v = \dfrac{1}{n}$,

the base of this system of logarithms is equal to $\left(1+\frac{1}{n}\right)^n$, when n is infinite. Now,

$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{1\,.\,2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)\,(n-2)}{1\,.\,2\,.\,3}\left(\frac{1}{n}\right)^3 + \&c$$

$$= 1+1+\frac{1}{1\,.\,2}\left(1-\frac{1}{n}\right) + \frac{1}{1.2.3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \&c.$$

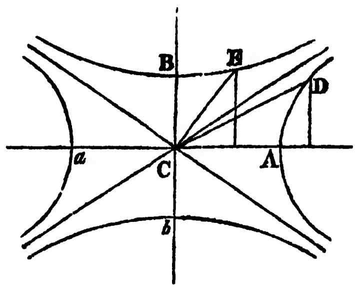
And when n is infinite, this series becomes

$$1 + 1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \&c. = 2.71828\ldots.$$

But this number is equal to e, the base of the Naperian system of logarithms (Alg. art. 399). Hence the area $HPMA$ is equal to the *Naperian logarithm* of the abscissa CP; for which reason these logarithms were at first called hyperbolic logarithms.

THE CONJUGATE HYPERBOLA.

189. In art. 129 there is an equation (H_2) of a different form from (H), but which we have shown may be reduced to this form by interchanging the letters x and y, and also a and b; and consequently this curve likewise is a hyperbola, placed in a different manner. If we examine the course of this curve, we shall find that $Bb = 2b$ is the real or transverse axis, and $Aa = 2a$ is the conjugate axis; and also that this curve has the same asymptotes as the original hyperbola. We shall now prove that all the conjugate or imaginary diameters of the curve AD are real diameters of the curve BE, and conversely. Hence this hyperbola is said to be conjugate to the other hyperbola, and the properties of the two curves, when taken together, bear a still stronger analogy to those of the ellipse.



190. Prop. *F.—If in the hyperbola* AD, CE *be a conjugate diameter to* CD, *it is required to find the locus of* E.

Let x, y, be the co-ordinates of the point E, and x', y', those of the point D. We have then (art. 166)

$$x'^2 + y'^2 - x^2 - y^2 = a^2 - b^2.$$

But (art. 165) $\qquad y' = \pm\frac{bx}{a}, \qquad x' = \pm\frac{ay}{b}$

$$\therefore\quad \frac{b^2x^2}{a^2} - x^2 + \frac{a^2y^2}{b^2} - y^2 = a^2 - b^2,$$

or $$\frac{b^2-a^2}{a^2}x^2 + \frac{a^2-b^2}{b^2}y^2 = a^2 - b^2$$

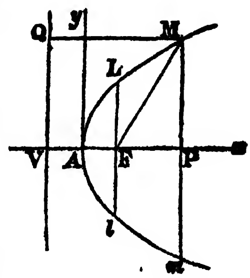
Hence, dividing by $a^2 - b^2$ and reducing, we get

$$a^2y^2 - b^2x^2 = a^2b^2,$$

which is the equation represented by (H_2); and therefore the extremities of all the conjugate diameters are in the conjugate hyperbola.

CHAP. V. — THE PARABOLA.

191. PROP. I.—*To determine the figure of the parabola from its equation.*



The equation to the parabola, when referred to rectangular axes, is

$$y^2 = 2px \ldots\ldots (P).$$

When $x = 0$, y is also $= 0$; hence the curve passes through the origin A.

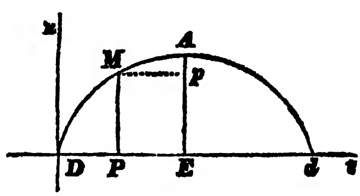
If p be positive; then for each positive value of x, there are two equal values of y with contrary signs. As x increases, y increases; and when x becomes indefinitely great, y also becomes indefinitely great. Hence there are two infinite arcs to the right of A.

If x be negative, y is imaginary; therefore no part of the curve lies to the left of A.

If p be negative, there will be two infinite arcs to the left of A, and no part of the curve will lie to the right of A.

192. *Cor.*—The line Ax is evidently an axis of the curve. The point A is called the vertex of the curve.

193. PROP. II.—*To find the equation to the parabola when the co-ordinates are measured from any point in the curve, and parallel to the former co-ordinates.*



Let $DE = Ed = a$, $DP = t$, $PM = u$; then (art. 191)

$$2p \times AE = a^2$$
$$2p \times Ap = Mp^2 = (a - t)^2;$$
$$\therefore\ 2p(AE - Ap) = a^2 - (a-t)^2.$$

Hence $\quad 2pu = 2at - t^2.$

194. *Cor.*—Because $2at - t^2 = t(2a - t) = DP \times Pd$;

$$\therefore\ DP \times Pd : DE \times Ed :: 2p \times PM : 2p \times AE$$

or, $\quad DP \times Pd : \quad DE^2 \quad :: PM : AE,$

that is, the ordinate PM varies as the rectangle $DP \times Pd$.

THE FOCUS.

195. The line $2p$ is called the *principal parameter*, or *latus rectum*; and from equation (P), it is a third proportional to any abscissa and its corresponding ordinate.

The *focus* F is a point in the axis, where the ordinate is equal to half the latus rectum.

196. *Cor.*—Since $FL = p$, we have at this point

$$p^2 = 2px;\quad \text{therefore}\quad x = AF = \tfrac{1}{2}p.$$

197. Prop. III.—*To determine the distance of the focus from any point in the curve.*

Let M be any point in the curve; put $AP = x$, $PM = y$; then

$$\begin{aligned} FM^2 &= FP^2 + PM^2 = (x - \tfrac{1}{2}p)^2 + y^2 \\ &= x^2 - px + \tfrac{1}{4}p^2 + 2px \\ &= x^2 + px + \tfrac{1}{4}p^2 = (x + \tfrac{1}{2}p)^2; \end{aligned}$$

$$\therefore \quad FM = x + \tfrac{1}{2}p.$$

Hence, if in FA produced, AV be taken equal to $\frac{1}{2}p$, and VQ be drawn at right angles to Ax, we shall have

$$FM = AP + AV = VP = MQ.$$

The line VQ is called the directrix; and by many writers this equality is taken for the definition of the parabola.

198. *Scholium.*—In the parabola, the distance of any point in the curve from the directrix is *equal to* its distance from the focus. In the ellipse, the distance of any point in the curve from the directrix is *greater* than its distance from the focus in a given ratio. And in the hyperbola, the distance from the directrix is *less* than its distance from the focus in a given ratio.

The equation to the parabola is $\quad y^2 = 2px$

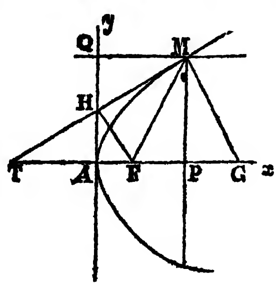
" ellipse $\quad y^2 = 2px - \dfrac{p}{a}x^2$

" hyperbola $\quad y^2 = 2px + \dfrac{p}{a}x^2.$

Apollonius gave the name of *hyperbola** to the last of these curves, because the square of the ordinate is greater than the rectangle of the abscissa and the latus rectum; and the name of *ellipse*† to the second curve, because this square is less than the same rectangle. The name of *parabola*‡ was afterwards given to the first curve, on account of the equality of this square and rectangle.

THE TANGENT AND NORMAL.

199. Prop. IV.—*To find the equation to a tangent to the parabola.*



Let x', y', be the co-ordinates of the given point, and x'', y'', those of any other point in the parabola. The equation to a straight line drawn through these two points is (art. 17)

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x').$$

And because these two points are in the curve, we have

$$y'^2 = 2px'; \qquad y''^2 = 2px'';$$

---

* From ὑπερβολή, excess. † ἔλλειψις, deficiency.
‡ παραβολή, comparison.

$$\therefore \quad y''^2 - y'^2 = (y'' + y')(y'' - y') = 2p(x'' - x');$$

$$\therefore \quad \frac{y'' - y'}{x'' - x'} = \frac{2p}{y'' + y'}.$$

Hence the equation to the secant becomes, by substitution,

$$y - y' = \frac{2p}{y'' + y'}(x - x').$$

Let, now, the point (x'', y'') move up to the point (x', y'), and ultimately coincide with it; then the secant becomes a tangent, and $x'' = x'$, $y'' = y'$. Hence the equation to the tangent is

$$y - y' = \frac{p}{y'}(x - x'),$$

or $\quad yy' - y'^2 = px - px'$; and because $y'^2 = 2px'$,

$$\therefore \quad yy' = p(x + x') \ldots\ldots\ldots (T),$$

which is the equation to the tangent generally used.

200. *Cor.* 1.—Let the tangent MT cut the axis in T, then at this point $y = 0$; hence

$$p(x + x') = yy' = 0; \quad \therefore \quad x + x' = 0;$$

$$\text{or,} \quad x = -x' = -AP.$$

Hence AT is measured in the opposite direction to AP, and is equal to it. Hence also the subtangent $PT = 2AP$.

201. *Cor.* 2.—Because $FT = AT + AF = x' + \frac{1}{2}p$, and $FM = x' + \frac{1}{2}p$ (art. 194); therefore $FT = FM$. Hence the angle $FMT = FTM = TMQ$; or the tangent M bisects the angle FMQ.

202. Prop. V.—*To find the trigonometrical tangent and sine which a tangent to the parabola makes with the axes.*

(1). By transposition and division, equation (T) becomes

$$y = \frac{p}{y'}x + p\frac{x'}{y'}.$$

And if this be compared with the general equation to the straight line,

$$y = Ax + B = x \tan\theta + B, \quad \text{we have}$$

$$A = \tan\theta = \frac{p}{y'}.$$

(2). We have likewise

$$\sin^2\theta = \frac{\tan^2\theta}{1 + \tan^2\theta} = \frac{p^2}{y'^2 + p^2} = \frac{p^2}{2px' + p^2};$$

$$\therefore \quad \sin\theta = \sqrt{\left(\frac{p}{p + 2x'}\right)}.$$

203. Prop. VI.—*To find the equation to the normal, at any point* (x', y'), *in the parabola.*

Since the normal passes through the point (x', y'), its equation will be of the form (art. 18)

$$y - y' = A'(x - x').$$

Also, if $y = Ax + B$ be the equation to the tangent at this point, then $A' = -\frac{1}{A}$, because the normal is perpendicular to the tangent (art. 83).

But $A = \frac{p}{y'}$ (art. 202); therefore $A' = -\frac{y'}{p}$.

Hence the equation to the normal is

$$y - y' = -\frac{y'}{p}(x - x') \dots\dots\dots (N).$$

204. *Cor.* 1.—If the normal MG cut the axis in G, then $y = 0$;

$$\therefore \quad -y' = -\frac{y'}{p}(x - x'); \quad \text{or,} \quad x - x' = p;$$

hence the subnormal $PG = x - x' = p$.

205. *Cor.* 2.—The length of the normal

$$MG = \sqrt{(p^2 + y'^2)} = \sqrt{p(p + 2x')}.$$

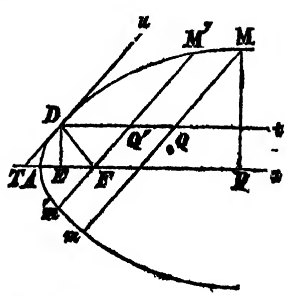
206. Prop. VII.—*To find the length of the perpendicular from the focus upon the tangent.*

$$FH = FT \sin\theta = (AF + AT)\sin\theta$$
$$= (\tfrac{1}{2}p + x')\sqrt{\left(\frac{p}{p + 2x'}\right)} = \sqrt{\tfrac{1}{2}p(\tfrac{1}{2}p + x')}.$$

Hence $FH^2 = AF \times FT$; and since FHT is a right angle, therefore FAH is a right angle (Geom. prop. 72). And because the tangent at the vertex is perpendicular to the axis Ax (art. 202), the point H is situated in the line Ay.

THE DIAMETERS.

207. Prop. VIII.—*The equation to the parabola, between the rectangular co-ordinates, may be changed into another of the same form, having a different origin and different axes.*



Let the origin of co-ordinates be transferred from the vertex A to any point, D, in the curve. Let Dt, Du, be the new axes; and let Dt be parallel to Ax, and the angle $tDu = \beta$.

Let $AP = x$, $PM = y$, be the rectangular co-ordinates of any point, M, in the parabola; $DQ = t$, $QM = u$, the new co-ordinates of M. Also, let $AE = a$, $DE = b$. We have then

$$y = PM = DE + MN = b + u\sin\beta,$$
$$x = AP = AE + DQ + QN = a + t + u\cos\beta.$$

Substituting these values in the equation $y^2 = 2px$, we get

$$(b + u\sin\beta)^2 = 2p(a + t + u\cos\beta);$$

$$\therefore \quad u^2 \sin^2\beta + 2bu \sin\beta + b^2 = 2pa + 2pt + 2pu\cos\beta;$$

and because D is a point in the curve $b^2 = 2pa$,

$$\therefore \quad u^2 \sin^2\beta + 2u(b\sin\beta - p\cos\beta) = 2pt.$$

Hence, putting $b\sin\beta - p\cos\beta = 0$, since β is indeterminate, we get

$$u^2\sin^2\beta = 2pt; \quad \text{or,} \quad u^2 = \frac{2p}{\sin^2\beta}\, t \ldots\ldots\ldots (2).$$

And since each value of t gives two equal values of u with contrary signs, the axis of t is a diameter of the curve.

208. *Cor.* 1.—Since $b\sin\beta - p\cos\beta = 0$, therefore

$$\tan\beta = \frac{p}{b} = \tan\theta \text{ (art. 202)}; \quad \therefore \quad \beta = \theta;$$

or the ordinates to the diameter Dt are parallel to the tangent at D.

209. *Cor.* 2.—Because $\sin^2\beta = \dfrac{\tan^2\beta}{1+\tan^2\beta} = \dfrac{p^2}{b^2+p^2} = \dfrac{p^2}{2ap+p^2}$;

$$\therefore \quad \frac{2p}{\sin^2\beta} = 4\left(a + \tfrac{1}{2}p\right) = 4FD.$$

Putting, therefore, $4FD = 2p'$, the equation becomes

$$u^2 = 2p't \ldots\ldots\ldots\ldots\ldots (p).$$

210. *Cor.* 3.—If the ordinate $Q'M'$ pass through the focus F, then

$$u^2 = 2p' \times DQ' = 2p' \times FT' = 2p' \times FD = 4FD^2.$$

Hence $\quad u = 2FD = p'$, and $M'm' = 2u = 2p'$.

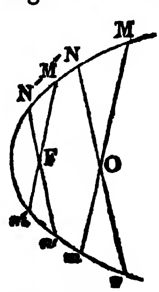
211. *Scholium.*—Since equation (p) has exactly the same form as the equation between rectangular co-ordinates, it is evident that all the properties which are independent of the inclination of the ordinates will be the same in the two systems. Hence the equation to the tangent, at any point, M, will be

$$yy' = p'(x + x');$$

the ordinates to the tangent being parallel to the axis of u. Also, the value of the subtangent

$$P'T' = 2t.$$

212. Prop. IX.—*If through any point,* O, *within or without a parabola, two straight lines,* MOm, NOn, *be drawn to meet the curve, parallel to two other straight lines given in position; the rectangle contained by the segments of the one will be to the rectangle contained by the segments of the other in a constant ratio.*



This proposition is proved exactly in the same manner as prop. 18, in the ellipse.

213. *Cor.*—If the point O coincide with the focus F, we have

$$FM' \times Fm' = \tfrac{1}{2}p \times M'm' \text{ (art. 216)} = \tfrac{1}{4}pP;$$

P being the parameter to the ordinates parallel to Mm. In like manner, if P' be the parameter to the ordinates parallel to Nn, $FN' \times Fn' = \frac{1}{4}pP'$. Hence

$$OM \times Om : ON \times On :: FM' \times Fm' : FN' \times Fn' :: P : P'.$$

POLAR EQUATIONS.

214. Prop. X.—*To find the polar equation to the parabola, the focus being the pole.* (See fig. to prop. I.)

Putting $FM = r$, and the angle $AFM = \phi$; we have

$$x = AF + FP = \tfrac{1}{2}p + r \cos PFM = \tfrac{1}{2}p - r \cos \phi,$$
$$y = PM = r \sin \phi.$$

Substituting these values in the equation $y^2 = 2px$, and reducing, we shall obtain the polar equation. We may, however, obtain it more simply, by substituting the value of x in the equation

$$r = FM = \tfrac{1}{2}p + x = p - r \cos \phi;$$

$$\therefore \quad r = \frac{p}{1 + \cos \phi} = \frac{p}{2 \cos^2 \frac{1}{2}\phi}.$$

215. *Cor.* 1.—If MF be produced to cut the parabola again in m, and $Fm = r'$, we shall have

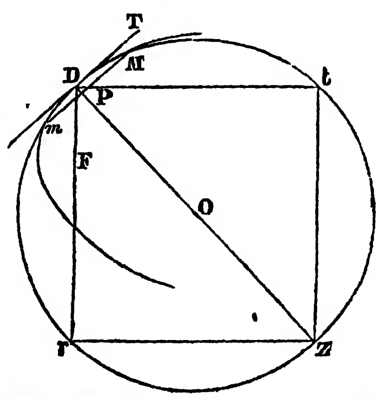
$$r' = \frac{p}{1 + \cos(\pi + \phi)} = \frac{p}{1 - \cos \phi} = \frac{p}{2 \sin^2 \frac{1}{2}\phi}.$$

216. *Cor.* 2.—Hence it follows that

$$\frac{1}{r} + \frac{1}{r'} = \frac{2}{p}, \quad \text{and} \quad \tfrac{1}{2}p\,(r + r') = rr'.$$

CURVATURE AND QUADRATURE OF THE PARABOLA.

217. Prop. XI.—*To find the radius of curvature at any point of a parabola.*



Let D be any point in the parabola; draw the diameter Dt, and take any point, M, in the curve near D. Draw the chord Mm parallel to the tangent at D, and through the points M, D, m let the circle Dtz be drawn; then, when the points M, m move up to D, this will ultimately be the circle of curvature. Draw Dz perpendicular to the tangent at D, this will ultimately be the diameter of curvature. Let Mm cut Dt in P; and put $DP = t$, the chord $Dt = C$, the chord of curvature parallel to the axis, or the value of Dt when M and m move up to $D = c$: also, put the parameter to the diameter of the parabola $Dt = 2p'$, the angle $TDt = \theta$, and the radius of curvature $= \rho$. Then, because PM is parallel to DT, it is an ordinate to the diameter, and the chord Mm is bisected in P; therefore

$$2p't = PM^2 = PM \times Pm = DP \times Pt = t\,(C - t);$$

$$\therefore \quad 2p' = C - t.$$

And when M and m move up to D, $t = 0$, and $C = c$; consequently the chord of curvature parallel to the axis is equal to $2p'$. Hence it follows that the radius of curvature $\rho = \frac{p'}{\cos ODt} = \frac{p'}{\sin\theta}$.

But $p' = 2FD = 2(x' + \frac{1}{4}p) = p + 2x'$ (art, 209),

and $\sin\theta = \sqrt{\left(\frac{p}{p + 2x'}\right)}$ (art. 202); therefore

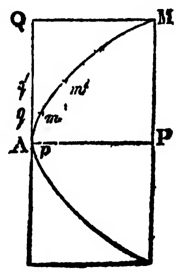
$$\rho = \frac{(p + 2x')^{\frac{3}{2}}}{\sqrt{p}}.$$

Cor. 1.—Since the normal $n = \sqrt{p(p + 2x')}$ (art. 205),

$$\therefore \quad \rho = \frac{n^3}{p^2}.$$

219. *Cor.* 2.—If Dr be the chord of curvature, passing through the focus F, then the angle $ODr = ODt$, therefore $Dr = Dt$; or the chord of curvature passing through the focus is equal to $2p' = 2p + 4x'$.

220. Prop. XII.—*The area of the parabola is equal to two-thirds of the circumscribing parallelogram.*



Let AM be the arc of a parabola, A the vertex, and AP the axis. Complete the rectangle $APMQ$, and suppose AQ to be divided into n parts, Aq, qq', &c., each equal to h; also complete the rectangles Am, qm', $q'm''$, &c.

Let $AP = x$, $PM = y$; then the sum of the rectangles $Am + qm' + q'm'' +$ &c. is equal to

$$h\,(qm + q'm' \ldots\ldots\ldots + QM).$$

And the equation to the curve gives

$$qm = Ap = \frac{h^2}{2p}; \quad q'm' = \frac{2^2h^2}{2p}; \ldots\ldots QM = \frac{n^2h^2}{2p}.$$

Hence the sum of the rectangles $Am + qm' +$&c. is equal to

$$h\left(\frac{h^2}{2p} + \frac{2^2h^2}{2p} \ldots + \frac{n^2h^2}{2p}\right) = \frac{h^3}{2p}\,\frac{n(n+1)(2n+1)}{6} \quad \text{(Alg. 197)}$$

$$= \frac{n^3h^3}{6p}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right)$$

$$= \frac{xy}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right).$$

Now let n be indefinitely great, then the sum of the rectangles will approach indefinitely near to the curvilineal area AMQ, and the fractions $\frac{1}{n}$, $\frac{1}{2n}$ become indefinitely small. Hence the

curvilineal area $AMQ = \frac{1}{3}xy$;

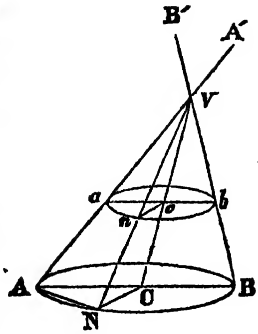
$\therefore$ the area $AMP = \frac{2}{3}xy$.

221. *Cor.*—The proof is nearly the same when AP is any diameter, and pm, PM, &c. are oblique ordinates.

CHAP. VI.—THE SECTIONS OF THE CONE AND CYLINDER.

DEFINITIONS.

222. If through the point V, without the plane of the circle ANB, a straight line, AVA', be drawn and produced indefinitely, both ways; and if the point remain fixed, while the straight line AVA' is moved round the whole circumference of the circle; two surfaces will be generated by its motion, each of which is called a *conical surface*,* and these mentioned together are called *opposite conical surfaces*.



The solid contained by the conical surface and the circle ANB is called a *cone*.

The fixed point V is called the *vertex* of the cone.

The circle ANB is called the *base* of the cone.

A straight line drawn from the vertex to the circumference of the base is called a *side* of the cone.

A straight line, VC, drawn from the vertex to the centre of the base, is called the *axis* of the cone. If the axis of the cone be perpendicular to the base, it is called a *right cone;* but if it be not perpendicular to the base, it is called an *oblique or scalene cone*.

223. If a straight line, AA' (see the figure to prop. 6), be moved round the circumference of the circle ANB, and kept always parallel to the line CC', which passes through the centre of the circle; the surface generated by the motion of AA' is called a *cylindrical surface*.

The solid contained by the cylindrical surface, the circle ANB, and a plane, $A'N'B'$, parallel to ANB, is called a *cylinder*.

The circle ANB is called the base of the cylinder.

Any line, NN', drawn from the circumference of the base parallel to CC', is called a side of the cylinder.

The line CC' is called the axis of the cylinder. If the axis of the cylinder be perpendicular to the base, it is called a *right cylinder;* but if it be not perpendicular to the base, it is called an *oblique cylinder*.

224. Prop. I.—*If a cone be cut by a plane passing through the vertex, the section will be a triangle.* (See the figure above.)

Let $VANB$ be a cone, of which VC is the axis; let AN be the common section of the base of the cone, and the cutting plane; join VA, VN. When the generating line comes to the points A and N, it is evident that it will coincide with the straight lines VA, VN; they are therefore in the surface of the cone, and they are in the plane which passes through the points V, A, N; therefore the triangle VAN is the common section of the cone, and the plane which passes through its vertex.

* By later writers these surfaces are frequently called *sheets*, a word taken from the French authors, who give them the name of *nappes*.

225. PROP. II.—*If a cone be cut by a plane parallel to its base, the section will be a circle whose centre is in the axis.*

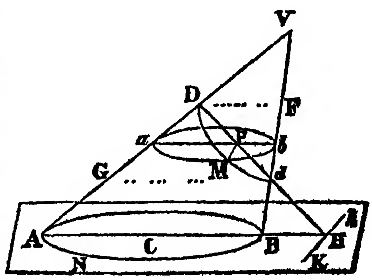
Let anb be a section parallel to the base of the cone; and VCA, VCN, two sections of the cone, made by any two planes passing through the axis VC. Let ca, cn, be the common sections of the plane anb, and the triangles VCA, VCN. Because the planes anb, ANB, are parallel, ca, cn are parallel to CA, CN (Geom. prop. 97); therefore

$$CA : ca :: VC : Vc :: CN : Cn.$$

But $CA = CN$, therefore $ca = cn$.

In like manner it may be shown that every other line drawn from the point c to the perimeter anb is equal to ca, and consequently the section anb is a circle of which c is the centre.

226. PROP. III.—*To find the equation to the curve formed by the intersection of a cone by a plane.*



Let any cone, VNB, be cut by a plane, DMd, which is not parallel to the base; and let this plane be produced, until it meet the plane of the base in the straight line Kk. Through C, the centre of the base, draw ABH perpendicular to Kk; and suppose a plane to pass through the points V, A, B, cutting the plane DMd in the line Dd, which we will take for the line of the abscissæ. The problem then will divide itself into three cases, according as Dd meets the line VB in the same surface AVB, or in the opposite surface $A'VB'$, or is parallel to VB. In the present proposition we shall consider the case when Dd meets VB, in the same surface AVB.

Let any plane, aMb, be drawn parallel to the base, cutting DMd in the line MP, and the plane AVB in the line aPb. Let $Dd = 2a$, $DP = x$, $PM = y$. Also draw DF, dG, parallel to AB, and put $DF = 2f$, $dG = 2g$. Because the plane DMd cuts the parallel planes aMb, ANB, their intersections PM, Kk, are parallel (Geom. prop. 97); for a similar reason, ab is parallel to AB. And since ab, PM, are parallel to AB, HK, the angle aPM is equal to AHK, and is therefore a right angle. Hence

$$y^2 = PM^2 = aP \times Pb.$$

But from the similar triangles DaP, DGd,

$$aP : x :: 2g : 2a; \quad \therefore \quad aP = \frac{gx}{a}.$$

Also from the similar triangles dPb, dDF,

$$Pb : 2a - x :: 2f : 2a; \quad \therefore \quad Pb = \frac{f}{a}(2a - x);$$

consequently the equation to the curve is

$$y^2 = \frac{fg}{a^2}(2ax - x^2).$$

Hence the conic section is an ellipse, whose diameters are $2a$, and $2\sqrt{fg}$.

227. *Cor.* 1.—If the plane VAB pass through the line drawn from V perpendicular to the base ANB, VAB will be perpendicular to NAB (Geom. prop. 102). Hence HK will be perpendicular to the plane VAB (Geom. prop. 103), and consequently perpendicular to DH. In this case, therefore, the ordinate PM will be perpendicular to the diameter Dd. Hence Dd is an axis of the ellipse.

228. *Cor.* 2.—If the angle $KdD = VAB$, the triangles DFd, DGd, are similar; therefore

$$2f : 2a :: 2a : 2g, \quad \text{and} \quad fg = a^2;$$

$$\therefore \quad y^2 = 2ax - x^2.$$

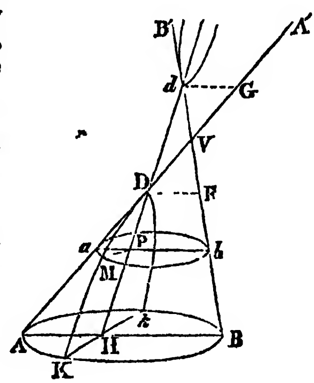
If, likewise, the plane VAB be perpendicular to the base, and consequently PM perpendicular to Dd, the curve in this case will be circle.

This section is called a *subcontrary section.*

229. Prop. IV.—*To determine the curve, when the section* DKk *meets* VB *in the opposite surface.*

Let the plane DMP meet the plane of the base, as before, in the line Kk; draw the diameter AB perpendicular to Kk, and let the figure be constructed as in the last proposition.

Let $DP = x$, $PM = y$, $Dd = 2a$, $DF = 2f$, $dG = 2g$. We then find, as before,



$$y^2 = aP \times P\,;$$

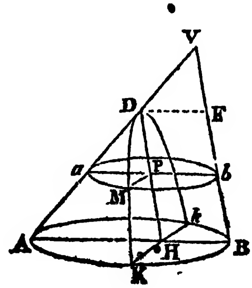
$$aP = \frac{gx}{a}; \qquad Pb = \frac{f}{a}(2a + x);$$

$$\therefore \quad y^2 = \frac{fg}{a^2}(2ax + x^2).$$

Hence the conic section is a hyperbola, whose diameters are. $2a$, $2\sqrt{fg}$.

230. Prop. V.—*To determine the curve when the section* DKk *is parallel to* VB.

Constructing the figure as before, and putting $DF = 2f$, $VF = d$; we have in this case, since DH is parallel to VB, and the triangles DaP, VDF, are similar,



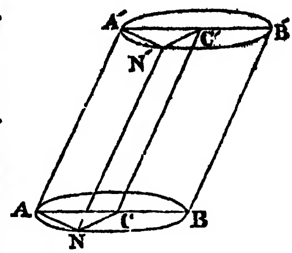
$$aP : x :: 2f : d;$$

$$\therefore \quad aP = \frac{2fx}{d}; \quad \text{also} \quad Pb = 2f.$$

$$\text{Hence} \quad y^2 = \frac{4f^2}{d}x\,;$$

and therefore the conic section is a parabola.

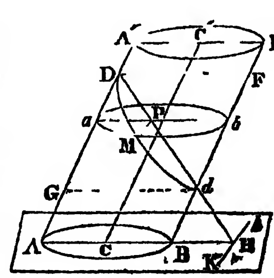
231. Prop. VI.—*If a cylinder be cut by a plane passing through the axis or parallel to it, the section will be a parallelogram: and if it be cut by a plane parallel to the base, the section will be a circle.*



(1.) Let the plane AN' be parallel to the axis CC', and let it cut the base of the cylinder in the line AN. Let AC', NC', be two planes passing through the points A, N, and the axis CC'; and let AA', NN', be the common intersections of these planes with the plane AN'. Because CC' is parallel to the plane AN', and AA' is the intersection of this plane with the plane AC', the lines AA', CC' are parallel, since they do not meet each other. Hence AA' is in the surface of the cylinder. In like manner, NN' also is in the surface of the cylinder. And because AA', NN', are both parallel to CC', they are parallel to each other (Geom. prop. 95, cor. 2). Also, since the planes CAN, $C'A'N'$, are parallel, and the plane AN' meets them, their intersections AN, $A'N'$, are parallel. Hence the figure $ANN'A'$ is a parallelogram.

(2.) Because AA' is parallel to CC' and $C'A'$ parallel to CA, therefore AC' is a parallelogram, and $CA = C'A'$. For the same reason, $CN = C'N'$. But because C is the centre of the circle ANB, $CA = CN$, therefore $C'A' = C'N'$. In like manner it may be shown that every other line drawn from C' to the perimeter $A'N'B'$ is equal to $C'A'$; consequently the section $A'N'B'$ is a circle of which C' is the centre.

232. Prop. VII.—*To find the equation to a curve formed by the intersection of a cylinder by a plane.*



Let any cylinder, AB', be cut by a plane, DMd, which is not parallel to the base, and let this plane be produced until it meet the plane of the base in the straight line Kk. Draw the diameter ABH perpendicular to Kk, and suppose a plane to pass through AB, and the axis CC', cutting the plane DMd in the line Dd, which we will take for the line of abscissæ. Let any plane, aMb, be drawn parallel to the base, cutting the plane DMd in MP, and the plane $C'AB$ in aPB.

Put $Dd = 2a$, $DP = x$, $PM = y$: draw DF, dG, parallel to AB, and put $AB = 2r$; then it may be proved, precisely in the same manner as in prop. 5, that

$$y^2 = \frac{r^2}{a^2}(2ax - x^2),$$

which is the equation to an ellipse.

233. *Cor.*—If the plane AB' be perpendicular to the base, and the angle DdB' be equal to DAB, then will $Pa = PD$, and $Pb = Pd$; therefore $Dd = ab = 2r$. Hence, in this case,

$$y^2 = 2rx - x^2.$$

And because PM is perpendicular to Dd (art. 60), therefore the *subcontrary* section DMd is a circle equal to the base.

CHAP. VII.—PROBLEMS ON LINES OF THE SECOND ORDER.

234. PROB. I.—*To determine the equation to a conic section which passes through five different points.**

Let $Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0 \ldots\ldots (1)$
be the general equation to the conic sections. Dividing by A, and putting $\frac{B}{A} = b$, $\frac{C}{A} = c$, &c., we have

$$y^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots\ldots (2);$$

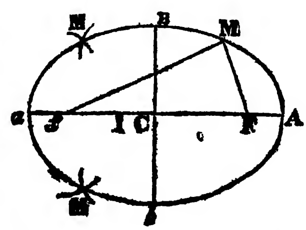
in which expression there are five indeterminate coefficients, and therefore we must have five equations to find their value. Hence it follows that five points must be given, to determine the particular form of the conic section.

Let α, β; α', β', &c., be the co-ordinates of the five given points. By substituting these values in equation (2), we get five independent equations, from which the values of the coefficients b, c, d, e, f, may be found.

235. *Scholium.*—In the preceding problem we have supposed the species of the curve to be unknown. If the curve be a circle, then three points are sufficient for this purpose, since $b = 0$, $c = 1$; and d, e, f, are the only three coefficients which remain undetermined. If the curve be a parabola, four points are sufficient; for in this case $b^2 = 4c$, and therefore there are only four independent coefficients.

236. PROB. II.—*The transverse and conjugate axes of an ellipse being given, to describe the ellipse.*

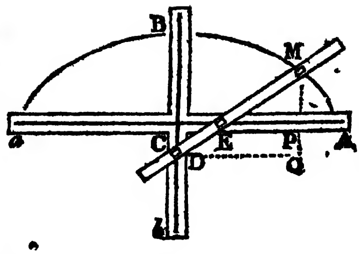
1. *By mechanical description.*—Let Aa, Bb be the transverse and conjugate axes, and C the centre. With the centre B and radius $= CA$ describe a circle cutting the transverse axis in F and f; these points will be the foci of the ellipse (art. 73).



Let the ends of a string equal in length to Aa be fastened at the points F, f; and while the string is kept uniformly stretched by a pencil, M, let the point of the pencil be carried round in the plane ABa. By this motion it will describe a curve line, which is the ellipse required, as is evident from art. 77.

2. *With the elliptic compasses or trammels.*—Aa, Bb are two cross-bars with grooves in them, at right angles. DEM is a ruler about a foot long, bearing three cursors, to one of which may be screwed points of any kind, and to the bottom of the other two are rivetted two

sliding dovetails, adjusted to the grooves in Aa, Bb. The dovetails being fixed on the ruler DM, and running along the grooves, the point M will describe an ellipse.



For if DQ, MQ be drawn parallel to CA, CB; and $DM = a$, $EM = b$, $CP = x$, $PM = y$

then $\quad MQ = \frac{a}{b} MP = \frac{a}{b} y.$

$$\text{Hence} \quad MQ^2 + DQ^2 = DM^2; \quad \text{or,} \quad \frac{a^2}{b^2} y^2 + x^2 = a^2;$$

$$\therefore \quad a^2y^2 + b^2x^2 = a^2b^2,$$

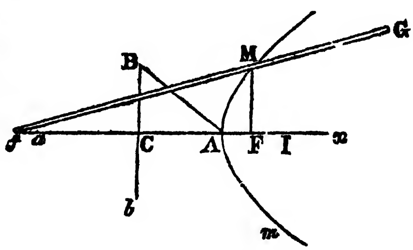
which is the equation to an ellipse, of which DM is the semi-transverse, and EM the semi-conjugate axis.

It is upon this principle that the engines for turning ovals are constructed.

3. *By finding a number of points in the curve.* (See the first figure in this problem.)—Find the two foci F, f, as before; and in the transverse axis take any point, I. Then with the radii AI, BI, and centres F, f, describe arcs intersecting in M, M; these will be points in the curve. And in the same manner any number of points may be found.

237. PROB. III.—*The transverse and conjugate axes of a hyperbola being given, to describe the curve.*

(1.) *By mechanical description.*—Let Aa, Bb be the two axes; join AB, and in Aa produced take CF, Cf, each equal to AB. The points F, f will be the foci of the hyperbola.



Let one end of a string be fastened at F, and the other to G, the extremity of a ruler, fMG; and let the difference between the length of the ruler and the string be equal to Aa. Let the other end of the ruler be fixed to the point f, and let the ruler be made to revolve about f as a centre, in the place of ACB, while the string is stretched by means of a pencil, M, so that the part of it between M and G is applied close to the edge of the ruler; the point of the pencil will by its motion trace a curve line, MAm, upon the plane, which is one of the hyperbolas required.

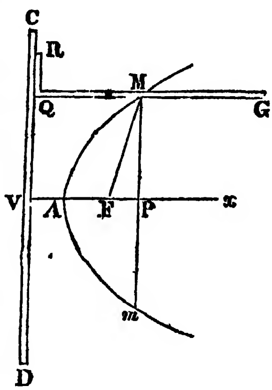
If the ruler be made to revolve about the other focus F, while the end of the string is fastened to f, the opposite hyperbola will be described by the moving point M.

(2.) *By finding a number of points in the curve.*—Find the two foci F, f, as before; and in the transverse axis Ax, take any point, I. Then with the radii AI, aI, and the centres F, f, describe arcs intersecting in M, m; these will be points in the curve. And in the same manner any number of points may be found.

Again, if in the axis ax' any point be taken, and arcs be described

as before, the intersections of these arcs will give the opposite hyperbola.

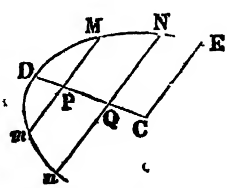
238. Prob. IV.—*The direction and focus of a parabola being given in position, to describe the parabola.*



(1.) *By mechanical description.*—Let VC be the given directrix, and F the focus. Place the edge of the ruler CD along the directrix, and keep it fixed in that position. Let RQG be another ruler, of such a form that the part QR may slide along CD, the edge of the fixed ruler, and the part QG may have its edge constantly perpendicular to CD. Let FMG be a string of the same length as QG, the edge of the moveable ruler; let one end of the string be fastened at F, and the other end fastened to G, the end of the moveable ruler. By means of a pencil, M, let the string be stretched so that the part of it between G and M may be applied close to the edge of the fixed moveable ruler, while at the same time it slides along CD, the edge of the fixed ruler. The pencil M will thus move along QG, the edge of the ruler, and its point will trace upon the plane CVF a curve line, which is the parabola required. For since the string $GMF = GQ$, the part $MF = MQ$; therefore the pencil describes a parabola whose focus is F, and directrix is VC.

(2.) *By finding a number of points in the curve.*—Through the focus F draw Vx perpendicular to the directrix, and Vx will be the axis of the curve. Draw any straight line, Mm, parallel to the directrix, cutting Vx in P; and from the centre F, with the radius VP, describe arcs cutting Mm in M and m; these will be two points in the parabola required. And in the same manner any number of points may be found.

239. Prob. V.—*An arc of a conic section being given, to determine to which of the curves it belongs; and also the axes and foci of the curves.*



Draw any two parallel chords, MPm, NQn; and draw a line, DPQ, bisecting these chords. DP is a diameter of the curve. Draw any two other parallel chords, and a line, $D'P'$, bisecting them. Then if the lines DP, $D'P'$ meet on the concave side of the arc, the curve is an ellipse, and the point of intersection is the centre. If the lines meet on the convex side of the arc, the curve is a hyperbola; and if they are parallel, the curve is a parabola.

(1.) Let the curve be an ellipse. To find the diameter conjugate to CD, draw CE parallel to Mm; and putting $CD = a'$, $CE = b'$, $CP = x$, $PM = y$, we have

$$a'^2y^2 + b'^2x^2 = a'^2b'^2;$$

in which equation a', x, y, are known, and therefore b' may be found.

To find the axes $2a$, $2b$, we have, putting the angle $DCE = \varphi$,

$$a^2 + b^2 = a'^2 + b'^2; \qquad ab = a'b' \sin\varphi;$$

from which equations a and b may be determined.

To find the *position* of the axes: let α, β be the angles which the diameters CD, CE make with the transverse axis; then putting $\tan\varphi = t$, which is supposed to be known, we have

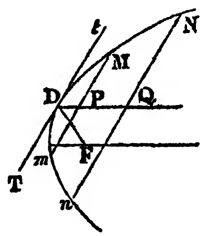
$$t = \tan\varphi = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta}.$$

But $\tan\beta = -\dfrac{b^2}{a^2\tan\alpha}$ (art. 101); hence

$$-\frac{b^2}{a^2\tan\alpha} - \tan\alpha = t\left(1 - \frac{b^2}{a^2}\right).$$

And the solution of this equation will give the value of $\tan\alpha$.

(2.) When the curve is a *hyperbola*, the diameters and axes may be found in the same manner.

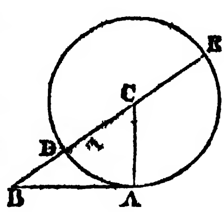


(3.) If the curve be a *parabola*: let $2p$ be a third proportional to DP and PM, then will $2p$ be the parameter to the diameter DP. Through D draw tDT, parallel to Mm; tT will be a tangent to the curve. Make the angle $TDF = PDt$, and take $DF = \frac{1}{2}p$; F will be the focus: and a line drawn through F, parallel to DP, will be the axis.

Prob. VI.—*To determine the roots of a quadratic equation by means of a geometrical construction.*

240. If two curves can be constructed whose equations are known, the intersections of these curves will give values of x and y, which will satisfy both equations. If, therefore, y be eliminated from these equations, the values of x, in the resulting equation, will correspond to the intersections of the two curves; and conversely, the intersections of these curves will give the roots of the resulting equation. This method, known by the name of the *construction of equations*, was formerly much used by mathematicians; but from the great improvements which have since been made in the solution of equations, it can now only be considered as an object of curiosity.

The roots of a quadratic equation may be found from the intersection of a straight line with a line of the second order. For since the equations to these two lines are of the first and second degrees, the equation resulting from the elimination of y will be of the second degree (Alg. art. 120), and may be made to coincide with any proposed quadratic. As the circle is more easily described than any other curve of the second order, it is always taken to construct the roots of a quadratic equation.



241. The four forms of a quadratic equation are—

(1) $x^2+px = q = a^2$; (2) $x^2-px = a^2$;

(3) $x^2-px = -a^2$; (4) $x^2+px = -a^2$.

Construction of the first and second forms.—Let a circle, ADE, be described, with a radius $CA = \frac{1}{2}p$. From any point, A, in the circum-

ference draw the tangent $AB = a$, or a mean proportional between 1 and q; and through B draw the diameter BDE; then will BD, BE, represent the positive and negative roots of the first and second equations; the less line being the positive root in the first case, and the greater line in the second case.

For by construction $DE = p$; and if BD be put $= x$, then $BE = x + p$; and by Geometry,

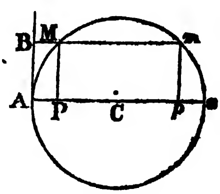
$$BD \times BE = BA^2; \text{ or, } x^2 + px = a^2.$$

Or if $BE = -x$, then $BD = BC - DE = -x - p$; therefore

$$(-x)(-x-p) = x^2 + px = a^2.$$

Hence BD and $-BE$ are the two roots of the first equation.

In the same manner it may be shown that BE and $-BD$ are the two roots of the second equation.



Construction of the third form.—Let a circle, AMm, be described, with a radius $CA = \frac{1}{2}p$, as before. Draw any diameter, Aa and AB, perpendicular to it, and equal to a. Draw BMm parallel to the diameter Aa, cutting the circumference in the points M, m; then will BM, Bm, represent the two positive roots of the third equation. Draw MP, mp perpendicular to Aa. Then if AP or $Ap = x$, aP or ap will be $= p - x$; therefore

$$AP \times Pa = PM^2, \text{ or } px - x^2 = a^2.$$

Hence AP and Ap are the two roots of this equation.

If BM does not meet the circumference, the two roots of the equation are impossible.

Construction of the fourth form.—The roots of this equation are the same as those of the third form, except that they are affected with a negative sign.

242. Prob. VII.—*To determine the roots of a biquadratic equation by a geometrical construction.*

It appears from Algebra (art. 122), that if we have two general equations of the second degree in x and y, and one of the unknown quantities be eliminated, the resulting equation will be of the fourth degree. If, therefore, this equation be made to coincide with any proposed biquadratic, the intersections of the two curves which represent the two equations of the second degree will give the roots of this biquadratic equation.

Since the circle is the most easy to be described of all curves of the second degree, we should first take the equation to two circles, and eliminate y; but it will immediately be found that the resulting equation rises only to the second degree. This agrees with what is proved in Geometry, that the circumferences of two circles cannot intersect one another in more than two points (Geom. prop. 36, cor.)

The most simple curve, after the circle, is a parabola. Let, therefore,

$$y^2 = ax; \quad y^2 + x^2 + Dy + Ex + F = 0,$$

be the equations to a parabola and a circle. Substituting, in the second equation, the value of x taken from the first, we get

$$y^4 + (a^2 + Ea)y^2 + Da^2y + Fa^2 = 0;$$

and this may be made to coincide with any equation,

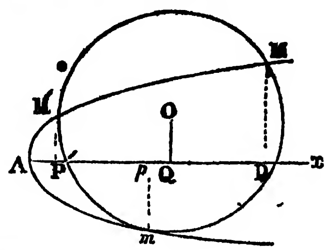
$$y^4 + qy^2 + ry + s = 0,$$

of the fourth degree, which wants the second term.

Since we have four indeterminate quantities, a, D, E, F, and only three equations, we may assume one of these quantities, for example, $a = 1$; thus, the same parabola whose equation is $y^2 = x$ will serve for all equations.

243. *Cor.*—The roots of a cubic equation may be constructed in the same manner; for if we multiply the equation by y, it will become a biquadratic equation, one of whose roots is equal to nothing.

244. *Ex.*—Let it be required to construct the roots of the equation



$$y^4 - 9y^2 - 4y + 12 = 0.$$

If this be compared with the equation $y^4 + (1 + E)y^2 + Dy + F = 0$, we have

$$D = -4, \quad F = 12, \quad 1 + E = -9,$$
$$\text{or} \quad E = -10.$$

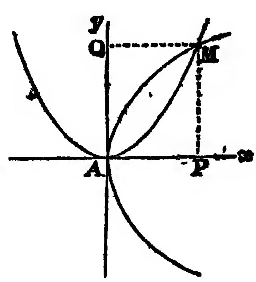
Hence the equation to the circle becomes

$$y^2 + x^2 - 4y - 10x + 12 = 0; \quad \text{or,} \quad (y-2)^2 + (x-5)^2 = 17.$$

Let the parabola MAm be described, whose parameter is unity. Draw the axis AP, and take $AQ = 5$, and QO (perpendicular to AQ) $= 2$. With the centre O and radius $= \sqrt{17}$ describe the circle $MM'm$, meeting the parabola in the points M, M', m. Draw the ordinates MP, $M'P'$, mp; and measuring these values, we shall find the positive root $MP = +3$, $M'P' = +1$; and since the circle touches the parabola at m, there are two negative roots, pm, each $= -2$.

245. PROB. VIII.—*To find two mean proportionals between two given lines,* a *and* b.

Let x and y be the required lines. Then



$$a : x :: x : y :: y : b;$$

$$\therefore \quad x^2 = ay, \quad \text{and} \quad y^2 = bx.$$

Hence $x^4 = a^2y^2 = a^2 \times bx$; or $x^4 - a^2bx = 0$.

The roots of this equation may be found by the intersection of a circle and a parabola, as in the last problem. The following solution, however, is interesting, as being the first known instance of the application of geometrical foci to plane problems. Menechme, of the school of Plato, was the author of it.

With the axis Ax, and parameter b, describe the parabola MA.

With the axis Ay, and parameter a, describe the parabola MA.

Then if $AP = x$, $PM = y$, we shall have $y^2 = bx$, $x^2 = ay$; or

$$a : x :: x : y :: y : b;$$

and consequently a, x, y, b are in continued proportion.

246. *Scholium.*—Another problem, equally celebrated, was to find the side of a cube which should be double of a given cube. But we have shown in the scholium, vol. i. p. 406, that this problem is only a particular case of the one above; and that if in the last problem $b = 2a$, the two solutions will be the same.

247. Prob. IX.—*To trisect an angle.*

Let the given angle $BAC = 3\alpha$; then the problem will be solved, if $\cos \alpha$ be found in terms of $\cos 3\alpha$. Now (Trig. art. 67),

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha:$$

and putting $\cos \alpha = y$, $\cos 3\alpha = a$, we get

$$y^3 - \tfrac{3}{4}y - \tfrac{1}{4}a = 0 \ldots\ldots\ldots (1);$$

from which equation we are to determine the value of y.

Multiplying by y, this equation becomes

$$y^4 - \tfrac{3}{4}y^2 - \tfrac{1}{4}ay = 0 \ldots\ldots\ldots (2);$$

and this equation may be derived by eliminating x, either from the two equations

$$y^2 = x; \qquad x^2 - \tfrac{3}{4}x - \tfrac{1}{4}ay = 0;$$

or from the two equations

$$y^2 = x; \qquad y^2 + x^2 - \tfrac{1}{4}ay - \tfrac{7}{4}x = 0.$$

In the first case, the roots of equation (2) will be given by the intersection of two parabolas; and in the second case, by the intersection of a circle and a parabola.

Problems for Practice in Conic Sections.

1. A parabola being given in position, to find its focus, axis, and directrix.

2. An ellipse and a hyperbola being given in position, to find their axes and foci.

3. From a given point, either in the curve or without it, to draw a tangent to the parabola, when the focus is given.

4. From a given point, either in the curve or without it, to draw a tangent to an ellipse or a hyperbola, when the two foci are given.

5. Construct accurately, on paper, a parabola whose abscissa and double ordinate shall each be 6 inches.

6. Construct an ellipse whose transverse axis shall be 8 inches, and conjugate axis 4 inches.

7. Construct the four conjugate hyperbolas whose two axes are $1\frac{1}{2}$ and 3 inches.

8. If the axes of an ellipse be 60 and 80; what are the lengths of two conjugate diameters, one of which makes an angle of 20° with the transverse axis? Ans. 64·2941 and 76·5915.

9. If the axes of an ellipse be 60 and 100 inches; it is required to find the radius of a circle which will touch the curve, when the centre is in the transverse axis, at the distance of 16 inches from that of the ellipse. Ans. 27·49545 inches.

10. Required the radius of curvature to the parabola and the ellipse, in problems 5 and 6, at the extremities of the axes.

11. If VAB be a right cone in art. 226; prove that DG is equal to the distance between the foci of the ellipse DMd.

12. Prove that the polar equation to a circle, any point within or without the circle being the pole, is

$$r^2 - 2r(\beta \sin \varphi + \alpha \cos \varphi) r + \alpha^2 + \beta^2 - a^2 = 0;$$

r being the radius vector, a the radius of the circle, and α, β, the co-ordinates of the centre.

Construct the figures to which the following equations belong:—

13. $4y^2 + 9x^2 = 36$.

14. $y^2 - 6y + 8 = 0$.

15. $y = 9 + 6x + x^2$.

16. $y^2 - xy + x^2 + y + x - 1$.
Ans. An ellipse whose axes are 4 and $\frac{4}{3}\sqrt{3}$.

17. $3y^2 - 4xy + 3x^2 + y - x - \frac{9}{10} = 0$.
Ans. An ellipse whose axes are 2 and $\frac{2}{5}\sqrt{5}$.

18. $y^2 - 2xy + x^2 - 6y - 6x + 9 = 0$.
Ans. A parabola whose latus rectum is $3\sqrt{2}$.

19. $y^2 - 2xy - x^2 - 2 = 0$.
Ans. A hyperbola whose axes are each $2\sqrt[4]{2}$.

20. $y^2 - 10xy + x^2 + y + x + 1 = 0$.
Ans. A hyperbola whose axes are $\frac{1}{2}\sqrt{3}$, and $\frac{1}{3}\sqrt{2}$.

CHAP. VIII.—CURVES OF THE HIGHER ORDERS, AND TRANSCENDENTAL CURVES.

248. We have seen, in the preceding chapters, that an equation of the first degree always represents a straight line; and an equation of the second degree represents three different species of curve lines; besides the circle, which is a particular kind of ellipse, and the straight line, which is included in all equations of a higher degree.

When we come to equations of a higher degree, we find that eighty different kinds of curves are represented by the general equation of the third degree, and more than 5000 different species are included in that of the fourth degree. It would be utterly impossible, therefore, to give a general investigation of all the different curves in the higher orders. We can only therefore select, in a work of this limited extent, some particular examples, which, either from their history or their utility, deserve more especially the attention of the student.

CURVES OF THE PARABOLIC CLASS.

249. The general equation to curves of this class is

$$y = mx^n + px^{n-1} \ldots . + tx + u;$$

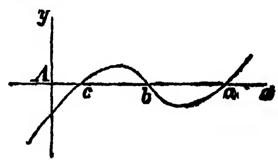
and the curve represented by this equation is called a parabola of the nth order.

250. Prop. I.—*To trace the parabola of the third degree, whose equation is* $y = mx^3 + px^2 + qx + r$.

(1). Suppose the three roots of the equation $mx^3+px^2+qx+r = 0$ to be real, and equal to a, b, c, taken in order; then (Alg. art. 270)

$$y = m(x-a)(x-b)(x-c)$$

If x be greater than a, y is positive; and if x be indefinitely great, y is indefinitely great. If $x < a$ and $> b$, then y is negative; and if x be $< b$ and $> a$, y is again positive. If $x < c$, y is negative; and if x be negative and indefinitely great, y is also negative and indefinitely great. Hence the curve has the form of the annexed diagram.



(2). If two of the roots be equal, one of these semi-ovals will disappear; and if two of the roots be impossible, both semi-ovals will disappear, and the curve will only cut the axis of x once.

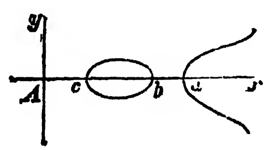
(3). If three roots are equal, the equation is of the form $y = m(x-a)^3$; and the curve is then called the *cubical parabola.*

251. Prop. II.—*To trace the curve whose equation is*

$$y^2 = mx^3 + px^2 + qx + r.$$

(1). Suppose the roots of the equation $mx^3 + px^2 + qx + r = 0$ to be real and unequal. Then, if these roots be a, b, c, taken in order,

$$y = \sqrt{m}\,\sqrt{[(x-a)(x-b)(x-e)]}.$$

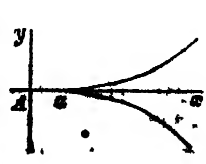


Hence, if $x > a$, y has two real and equal values, with different signs. If $x < a$ and $> b$, y is impossible. If $x < b$ and $> c$, y has two equal values, with contrary signs. If $x < c$, y is again impossible. Hence the curve consists of an oval and two infinite branches, as in the adjoining figure.

(2). If two roots be equal, or $a = b$, the figure will be the same as that above, when the points a and b coincide. If $b = c$, the oval will be reduced to a point, the other part of the figure remaining the same as before.

(3). If two roots be impossible, the oval disappears altogether.

(4). If $a = b = c$, $y^2 = m(x-a)^3$. The figure in this case has two branches, with their convexity towards the axis. This curve is called the *semicubical parabola.* If the origin be at a, the equation becomes $y^2 = mx^3$.



CURVES OF THE HYPERBOLIC CLASS.

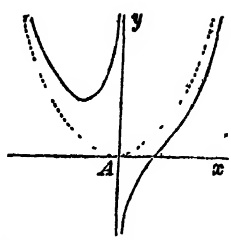
252. These curves are represented by the general equation

$$x^m y^n = a,$$

and their forms are similar to the common hyperbola (art. 188).

253. Prop. III.—*To trace the curve whose equation is*

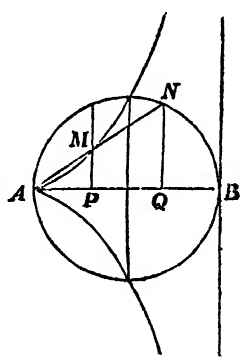
$$xy = mx^3 + px^2 + qx + r.$$



The axis of y is evidently an asymptote; the other part of the figure will depend upon the nature of the roots.

If $y = \frac{x^3 - a^3}{ax}$ the curve is called a *trident*, from its form. If a parabola be described, as in the dotted part of the annexed figure, whose equation is $y = \frac{x^2}{a}$, the branches of the parabola will approach indefinitely near to the branches of the trident. These branches of the trident are called *parabolic* branches; whereas the other branches, which approach indefinitely near to a hyperbola and the asymptotes of a hyperbola, are called *hyperbolic* branches.

THE CISSOID OF DIOCLES.



254. Let AB be the diameter of a circle, ANB; from the points P and Q, taken always at equal distances from A and B, draw PM, QN at right angles to AB, and join AN, meeting PM in M; the point M will trace out a curve called the *cissoid of Diocles*.*

255. Prop. IV.—*To find the equation to the cissoid.*

Let $AP = x$, $PM = y$, $AB = a$; then, from the similar triangles, APM, AQN,

$$AP^2 : PM^2 :: AQ^2 : QN^2, \text{ or, } AQ \times QB;$$

$$\therefore\ x^2 : y^2 :: a - x : x.$$

Hence the equation to the curve is

$$y^2 = \frac{x^3}{a - x}.$$

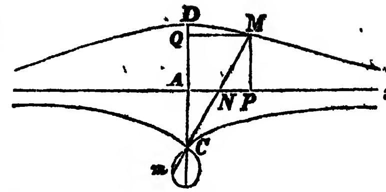
Scholium.—This curve was invented by Diocles to solve the celebrated problem of the *Duplication of the cube*, and the *Insertion of two mean proportionals between two given lines.*

THE CONCHOID OF NICOMEDES.

256. Def.—Let Ax be a line given in position, and about any point C, taken without it, let the indefinite line CM revolve, and cut Ax in

* From κισσο, *ivy*, because the curve climbs up its asymptote, like ivy up a tree.

N; then, if NM be taken always of the same length, the point M will trace out a curve called the *conchoid of Nicomedes.**



The point C is called the *pole of the conchoid.* Ax is called the *directrix*, and AD the *modulus.*

257. Prop. V.—*To find the equation to the conchoid.*

Draw CAD and MP at right angles to AB, and MQ parallel to it; let $CA = a$, $AD = NM = b$, $AP = x$, $PM = y$; then, from the similar triangles CQM, MPN,

$$CQ^2 : QM^2 :: MP^2 : PN^2,$$

or, $$(a+y)^2 : x^2 :: y^2 : b^2 - y^2;$$

$$\therefore \quad x^2y^2 = (a+y)^2(b^2-y^2)$$

is the equation to the curve.

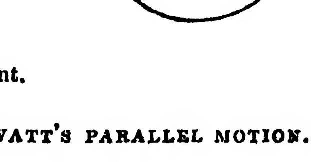
If Nm be measured in the opposite direction equal to b, the equation to the curve is

$$x^2y^2 = (a-y)^2(b^2-y^2).$$

DM is called the superior, and dm the inferior conchoid; if b be $> a$, the inferior conchoid will have the form which is shown in the figure; if $b = a$, there will be no oval, and the points C and D will coincide; if b be $< a$, the curve will have the same form as the superior conchoid; but the point C, although an insulated point, belongs to the curve, and is called a *conjugate* point.

258. *Scholium.*—This curve was invented by Nicomedes to solve the problems of the *duplication of the cube*, and the *trisection of an angle.* The latter problem may be briefly investigated as follows:

Let COB be the angle to be trisected. With the centre O, and any radius OC, describe the circle BCE. With the pole C, the directrix AF, and modulus OC, describe the conchoid CE, cutting the circle in E. Join BE and produce it to F, then will the angle EOF be a third part of the angle COB, or the arc DE = a third of the arc BC.



The demonstration is sufficiently evident.

THE LEMNISCATE,† GENERATED BY WATT'S PARALLEL MOTION.

259. Prop. VI.—*Let* BC *be a straight line of given length, having its extremities always in the circumference of two equal circles; to find the locus of the middle point,* M, *of the line* BC.

Let O, O' be the centres of the two circles; bisect OO' in A; and

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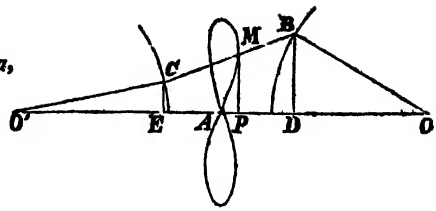
* From κογχη, signifying a shell.

† From *lemniscus*, a ribbon. Any curve of the form of the figure 8 is called a *lemniscate.*



draw the ordinates MP, BD, CE. Put
$OB = O'C = r$, $OA = O'A = a$,
$BM = CM = d$. Also, put

$AD = t$, $DB = u$;
$AE' = t'$, $EC = u'$;
$AP = x$, $PM = y$.

We have then the following equations:

$$u^2 + (a-t)^2 = r^2; \qquad u'^2 + (a-t')^2 = r^2 \ldots (1, 2),$$

$$(u-u')^2 + (t+t')^2 = 4d^2 \ldots\ldots\ldots\ldots (3),$$

$$2y = u + u'; \qquad 2x = t - t' \ldots\ldots\ldots (4, 5);$$

and, to get the equation to the curve, we must eliminate the four quantities, t, t', u, u'.

We shall first eliminate the rectangles tt', uu', by adding equation (3) to the squares of equations (4) and (5); we then get

$$u^2 + u'^2 + t^2 + t'^2 = 2d^2 + 2y^2 + 2x^2.$$

Also, adding equations (1) and (2) together, we get

$$u^2 + u'^2 + t^2 + t'^2 = 2r^2 - 2a^2 + 2a(t + t').$$

Putting these two values equal to each other, we obtain

$$a(t + t') = y^2 + x^2 + a^2 + d^2 - r^2;$$

or, substituting m^2 for $a^2 + d^2 - r^2$,

$$a(t + t') = y^2 + x^2 + m^2 \ldots\ldots\ldots (6).$$

We have also, from equations (1) and (2),

$$u^2 - u'^2 + t^2 - t'^2 - 2a(t - t') = 0;$$

therefore, substituting the values of $u + u'$, and $t - t'$, from equations (4) and (5),

$$(u - u')\,2y + (t + t')\,2x - 4ax = 0.$$

Hence $(u - u')\,y = x[2a - (t + t')] = x\left[2a - \dfrac{y^2 + x^2 + m^2}{a}\right]$

$$= \frac{x}{a}[n^2 - (y^2 + x^2)] \ldots\ldots\ldots\ldots (7).$$

Putting $2a^2 - m^2 = a^2 - d^2 + r^2 = n^2$.

Hence, substituting the values of $(t + t')$, $(u - u')$, derived from (6) and (7), in equation (3), we obtain

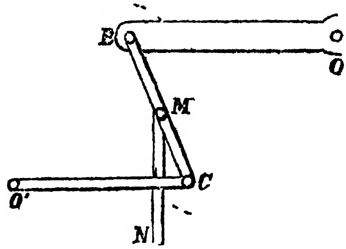
$$\frac{x^2}{y^2}[n^2 - (y^2 + x^2)]^2 + [y^2 + x^2 + m^2]^2 = 4a^2d^2;$$

and therefore the final equation to the curve is

$$y^2(y^2 + x^2 + m^2)^2 + x^2(y^2 + x^2 - n^2)^2 = 4a^2d^2y^2.$$

Cor.—If the circles be unequal, and M be any point in the line BC, the curve will be of the same nature; but the investigation is longer, and the equation more complicated.

260. *Scholium.*—If the lemniscate be constructed by points, it will be found that the curve is a double oval, such as is represented in the preceding diagram. At A there is a point of contrary flexure, and the curve, for a considerable length, differs insensibly from a straight line. The beautiful contrivance of Watt, to reduce a circular to a rectilineal motion in the steam-engine, is founded upon this property.



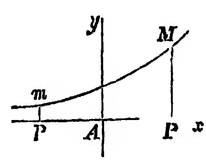
Let OB be the extremity of an *engine-beam,* movable about its centre O; and $O'C$ the *radius-rod,* which moves about the centre O'. The two extremities, B, C, are joined together by a bar, BC; and the piston, MN, is fixed at M, the middle of the bar. Since the extremities of the bar BC describe the circumference of two circles, which have their convexities turned opposite ways, or towards each other, it is evident that there is some point in the bar that will have the convexity of its path turned neither way; that is, will be neither concave nor convex, but insensibly straight, for a certain portion of its ascent and descent. And it was, probably, from considerations of this kind that Watt was led to the discovery of this contrivance.

TRANSCENDENTAL CURVES.

261. Def.—Those curves whose equations cannot be expressed in a finite number of terms, containing only integral powers of x and y, are called *transcendental curves* (art. 6).

THE LOGARITHMIC CURVE.

262. Def.—The curve MCm, of which the abscissa AP is the logarithm of the corresponding ordinate PM, is called the *logarithmic curve.*



263. Prop. VII.—*To trace the logarithmic curve.*

Let $AP = x$, $PM = y$, then $x = \log y$; or if a be the base of the system of logarithms, $y = a^x$.

When $x = 0$, $y = a^0 = 1$; as x increases from 0 to ∞, y increases from 1 to ∞; as $-x$ increases from 0 to ∞, y decreases from 1 to a quantity indefinitely small; hence CM continually approaches the axis Ap, which is therefore an asymptote.

THE CYCLOID.

264. Def.—If a circle be made to roll in a plane upon a straight line AB, the point M in the circumference, which was in contact with AB at the beginning of the motion, will, in a revolution of the circle, describe a curve, $AMDB$, called a *cycloid.**

The line AB is called the *base* of the cycloid.

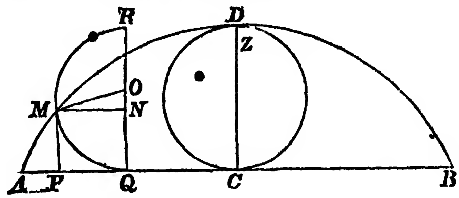
* From κυκλος, a circle, and ειδος, figure.

The circle QMR is called the *generating circle.*

The line CD, which is drawn bisecting AB at right angles, and produced until it meets the curve in D, is called the *axis*, and the point D the *vertex*.

265. *Cor.*—It follows, from the description of the cycloid, that the base, AB, is equal to the circumference of the generating circle; and AC to half the circumference. Also, when the generating circle comes to C, draw the diameter Cz, which will be perpendicular to AB. And because the circle has completed half a revolution, z is the generating point.

266. Prop. VIII.—*To find the equation to the cycloid.*



Since the circle in rolling applies every point in its circumference to the line AB, it is evident that in any situation, as QMR, the distance AQ is equal to the arc MQ contained between the point M, which first touched the line AB in A, and the point Q, which is in contact with it in its present position.

Draw the diameter QR, which will be perpendicular to AB; also, draw MP perpendicular and MN parallel to AB; and draw the radius of the generating circle MO.

Let $QO = a$, $AP = x$, $PM = QN = y$, $\angle MOQ = \theta$.

Then $y = OQ - ON = a - a\cos\theta = a \operatorname{vers}\theta$;

$$\therefore \quad \operatorname{vers}\theta = \frac{y}{a}, \quad \text{and} \quad \theta = \operatorname{vers}^{-1}\frac{y}{a}.$$

Also, $x = AQ - PQ = \text{arc } MQ - MN = a\theta - \sqrt{2ay - y^2}$.

Hence the equation to the cycloid is

$$x = a \operatorname{vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2}.$$

SPIRALS.

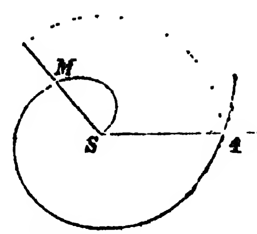
267. Def.—Spirals are transcendental curves, which derive their name from making several revolutions round a fixed point, and receding at the same time continually from this point.

THE SPIRAL OF ARCHIMEDES.

268. Def.—If a straight line, SA, move uniformly round a point, whilst a point, M, also moves uniformly along the line, then the point M will trace out a curve line called the *Spiral of Archimedes.*

269. Prop. IX.—*To find the polar equation to the spiral of Archimedes.*

Let the point M start from S, at the same time that the line SM com-



mences its motion from S; and let A be the position of the point M, when SA has made one revolution.

Put $SM = r$, $\angle ASM = \theta$;

then, since the increase of r and θ is uniform,

$$SM : SA :: \angle ASM : \text{four right-angles}$$
$$:: \theta : 2\pi.$$

Hence, putting $\dfrac{SA}{2\pi} = a$, the equation to the curve is

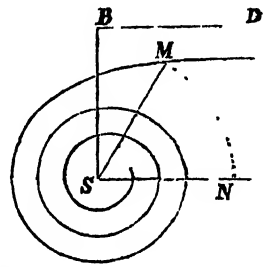
$$r = a\theta.$$

Scholium.—The spiral of Archimedes is sometimes used by architects for the volutes of the capitals of columns.

270. Prop. X.—*To trace the spirals when* r *varies as* θ^n.

The general equation to the spiral, in this case, is

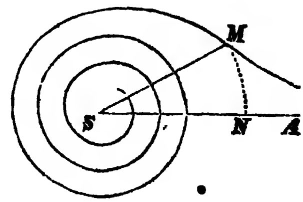
$$r = a\theta^n.$$



(1). If $n = -1$, the equation is $r = a\theta^{-1}$, or, $r\theta = a$. This curve is called the *reciprocal* or *hyperbolic* spiral, from the similarity of its equation to that of the hyperbola referred to its asymptotes (art. 180).

When $\theta = 0$, r is infinite. If $r = 0$, θ is infinite; the spiral therefore makes an infinite number of revolutions round S.

Since $MN = r\theta = a$, the value of MN is the same for all positions of SM. Hence it is evident, if SB be drawn at right angles to SA, and taken $= a$, that MN will approximate to a straight line parallel and equal to SB. Hence the line BD, parallel to SA, is an asymptote to the curve.



(2). If $n = -\frac{1}{2}$, we get immediately $r^2\theta = a^2$. This curve is called the *lituus*.* If $\theta = 0$, r is infinite. If $r = 0$, θ is infinite. This spiral, therefore, makes an infinite number of convolutions round S.

Since $MN = r\theta = \dfrac{a^2}{r}$, when r becomes indefinitely great, MN becomes indefinitely small. Hence it is evident that SA is an asymptote to the curve.

THE LOGARITHMIC SPIRAL.

271. Def.—If the angle of ASM be proportional to the logarithm of the radius vector, the curve is called the *logarithmic spiral*.

* From the Latin *lituus*, a trumpet.

272. PROP. XI.—*To find the equation to the logarithmic spiral.*

Let the angle $ASM = \theta$, $SM = r$; then, if a be the base of the system of logarithms, the equation to the spiral is

$$\theta = \log r, \quad \text{or,} \quad r = a^{\theta}.$$

273. *Scholium.*—The logarithmic spiral was invented by Descartes, and several of its curious properties investigated by James Bernoulli; who was so delighted with its singular property of reproducing itself in its involute, evolute, &c., that he desired to have it engraved on his tomb, with this inscription—*Eadem numero mutata resurgo.*

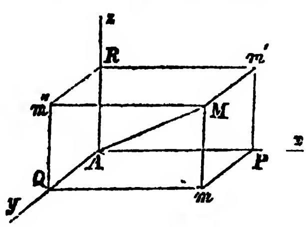
PART II.—ANALYTICAL GEOMETRY OF THREE DIMENSIONS.

274. In the preceding part of this Treatise we have supposed all the lines and points to be situated in the plane of the co-ordinate axes of x and y; in which case the Geometry is said to be of two dimensions. We now come to the consideration of lines and surfaces situated any where in space; and therefore this branch is entitled Solid Geometry, or Geometry of Three Dimensions.

CHAP. I.—THE POINT, THE STRAIGHT LINE, AND THE PLANE.

THE POINT.

275. The position of a point in a plane was shown to be determined, when its distance from two given lines, the axes of x and y in that plane, is known (art. 5). When a point is given in space, we conceive two lines to be drawn at right angles to each other, which we may call the axes of x and y; and, for the sake of distinctness, we will suppose the plane passing through these lines to be horizontal. We then suppose a third axis, Az, to be at right angles to the plane xAy, and consequently perpendicular to the axes Ax, Ay. The situation of any point, M, then will be determined, if we know the length of the perpendicular, Mm, drawn from M upon the plane xAy, and also the situation of the point m, or the co ordinates of the point where the perpendicular Mm meets the plane xAy.



276. Suppose the two planes zAx, zAy to be drawn through Az; these planes will be perpendicular to the plane xAy; also, suppose the

parallelopiped AM to be completed. Then the perpendiculars from M upon the three planes xAy, xAz, yAz, namely, Mm, Mm', Mm'', are respectively equal to AR, AQ, AP, or equal to Mm, Pm, AP; and these three lines are called the co-ordinates of M.

The point A is called the *origin* of co-ordinates.

The three planes xAy, xAz, yAz, are called the planes of xy, xz, and yz, respectively.

The three points m, m', m'', in which the perpendiculars Mm, Mm', Mm'' meet the three co-ordinate planes, are called the *projections* of the point M on the planes of xy, xz, and yz, respectively.

277. If $AP = a$, $Pm = b$, $mM = c$, the position of the point M is completely determined by the equations $x = a$, $y = b$, $z = c$; and therefore these are called the equations to the point M.

278. The three co-ordinate planes, when produced indefinitely, form eight different solid angles at the point A, of which four are situated above the plane of xy, and four beneath it. The same rules also are applicable to the signs of the co-ordinates, as in Plane Geometry: thus, if the values of z when measured above the horizontal plane be positive, when they are below this plane they must be considered negative. Hence we are enabled to determine immediately in which of these eight solid angles any point, M, is situated.

279. Prop. I.—*To find the distance of any point,* M, *from the origin of co-ordinates.*

Let Ax, Ay, Az be the three rectangular axes. Draw Mm perpendicular to the plane xy; mP perpendicular to Ax, and mQ to Ay. Put $AP = x$, $Pm = y$, $mM = z$; and also $AM = d$.

Because Mm is perpendicular to the plane of xy, the angle MmA is a right angle. Hence

$$AM^2 = Am^2 + Mm^2 = AP^2 + Pm^2 + Mm^2;$$

$$\therefore \quad d^2 = x^2 + y^2 + z^2.$$

280. *Cor.* 1.—Let α, β, γ be the angles which AM makes with the axes Ax, Ay, Az, respectively. Because AP is perpendicular to the plane MP, it is perpendicular to the line MP; therefore

$$AP = AM \cos MAP; \quad \text{or,} \quad x = d \cos \alpha.$$

In like manner $y = d \cos \beta$, $\quad z = d \cos \gamma$.

Hence $\quad d^2 = d^2 \cos^2 \alpha + d^2 \cos^2 \beta + d^2 \cos^2 \gamma;$

$$\therefore \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

281. *Cor.* 2.—We have also,

$$d^2 = x^2 + y^2 + z^2 = x \cdot d \cos \alpha + y \cdot d \cos \beta + z \cdot d \cos \gamma;$$

$$\therefore \quad d = x \cos \alpha + y \cos \beta + z \cos \gamma.$$

282. Prop. II.—*To find the distance between two points,* M, N, *whose co-ordinates are known.*

Let x, y, z; x', y', z' be the co-ordinates of two points, M, N, situated any where in space. Then, if three lines be drawn through each of the points M and N, parallel to the axes of x, y, z, and the parallelopiped MN be completed, the distance MN will be evidently the diagonal of this parallelopiped, the three contiguous sides of which are

$$x' - x, \quad y' - y, \quad z' - z.$$

Hence we have, from art. 279,

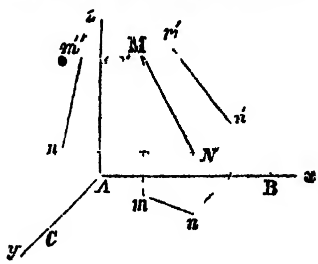
$$MN^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2.$$

Also, if d, d' be the distances of the points M, N from the origin A, then $d^2 = x^2 + y^2 + z^2$, $\quad d'^2 = x'^2 + y'^2 + z'^2$;

$$\therefore \quad MN^2 = d^2 + d'^2 - 2(xx' + yy' + zz').$$

THE STRAIGHT LINE.

283. PROP. III.—*To find the equations to a straight line in space.*



Let MN be any straight line in space. Draw Mm, Nn perpendicular to the plane of xy; these will evidently be in one plane, perpendicular to the plane of xy. In like manner, draw Mm', Nn' perpendicular to the plane of xz, and Mm'', Nn'' perpendicular to the plane of yz. The line MN is manifestly in the intersection of any two of these planes Mn', Mn'': and its position will be known, if the situation of these two planes be known, or if the lines $m'n'$, $m''n''$ be given.

If now, in the line MN, we take any point, M, whose co-ordinates are x, y, z, the co-ordinates of this point x, y will evidently be the same as those of the point m; and since the same may be said of all corresponding points in the lines MN, mn, the co-ordinates x, y of the line MN will be connected together by the same equation that belongs to mn: and the same remarks are equally true for the lines $m'n'$, $m''n''$.

Let $\quad x = mz + \mu, \quad y = nz + \nu,$

be the equations to the lines $m'n'$, $m''n''$; these, therefore, will be the equations which connect the co-ordinates x, y, z of the line MN, and are consequently said to be the equations of that line.

284. *Cor.* 1.—If we eliminate z between these two equations, we get $\quad y - \nu = \dfrac{n}{m}(x - \mu)$. This is the equation to the line MN between the co-ordinates x and y; and is therefore the same as the equation to the line mn.

285. *Cor.* 2.—In the preceding equations μ is the distance of the origin from the intersection of $m'n'$ with Ax; or, $\mu = AB$. Similarly, ν is the distance of the origin from the intersection of $m''n''$ with Ay: that is, $\nu = AC$.

Also, m and n are the trigonometrical tangents of the angle which $m'n'$ and $m''n''$ make with Az.

286. *Cor.* 3.—The equations to a straight line, which passes through the origin and is parallel to MN, are $x = mz$, $y = nz$; m and n having the same values as before.

PROBLEMS ON THE STRAIGHT LINE.

287. PROB. I.—*To find the equation to a straight line passing through two given points*, M, N.

Let $x = mz + \mu$, $y = nz + \nu$ be the equations to MN. Also, let a, b, c; a', b', c', be the co-ordinates to the points M, N respectively, which are supposed to be given. Then, since the equation $x = mz + \mu$ is true for every point in the straight line MN, we have

$$x = mz + \mu; \qquad a = mc + \mu; \qquad a' = mc' + \mu;$$

and the values of m and μ are to be determined from the two last of these equations, and substituted in the first. We shall find, therefore, in the same manner as in art. 17,

$$x - a = m(z - c); \qquad a' - a = m(c' - c).$$

Substituting the value of m, deduced from the last equation in the preceding one, we get

$$x - a = \frac{a' - a}{c' - c}(z - c);$$

and similarly, $$y - b = \frac{b' - b}{c' - c}(z - c);$$

which are the equations to the straight line MN.

288. PROB. II.—*To find the intersection of two given straight lines.*

When two straight lines are situated in the same plane, they will always meet each other when produced, unless they are parallel; but this may not be the case with two straight lines situated any where in space. It will be necessary, therefore, to determine from the equations to the straight lines whether they will meet or not.

Let $x = mz + \mu$, $y = nz + \nu$ be the equations of the 1st line,
and $x = m'z + \mu'$, $y = n'z + \nu'$ do. 2nd do.;
in which the quantities m, μ, n, ν, &c., are all given. Now it is evident, if the two lines intersect each other in the point M, the co-ordinates of this point, or the values of x, y, z, will satisfy all the four equations. And as we have only three indeterminate quantities, x, y, z, the values of these co-ordinates, derived from any three of these equations and substituted in the fourth, ought to render it identical.

If these lines do not meet, there are no corresponding values of x, y, and z, which will satisfy all the equations.

From the 1st and 3rd equations, $(m - m')z + (\mu - \mu') = 0$;
from the 2nd and 4th, $(n - n')z + (\nu - \nu') = 0$;
and the value of z, derived from the first of these equations, ought to satisfy the last equation. Hence, substituting this value, we get

$$(m - m')(\nu - \nu') = (n - n')(\mu - \mu'),$$

which is the equation of condition, or the relation that must exist between the constants, in order that the two lines may meet.

We have now, from the preceding equations,

$$z = -\frac{\mu - \mu'}{m - m'}, \quad \text{or} \quad z = -\frac{\nu - \nu'}{n - n'},$$

$$x = mz + \mu = \frac{m\mu' - m'\mu}{m - m'},$$

$$y = nz + \nu = \frac{n\nu' - n'\nu}{n - n'},$$

which are the co-ordinates of the intersection of the two given straigh lines.

289. Prob. III.—*To find the angles which a given straight line passing through the origin makes with the co-ordinate axes.*

Since the line passes through the origin of co-ordinates, its equation will be of the form (art. 286),

$$x = mz, \qquad y = nz.$$

Let r be the distance of any point (x, y, z) from the origin; we have then, from art. 279,

$$r^2 = x^2 + y^2 + z^2 = (m^2 + n^2 + 1)z^2.$$

Now, if α, β, γ be the angles which this line makes with the axes of x, y, z, , we have (art. 280),

$$\cos\alpha = \frac{x}{r} = \frac{m}{\sqrt{(m^2 + n^2 + 1)}},$$

$$\cos\beta = \frac{y}{r} = \frac{n}{\sqrt{(m^2 + n^2 + 1)}},$$

$$\cos\gamma = \frac{z}{r} = \frac{1}{\sqrt{(m^2 + n^2 + 1)}}.$$

290. *Cor.* 1.—The angles which this line makes with the planes of yz, xz, and xy, are the complements of the angles which it makes with the axes of x, y, z; and therefore the preceding expressions represent the sines of these angles respectively.

291. *Cor.* 2.—We shall have, also, the same expressions for the angle which any straight line, whose equations are $x = mz + \mu$, $y = nz + \nu$, makes with the axes of x, y, z. For, since the coefficients of z in these equations are the same as those in the proposition, the two lines are parallel to each other (art. 286).

292. *Cor.* 3.—Since $\frac{\cos\alpha}{\cos\gamma} = m$ and $\frac{\cos\beta}{\cos\gamma} = n$, the equations to the straight line in the last article are

$$x = \frac{\cos\alpha}{\cos\gamma}z + \mu; \qquad y = \frac{\cos\beta}{\cos\gamma}z + \nu$$

293. Prob. IV.—*To find the angle contained between any two given lines passing through the origin.*

Let $x = mz$, $y = nz$; $x = m'z$, $y = n'z$, be the equations to the two given straight lines, which pass through the origin of co-ordinates. Let a, b, c; a', b', c', be the co ordinates of any two points, M, N, in

these two lines, respectively. Let $AM = d$, $AN = d'$, and the angle $MAN = \phi$; then

$$MN^2 = d^2 + d'^2 - 2dd' \cos\phi \text{ (Trig. art. 109)};$$

also $$MN^2 = (a' - a)^2 + (b' - b)^2 + (c' - c)^2 \text{ (art. 282)}.$$

And because $a^2 + b^2 + c^2 = d^2$; $a'^2 + b'^2 + c'^2 = d'^2$, we obtain, by putting these values of MN^2 equal to each other,

$$dd' \cos\phi = aa' + bb' + cc'.$$

Also, since $a = mc$, $b = nc$, $d = \sqrt{(a^2+b^2+c^2)} = c\sqrt{(m^2+n^2+1)}$;

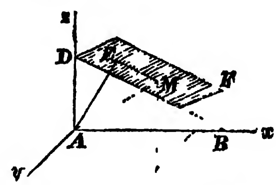
$$\therefore \quad \cos\phi = \frac{mm' + nn' + 1}{\sqrt{[(m^2 + n^2 + 1)\ (m'^2 + n'^2 + 1)]}}.$$

294. *Cor.* 1.—Since $a = d\cos\alpha$, $b = d\cos\beta$, $c = $ &c.;

$$\therefore \quad \cos\phi = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

THE PLANE.

295. Prop. I.—*To find the equation to a plane.*



Let DF represent any plane. Take any point, M, in this plane, and let AE be drawn from the origin, A, perpendicular to DF. Join ME. Let x, y, z be the coordinates of the point M, and a, b, c those of the point E; also, let $AE = d$. Then, because AE is perpendicular to the plane DF, it is perpendicular to every line, EF, drawn through E in the plane (Geom. prop. 93). Hence $AM^2 = AE^2 + EM^2$.

But $AM^2 = x^2 + y^2 + z^2$; $AE^2 = d^2 = a^2 + b^2 + c^2$; and $EM^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$. Substituting these values above,

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2 + (x - a)^2 + (y - b)^2 + (z - c)^2.$$

Transposing and reducing, we get

$$ax + by + cz = a^2 + b^2 + c^2 = d^2.$$

296. *Cor.* 1.—If α, β, γ be the angles which AE makes with the axes of x, y, z, then $a = d\cos\alpha$, $b = d\cos\beta$, $c = d\cos\gamma$.

Substituting these values in the last equation, we get

$$x\cos\alpha + y\cos\beta + z\cos\gamma = d.$$

Here are four constant quantities, α, β, γ, d; but three of them only are necessary, since the three angles are connected together by the equation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Hence it appears that there are only three independent quantities in the equation to a plane.

297. *Cor.* 2.—If the plane passes through the origin, $d = 0$, and the equation to the plane is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = 0.$$

As this equation, however, was deduced on the supposition that d was finite, we will give the following independent proof.

From the origin, A, suppose AH to be drawn perpendicular to the given plane. Let a, b, c be the co-ordinates of any point, H, in this perpendicular; we have then

$$HM^2 = AH^2 + AM^2; \quad \text{that is,}$$

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = (a^2 + b^2 \quad c^2) + (x^2 + y^2 + z^2).$$

$$\text{Hence} \qquad ax + by + cz = 0;$$

$$\therefore \qquad x\cos\alpha + y\cos\beta + z\cos\gamma = 0.$$

298. *Cor.* 3.—The general equation of the first degree between the co-ordinates x, y, z, is $Ax + By + Cz = D$; or dividing by D, and substituting A, B, C for $\frac{A}{D}$, $\frac{B}{D}$, $\frac{C}{D}$;

$$Ax + By + Cz = 1.$$

Here we have also three independent quantities; and therefore this equation may always be made to coincide with the former. Hence it appears that an equation of the first degree always represents a plane surface.

299. *Scholium.*—The line BD, where the plane DF intersects the plane of xz, is called the *trace* of this plane. The equation to the trace BD is found by putting $y = 0$ in the equation to the plane; we then get

$$x\cos\alpha + z\cos\gamma = d; \qquad \text{or,} \qquad Ax + Cz = 1,$$

which is the equation to BD.

In like manner the traces on the planes of xy, yz are found by putting $z = 0$, $x = 0$, in the equation to the plane.

300. Prop. II.—*To find the angles which any given plane makes with the axes of* x, y, z.

Let $Ax + By + Cz = 1$ be the equation to the given plane, in which the coefficients A, B, C are supposed to be known. Let d be the length of the perpendicular from the origin on this plane, and α, β, γ the angles which d makes with the axes of x, y, z. The equation to this plane, then, is (art. 296)

$$x\frac{\cos\alpha}{d} + y\frac{\cos\beta}{d} + z\frac{\cos\gamma}{d} = 1.$$

And, since these equations are identical, we have

$$A = \frac{\cos\alpha}{d}, \qquad B = \frac{\cos\beta}{d}, \qquad C = \frac{\cos\gamma}{d};$$

$$\therefore \quad A^2 + B^2 + C^2 = \frac{\cos^2\alpha \quad \cos^2\beta + \cos^2\gamma}{d^2} = \frac{1}{d^2}.$$

$$\text{Hence } d = \frac{1}{\sqrt{(A^2 + B^2 + C^2)}}, \quad \text{consequently}$$

$$\cos\alpha = \frac{A}{\sqrt{(A^2 + B^2 + C^2)}}, \quad \cos\beta = \frac{B}{\sqrt{(A^2 + B^2 + C^2)}},$$

$$\cos\gamma = \frac{C}{\sqrt{(A^2 + B^2 + C^2)}}.$$

But α, β, γ are evidently the complements of the angles which the plane DF makes with the axes of x, y, z; and consequently the preceding expressions will represent the sines of these respective angles.

301. *Cor.*—The same result will be obtained if the plane passes through the origin, and its equation is $Ax + By + Cz = 0$.

302. Prop. III.—*To find the equation to a plane which is parallel to a given plane, and passes through a given point.*

Let the equation to the given plane be $Ax + By + Cz = 1$,
and to the required plane $A'x + B'y + C'z = 1$.
Also, let x', y', z' be the co-ordinates of the given point. Since this is a point in the plane, we have

$$A'x' + B'y' + C'z' = 1;$$

and, subtracting this equation from the preceding one, we get

$$A'(x - x') + B'(y - y') + C'(z - z') = 0.$$

Let d be the length of the perpendicular from the origin upon the given plane, and α, β, γ the angles which this line makes with the axes of x, y, z. Then, because the required plane is parallel to the given plane, the same line will also be perpendicular to both the planes; and if we suppose d' to be the length of the perpendicular from the origin upon the required plane, we shall have

$$A = \frac{\cos\alpha}{d}, \quad A' = \frac{\cos\alpha}{d'}, \quad B = \frac{\cos\beta}{a}, \quad B' = \&c.$$

$$\therefore \frac{A'}{A} = \frac{d}{d'}; \quad \frac{B'}{B} = \frac{d}{d'}; \quad \frac{C'}{C} = \frac{d}{d'};$$

$$\text{or} \quad A' = A\frac{d}{d'}, \quad B' = B\frac{d}{d'}, \quad C' = C\frac{d}{d'}.$$

Substituting these values above, we have, for the required equation,

$$A(x - x') + B(y - y') + C(z - z') = 0.$$

303. *Cor.*—If it be required to draw the plane parallel to the given plane, and at a given distance from it, we find immediately, by substituting these values in the equation to the required plane,

$$Ax + By + Cz = \frac{d'}{d}.$$

304. Prop. IV.—*To find the equation to a straight line which is perpendicular to a given plane, and passes through a given point.*

Let $Ax + By + Cz = 1$ be the equations to the given plane; $x = mz + \mu$, $y = nz + \nu$ the equations to the required line; and x', y', z', the co-ordinates of the given point.

Since the required line passes through this point, we have

$$x' = mz' + \mu, \qquad y' = nz + \nu;$$

therefore, by subtraction, $x - x' = m(z - z')$, $y - y' = n(z - z)$.

But if α, β, γ be the angles which this line makes with the axes of x, y, z, we have (art 292) $m = \dfrac{\cos\alpha}{\cos\gamma}$, $n = \dfrac{\cos\beta}{\cos\gamma}$.

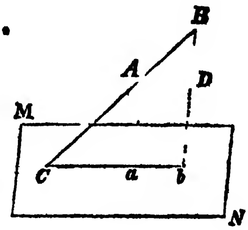
We have also (art. 300) $A = \frac{\cos\alpha}{d}$, $B = \frac{\cos\beta}{d}$, $C = \frac{\cos\gamma}{d}$.

Hence $m = \frac{A}{C}$, $n = \frac{B}{C}$; and the equations to the required line are

$$x - x' = \frac{A}{C}(z - z'), \qquad y - y' = \frac{B}{C}(z - z').$$

PROJECTIONS.

305. PROP. I.—*To find the length of the projection of a straight line,* AB, *upon a given plane,* MN.



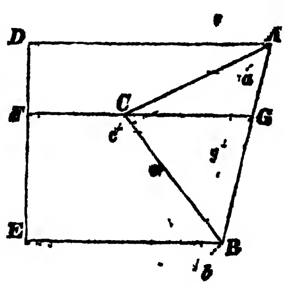
Let AB be any straight line in space; draw Aa, Bb perpendicular to the plane MN; then will the points a, b be the projections of A, B. And because Aa, Bb are perpendicular to the plane MN, they are in one plane perpendicular to MN, and every point in the line AB is projected into a point in the line ab (Geom. prop. 103); hence the straight line ab is the projection of AB.

Produce AB, ab until they meet the plane of MN in the point C; then will

$$ab = AD = AB\cos\theta,$$

θ being put for the angle ACa, or the inclination of AB to the plane MN.

306. PROP. II.—*To find the projection of an area of a triangle* ABC, *in any plane* DEba.



Suppose the plane of the triangle ABC to be produced until it meet the plane $DEba$ in the line DE. Let abc be the projection of ABC on the plane $DEba$. Draw aD, bE, cF perpendicular to DE; join AD, BE, CF, and produce FC, Fc until they meet AB, ab in the points G, g.

Because Aa is perpendicular to the plane $DEba$, and aD is perpendicular to DE, therefore DA is also perpendicular to DE (Geom. prop. 94), and the angle ADa is the inclination of the two planes ADE, aDE; call this angle θ. Now the area of the triangle

$$ABC = ACG + GCB = \tfrac{1}{2}CG\times DF + \tfrac{1}{2}CG\times FE = \tfrac{1}{2}CG\times DE.$$

Similarly, the area of the triangle $abc = \tfrac{1}{2}cg\times DE$.

Hence area ABC : area abc :: CG : cg : FC : Fc
:: rad : $\cos\theta$;

$\therefore$ area abc = area $ABC \times \cos\theta$.

307. *Cor.*—Hence it appears that the area of any plane rectilineal

figure is to its projection on any plane as the radius to the cosine of the angle contained between these two planes. Hence, also, it follows that the same proportion is true for any plane figure whatever.

308. Prop. III.—*The square of the number which represents the area of any plane figure, is equal to the sum of the squares of the numbers which measure the areas of its projections on the three co-ordinate planes.*

Let D be the area of the plane figure, and A, B, C the areas of the projections on the planes of yz, xz, xy. Also, let α, β, γ be the angles contained between the plane of D and these planes, respectively. We have then

$$A = D\cos\alpha, \qquad B = D\cos\beta, \qquad C = D\cos\gamma;$$

$$\therefore \quad A^2 + B^2 + C^2 = D^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = D^2.$$

CHAP. II.—SURFACES OF THE SECOND ORDER.

309. When an equation involves three unknown quantities, x, y, z, it may be always supposed to represent a surface indefinite in extent, either plane or curved. For if we draw the three axes of x, y, z at right angles to each other, as in the last chapter, and assume any number of points in the plane of xy, these will correspond to so many distinct values of x and y, and the corresponding values of z will lie in one or more surfaces, which are said to be the locus of this equation.

310. Surfaces, as well as lines, are divided into orders, according to the degree of the equation which expresses the relation between x, y, and z. Thus, we have seen that the equation of the first degree,

$$Ax + By + Cz = D,$$

represents a plane, or a surface of the first order. In like manner, the equation between three co-ordinates of the second degree, namely,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Kz + L = 0,$$

represents a surface of the second order; and this, like the corresponding equation (1), in art. 36, may be shown to belong to different surfaces, according to the relation which these coefficients have to each other.

311. It would be impossible, in a work of this nature, to enter into a discussion of this equation. We will simply observe, that the rectangles xy, xz, yz, are first made to disappear, by passing from one system of rectangular co-ordinates to another, whilst the origin remains the same, in a manner analogous to that which has been pursued in art. 46. And again, the coefficients of x, y, z may be made to vanish precisely in the same manner as in art. 51. If none of the coefficients of x^2, y^2, z^2 vanish at the same time with those of the rectangles xy, xz, yz, the equation, thus reduced, will assume the form of

$$Mx^2 + Ny^2 + Pz^2 = 1.$$

But if any of these coefficients vanish, for example, that of x^2, the equation, when finally reduced, will be of the form

$$Ny^2 + Pz^2 + Rx = 0.$$

312. In the first of these equations, the three coefficients, M, N, P, may either be all positive; or two of them may be positive and one negative; or one may be positive and two negative; or all three may be negative. But, as in the last case, the roots are manifestly impossible, and the surface imaginary, there are only three distinct equations to be considered.

By substituting for M, N, P the constants $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$, with their proper signs, these three cases will be represented by the three equations,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1;$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

(1). The first of these equations represents a surface which is called an *ellipsoid*. If we make $z = 0$, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is a section of the surface made by the plane xy; and it is evidently an ellipse. Similarly, the sections of this surface, made by the planes xz and yz, are ellipses. Hence it is evident that the surface is limited in every direction, and that all the sections parallel to the principal sections are ellipses.

(2). The second equation represents a surface which is called a *hyperboloid of one sheet*. The sections parallel to the plane of xy are ellipses, and those parallel to the planes of xz, yz are hyperbolas. The student may perhaps form the best idea of this surface by taking the example when $b = a$, and the ellipse becomes a circle; in which case the figure is generated by the revolution of a hyperbola round the conjugate axis, and forms one continuous surface or sheet.

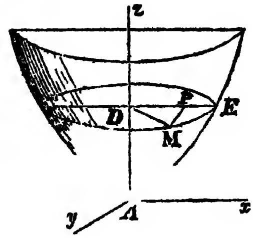
(3). The third equation represents a surface which is called a *hyperboloid of two sheets*. The sections parallel to the planes of xy and xz are hyperbolas, and those parallel to the plane of yz are either imaginary or ellipses. If $b = c$, the ellipses become circles; and the surface is formed by the revolution of a hyperbola about its transverse axis. Hence it is evident that there are two distinct surfaces or sheets.

313. The second equation in art. 311 admits also of two different forms, according as N and P have the same or different signs. In the first case, the surface is called an *elliptic paraboloid*. All the sections parallel to the planes of xy and xz are parabolas, and all the sections parallel to the plane of yz are ellipses. If $N = P$, it is a *paraboloid of revolution*.

If N and P have different signs, the sections parallel to the plane of yz are hyperbolas, and all the sections parallel to the planes of xy and xz are parabolas, as in the first case. Hence the surface is called a *hyperbolic paraboloid*.

SURFACES OF REVOLUTION.

314. When a surface is generated by the revolution of any plane curve about a line situated in that plane, the equation to the surface is readily found, if that to the generating line is known.



Let Ax, Ay, Az, be the axes of x, y, z; and, for the sake of distinctness, suppose Az to be the axis of revolution. Let M be any point in this surface; through M let the plane MDE be drawn perpendicular to the axis Az, or parallel to the plane of xy; the section EMD will evidently be a circle. Let DE be the intersection of this plane with the plane of xz; draw MP perpendicular to DE. Put $MP = x$, $PM = y$, $AD = z$, and $DM = DE = u$.

From the equation to the curve, we have $u^2 = f(z)$, a function of z. Also, in the triangle DPM, $u^2 = x^2 + y^2$; therefore

$$x^2 + y^2 = f(z)$$

is the general form for the equation to all surfaces of revolution.

THE SPHERE.

315. The surface of a sphere is generated by the revolution of a semicircle about its diameter as an axis (Geom. chap. vi., def. 12.)

If we suppose the centre of the sphere to be placed at the origin of co-ordinates, we shall evidently have $u^2 + z^2 = r^2$, r being the radius of the sphere. Hence, since $u^2 = x^2 + y^2$, the equation to the surface of the sphere is

$$x^2 + y^2 + z^2 = r^2.$$

316. Prop.—*To find the equation to the surface of a sphere, when the centre of the sphere is situated any where in space.*

Suppose the axis of revolution of the sphere to be parallel to Az; and let a, b, c be the co-ordinates of C, the centre of the sphere. Then, if $DE = u$, as before, we have

$$u^2 + (z - c)^2 = r^2; \quad \text{also} \quad (x - a)^2 + (y - b)^2 = u^2.$$

Hence the general equation to the surface of the sphere is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

SPHEROID.

317. The surface of a *prolate spheroid* is formed by the revolution of an ellipse about its *major* axis.

The surface of an *oblate spheroid* is formed by the revolution of an ellipse about its *minor* axis.

If the centre of the ellipse be at the origin of co-ordinates, and the major axis coincides with Az; then the equation to the ellipse is

$$a^2u^2 + b^2z^2 = a^2b^2.$$

Hence the equation to the surface of the prolate spheroid is

$$a^2(x^2 + y^2) + b^2z^2 = a^2b^2.$$

Similarly, the equation to the surface of the oblate spheroid is

$$b^2(x^2 + y^2) + a^2z^2 = a^2b^2.$$

318. We shall leave the following examples, which are sufficiently easy, to be investigated by the student.

(1). The surface of a *right cone* is formed by the revolution of the hypothenuse of a right-angled triangle about one of its sides as an axis (Geom. chap. vi. def. 9).

If A be the vertex, Az the axis of the cone, and m the tangent of half the vertical angle of the cone, the equation to the surface is

$$x^2 + y^2 = m^2z^2.$$

(2). The surface of a *right cylinder* is formed by the revolution of one of the sides of a rectangle about the opposite side as an axis (Geom. chap. vi. def. 8).

If r be one of the other sides of the rectangle, or the radius of the base of the cylinder, the equation to the surface is

$$x^2 + y^2 = r^2.$$

(3) The equation to the surface described by the revolution of a hyperbola about its transverse axis, when the origin is at the centre, is

$$a^2(x^2 + y^2) - b^2z^2 = -a^2b^2.$$

And the equation to the surface about the conjugate axis is

$$b^2(x^2 + y^2) - a^2z^2 = a^2b^2.$$

(4). The equation to the surface described by the revolution of a parabola about its axis, when the origin is at the vertex, is

$$x^2 + y^2 = 2pz.$$

SURFACES FORMED BY A GENERATRIX AND DIRECTRIX.

There is a large class of surfaces which may be supposed to be generated by means of a line being constrained to move in certain given directions. In this treatise we can only notice two general cases, cylindrical and conical surfaces.

319. Def. 1.—A *cylindrical surface* is generated by a straight line which moves parallel to itself, and always passes through a given curve.

The straight line which moves is called the *Generatrix*, and the given curve the *Directrix*.

If the given curve be a circle, the surface is the oblique cylinder described in art. 223.

320. Def. 2.—A *conical surface* is generated by the movement of a straight line which passes constantly through a given point, and also through a given curve.

The given point is called the *vertex* or *centre* of the surface; the straight line which moves is called the *Generatrix*; and the given curve is called the *Directrix*.

If the given curve be a circle, the surface is the conical surface defined in art. 222.

321. PROP. II.—*To determine the equation to a cylindrical surface, when the directrix is a circle in the plane of* xy.

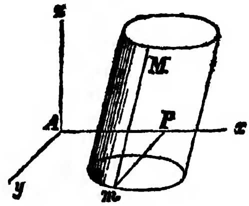
Let the equations to the generating line, in any position, be

$$x = mz + \mu; \qquad y = nz + \nu \ldots\ldots\ldots (1);$$

where m and n are constant, but μ and ν vary with different positions of the generatrix, and must therefore be made to disappear from the final equation of the surface. Also, let

$$(t - a)^2 + (u - b)^2 = r^2$$

be the equation to the directrix.



Since the generatrix passes through the directrix in every position, we have, at the point of intersection of these two lines,

$$x = t, \qquad y = u, \qquad z = 0.$$

Substituting these values in equations (1), we get

$$t = \mu, \qquad u = \nu;$$

that is, the values of μ and ν for every position of the generating line are the co-ordinates of the point m. Hence we have the three equations,

$$x = mz + t; \qquad y = nz + u;$$

and $$(t - a)^2 + (u - b)^2 = r^2;$$

in which the co-ordinates t, u, of the last equation, give any point m in the directrix; and the co-ordinates, x, y, z, any point M in the line Mm. If, therefore, we eliminate t and u from these three equations, we shall have an equation in terms of x, y, z, which will give us all the different points M in the cylindrical surface.

Since $t = x - mz$, and $u = y - nz$, we obtain

$$(x - mz - a)^2 + (y - nz - b)^2 = r^2,$$

for the equation to the cylindrical surface.

Cor. 1.—If the directrix be an ellipse, whose equation is

$$a^2u^2 + b^2t^2 = a^2b^2,$$

the centre of the ellipse being at the origin of the co-ordinates, then the equation to the cylindrical surface is

$$a^2(y - nz)^2 + b^2(x - mz)^2) = a^2b^2.$$

Cor. 2.—If the directrix be a straight line, whose equation is $u = at + b$, then the equation to the surface is

$$y - nz = a(x - mz) + b,$$

which being of the first degree, is the equation to a plane (art. 298).

322. PROP. III.—*To find the equation to a conical surface, when the directrix is a circle in the plane of* xy.

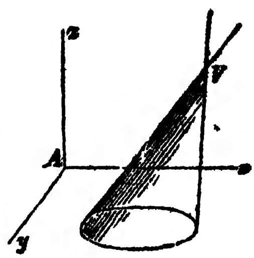
Let the equations to the generating line, in any position, be as before,

$$x = mz + \mu, \qquad y = nz + \nu \ldots\ldots\ldots (2);$$

where m, n, μ, ν, vary with every position of the generatrix, and must therefore be eliminated from the final equation to the surface. Also, let α, β, γ be the co-ordinates of V, the vertex of the cone, and

$$(t-a)^2+(u-b)^2 = r^2 \ldots\ldots\ldots\ldots (2),$$

the equation to the directrix.



Since the generating line, in every position, passes through the point V, whose co-ordinates are α, β, γ, we have, (art. 287),

$$\left.\begin{aligned} x-\alpha &= m(z-\gamma); \\ y-\beta &= n(z-\gamma) \end{aligned}\right\} \ldots\ldots (3).$$

Also, because the generatrix passes through the directrix in every position, we have, at the point of intersection of these two lines,

$$x = t, \quad y = u, \quad z = 0.$$

Substituting these values in equations (3), we get

$$t-\alpha = -m\gamma, \qquad u-\beta = -n\gamma;$$

therefore $\qquad m = -\dfrac{t-\alpha}{\gamma}, \qquad n = -\dfrac{u-\beta}{\gamma}.$

Hence the equations to the generatrix, in every position, are

$$x-\alpha = -\frac{t-\alpha}{\gamma}(z-\gamma), \qquad y-\beta = -\frac{u-\beta}{\gamma}(z-\gamma).$$

And if, from these equations, and equation (2), the co-ordinates t, u be eliminated, the resulting equation will evidently be the equation to the conical surface.

From the two last equations we have

$$t = \frac{\alpha z-\gamma x}{z-\gamma}, \qquad u = \frac{\beta z-\gamma y}{z-\gamma}.$$

Substituting these values in equation (2), the final equation to the surface is

$$\left(\frac{\alpha z-\gamma x}{z-\gamma}-a\right)^2+\left(\frac{\beta z-\gamma y}{z-\gamma}-b\right)^2 = r^2.$$

323. *Cor.* 1.—If the axis of the cone be parallel to the axis of z, or the figure be a right cone, then $a=\alpha$, $b=\beta$; hence

$$(x-\alpha)^2+(y-\beta)^2 = \frac{r^2}{\gamma^2}(z-\gamma)^2.$$

And if the axis of the curve coincide with Az, or $\alpha=0$, $\beta=0$,

$$x^2+y^2 = \frac{r^2}{\gamma^2}(z-\gamma)^2 = m^2(z-\gamma)^2,$$

putting $\dfrac{r^2}{\gamma^2} = m^2$.

324. *Cor.* 2.—If the directrix be an ellipse, whose centre is at the origin, and the vertex of the cone in the axis of z, then the equation to the surface is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{(z-\gamma)^2}{\gamma^2}.$$

CURVES OF DOUBLE CURVATURE.

325. DEF.—When the curve is not situated in one plane, like the spiral thread of a screw on the surface of a cylinder, it is called a *curve of double curvature.*

326. Curves of this kind may be supposed to be formed either by continued motion, or by the intersection of two curve surfaces; or, lastly, by tracing it from its equations, in the same manner as a number of points are found in a plane curve.

The equations to a curve of double curvature are always two, which we may suppose to be of the form

$$x = f(z); \qquad y = \varphi(z), \qquad \text{two functions of } z.$$

When any value is assigned to z in these equations, the values of x and y can easily be found; and therefore the corresponding point in the curve may be determined.

327. If the curve be formed from the intersection of two surfaces whose equations are

$$F(x, y, z) = 0; \qquad F_1 (x, y, z) = 0,$$

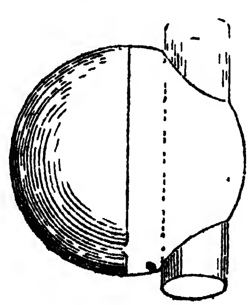
we may first eliminate y and then x from the equations, and we shall obtain two equations of the form

$$f_1 (x, z) = 0; \qquad \varphi_1 (y, z) = 0,$$

which will be the equations to the curve formed by the intersection of these surfaces.

The following example will serve to illustrate these remarks.

328. Ex.—*To find the curve arising from the intersection of a sphere and a right cylinder.*



Let the origin of co-ordinates be at the centre of the sphere, and the axis of the cylinder in the plane of xz, parallel to the axis of z. Then, if the radius of the sphere $= r$, the radius of the cylinder $= c$, and the distance between the centre of the sphere and the axis of the cylinder $= a$, we have the

equation to the sphere, $x^2+y^2+z^2=r^2$;

equation to the cylinder, $(x-a)^2+y^2=c^2$.

Eliminating y and x successively, we get

$$z^2 = r^2 + a^2 - c^2 - 2ax$$

$$z^2 = r^2 - a^2 - c^2 - 2a\sqrt{c^2 - y^2},$$

which are the two equations to the intersection of the surfaces.

If the two projections of this curve be made in the planes of xz, yz, these will also be the equations to these projections.

LINEAR PERSPECTIVE.

DEFINITIONS.

1. Linear perspective is the art of representing the visible outline of an object upon a given surface, that the representation and its original may produce the same effect on the eye when they are seen from a given point.

The surface, on which the object is delineated, is usually supposed to be plane; to be situated between the object and the eye; and to be perpendicular to the horizon. It is called the *perspective plane*, the *picture*, or the *plane of the picture*.

2. The *point of view*, or *point of sight*, is that point where the eye is supposed to be placed.

3. The *centre of the picture*, or the *principal point*, is that point in which the plane of the picture is met by a straight line, drawn perpendicular to it from the point of view.

This is usually, by artists, called the *point of sight*, because it is immediately opposite to the eye; but, for the sake of distinctness, we will call it the centre of the picture.

4. The distance between the eye and the centre of the picture is called the *distance of the picture*, or the *principal distance*.

5. The *ground plane* is a plane parallel to the horizon, in which the visible objects are supposed to be situated. This is also called the *geometrical plane*.

6. The *ground line* is the intersection of the picture and the ground plane. It is also called the *base line* and the *fundamental line*.

7. The *horizontal plane* is a plane passing through the point of view parallel to the horizon. It is therefore parallel to the ground plane, and perpendicular to the picture.

8. The *horizontal line* is the intersection of the picture and the horizontal plane.

In the figure to prop. 5, the plane XAB is the picture, E the point of view, C the centre of the picture, EY the horizontal plane, FG the horizontal line, ABZ the ground plane and AB the ground line.

9. The *vanishing point* of any straight line is the point where a line drawn from the eye parallel to this line meets the perspective plane.

Cor.—Hence all lines parallel to one another have the same vanishing point; and all lines perpendicular to the plane of the picture have the centre of the picture for their vanishing point. And all lines parallel to the picture have no vanishing points.

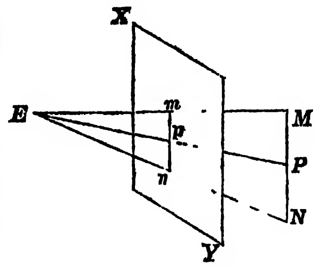
10. The *vanishing line* of any plane is the line in which the picture

is cut by a plane passing through the point of view, parallel to the original plane.

Scholium.—In general, the things to be drawn are called *objects* or *originals;* and their pictures when drawn are called *representations, figures, images, or projections.*

PROP. I.—THEOREM.

The representation of any point of an object is the point in which a straight line drawn from the eye to the original point cuts the plane of the picture.



Let E be the place of the eye, XY the plane of the picture, and M any point of an original object; join EM, and let it cut the picture in m; m is the representation of the point M.

For it appears, from optics, that light proceeds in straight lines from any object to the eye, and therefore the point M is seen by the eye at E, in the direction EM. If, therefore, light should proceed from the point m of the same colour and intensity as the light from M, it will enter the eye in the same manner, and therefore will produce the same effect. Hence m is the representation of M.

PROP. II.—THEOREM.

If a straight line, when produced, does not pass through the point of view, its representation is a straight line. (See the last figure.)

Let MN be the straight line, E the place of the eye, or the point of view; join EM, EN, cutting the plane of the picture in m and n; also join mn; then m and n are the representations of M and N (prop. 1). Likewise, if any other point, P, be taken im MN, and EP be joined, EP will be in the plane of EMN, and will cut the line mn in some point p. Thus every point in MN is represented by a corresponding point in mn, and therefore the line mn is the representation of MN.

Cor.—If MN, when produced, passes through E, its representation is the point where MN cuts the plane of the picture.

PROP. III.—THEOREM.

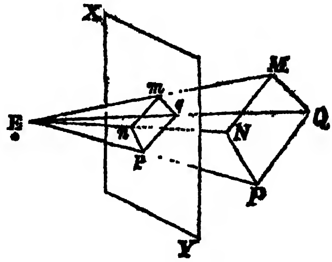
The representation of a straight line parallel to the picture is parallel to that line. (See the last figure.)

Let MN be a line parallel to the picture, E the place of the eye; join EM, EN, cutting the picture in m, n; then mn is the representation of MN. Now, if the lines MN, mn, situate in the same plane, be not parallel, they must meet if produced, therefore MN will meet the plane of the picture if produced, which is impossible, since MN is parallel to this plane. Hence mn is parallel to MN.

Cor.—If any number of parallel straight lines be all parallel to the picture, their representations will be parallel to each other. For the representation of any one of them is parallel to its original, and therefore parallel to all the rest, and also to their representations.

PROP. IV.—THEOREM.

The representation of any plane rectilineal figure parallel to the picture is similar to the original.



Let $MNPQ$ be a plane figure, parallel to the picture XY; E the place of the eye; then, if EM, EN, &c. be joined, these lines will form a pyramid whose base is $MNPQ$ and vertex E; and since the pyramid is cut by the plane of the picture XY parallel to the base, the section $mnpq$ will be similar to the base (Geom. prop. 117); but $mnpq$ is evidently the representation of the rectilineal figure $MNPQ$.

Cor. 1.—If two equal straight lines be situated in a plane parallel to the picture, their representations will be equal. Let MN, PQ be two equal straight lines in the plane MP, parallel to the picture; then, since mn, pq are parallel to MN, PQ (Geom. prop. 97),

$$MN : mn :: EN : En :: EP : Ep :: PQ : pq;$$

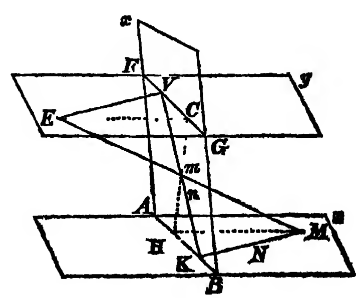
and since $MN = PQ$, therefore $mn = pq$.

Cor. 2.—If several lines be parallel to the picture and at equal distances from it, their representations will be to one another in the same proportion as the original lines.

Cor. 3.—The representation of any curvilineal figure parallel to the picture will be similar to the original; thus, the representation of a circle is a circle, the representation of an ellipse is a similar ellipse, and so on.

PROP. V.—THEOREM.

If M *be any point in the ground plane* ZAB, MK *a line in it making any angle with the ground line* AB; *and if* EV *be drawn parallel to* MK, *meeting the plane of the picture in* V, *and* EM, VK *be drawn; then the representation of the point* M *will be in the line* VK, *so situated that* EV : MK :: mV : mK.



For since EV, MK are parallel the figure $EVKM$ is in one plane cutting the picture in the straight line VK; therefore the triangles EmV MmK are similar, and

$$EV : MK :: mV : mK.$$

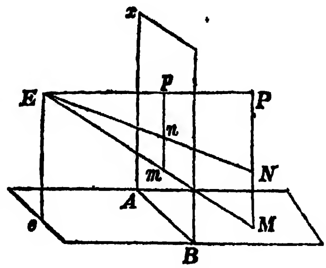
Cor. 1.—The line KM is represented by Km, and if KM be indefinitely extended beyond the picture, it will be represented by KV. It is on this account that V is called the vanishing point of the line KM.

Cor. 2.—The representation, mn, of any straight line, MN, in the ground plane, will pass, when produced, both through its vanishing point, and its intersection with the plane of the picture.

Cor. 3.—If MH be perpendicular to the picture, V will then coincide with C, the centre of the picture, and $EC : MH :: mC : mH$.

PROP. VI.—THEOREM.

The length of any vertical line standing on the ground plane, is to that of its picture as the height of the eye to the distance of the horizontal line from the picture of its foot.



Let MN be the vertical line standing on M, and let Ee be a vertical line passing through the point of view. Produce, if necessary, MN to P, making $MP = Ee$, and draw EM, EN, EP, cutting the plane of the picture in the points m, n, p. Then, since MP is vertical, it is parallel to the plane of the picture, and therefore is parallel to mp, the intersection of the two planes XAB, EMP. Hence, by similar triangles,

$$MN : mn :: EM : Em :: PM : pm;$$

but $PM = Ee$, the height of the eye, and pm is equal to the distance of m from EP, or is equal to the distance of m from the horizontal line.

PROP. VII —THEOREM.

The vanishing points of all straight lines, in any original plane, are in the vanishing line of that plane.

For since all the original lines are in the same plane, the lines which are parallel to them, passing through the point of view, will be all of them in a parallel plane (Geom. prop. 100), and therefore all the vanishing points will be in the intersection of this plane and the plane of the picture; that is, they are in the vanishing line.

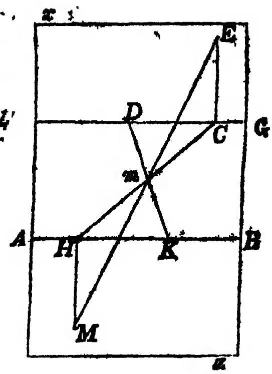
Scholium.—The preceding propositions are a sufficient foundation for the whole practice of perspective, whether in direct or inclined pictures, and serve to suggest all the various constructions, each of which has advantages suitable to particular cases. We will now proceed to exemplify these theorems by a few problems, which will enable the student to apply them to any cases which may afterwards occur to him in practice.

PROBLEMS.

All the constructions for putting any figure in perspective are performed on the plane of the picture, by making certain substitutions for the place of the eye, and the original picture. The general substitution is as follows:—

Let XZ represent the plane in which the picture is to be drawn; AB the ground line, FG the horizontal line, C the centre of the picture. Then the plane of the picture AZB, below AB, is first supposed to be the ground plane, and all the lines in this plane are to be laid on in the picture AZB, of their proper magnitude, and in their true

situation with respect to the ground line AB. Let CE also be drawn perpendicular to FG, and equal to the distance of the picture; then, if the planes AZB, XFG be conceived to turn round the axes AB, FG, in opposite directions, until they become perpendicular to the plane of the picture $ABFG$; or, in other words, if we conceive the figures AZB, XFG to be folded in the lines AB, FG, in opposite directions, until they become at right angles to $ABFG$, then will AZB coincide with the ground plane, and E with the place of the eye; and all the objects drawn on the ground plane AZB will be in their natural situation.



PROBLEM I.

Having given the centre and distance of the picture, to put in perspective any given point, M, *on the ground plane.* (See the last figure.)

First method.—Let AB be the ground line, FG the horizontal line, and C the centre of the picture. Let M be the given point in the plane AZB, considered as the ground plane, and E the place of the eye; that is, if H be the point where a perpendicular from the original point meets the line AB, let HM be drawn in the picture perpendicular to AB, and equal to the distance of the original point from AB; also, from C draw CE in the plane of the picture, perpendicular to FG, and equal to the distance of the picture from the point of view. Join CH, EM, cutting each other in m; m is the point required. For, by similar triangles, $EC : MH :: mC : mH$, and therefore (prop. 5, cor. 3) m is the picture of M.

Second method.—In the last figure take CD along the horizontal line equal to CE, and HK along the ground line in an opposite direction equal to HM; join CH, DK; the intersection of these lines will evidently be the point required.

This is one of the most simple and useful constructions for putting a point in perspective. The point D is called the *point of distance.*

Third method.—(See the fig. to prob. 3.) From M draw any two lines, MH, MK, to the ground line; and from E draw two lines, EW, EV, to the horizontal line, parallel to the other. Join VK, WH, cutting each other in m; then will m be the picture of M.

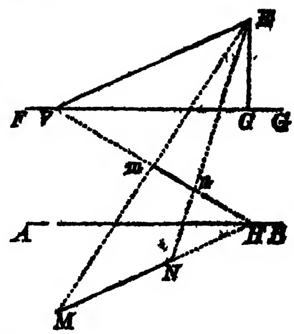
PROBLEM II.

To put in perspective any given straight line on the ground plane.

First method.—Find the representations m, n, of its extreme points, by any of the methods in the last problem, and join them; mn will be the projection required.

Second method.—Let AB be the ground line, and let the straight line MN be drawn in the plane AMB, considered as the ground plane; produce MN until it meet the ground line in H. Draw CE perpendi-

cular to FG, the horizontal line, and equal to the distance of the picture from the eye, and from E draw EV parallel to MN; join HV, and draw EM, EN, cutting HV in m, n; mn will be the projection required.



For, by similar triangles,

$$EV : MH :: mV : mH;$$

therefore (prop. 5) m is the image of M. In like manner n is the representation of N, and consequently mn is the representation of MN.

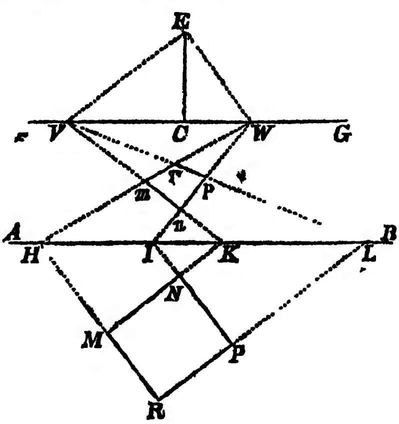
Cor. 1.—If MN be parallel to the ground line AB, its representation, mn, will also be parallel to AB.

Cor. 2.—If MN be perpendicular to AB, mn will be in a straight line passing through E.

PROBLEM III.

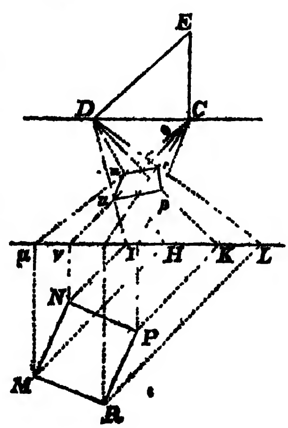
To put any rectilineal figure on the ground plane in perspective.

Put all the bounding lines in perspective by the last problem, and the figure formed by these lines is the picture required.



As an example, suppose that it were required to put the square $MNPR$ in perspective.

First method.—Let AB be the ground line, FG the horizontal line, C the centre of the picture; and let CE be drawn perpendicular to FG, and equal to the distance of the picture. From E draw EV, EW parallel to MN, MR; also, let MN, PN, RP, RM be produced to meet the ground line in the points K, I, L, H, and draw WH, WI, VK, VL, cutting the former lines in the points m, n, p, r; then the figure $mnpr$ is the representation of the square $MNPR$. The demonstration is evident from problem 1.



This construction, however, runs the figure to great distances on each side of the middle line, when any of the lines of the original figure are nearly parallel to the ground line. This inconvenience will be avoided by the following construction.

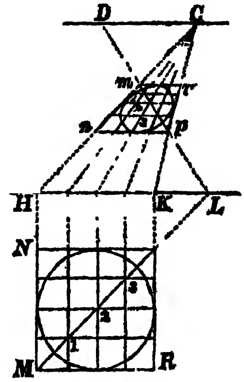
Second method.—Take CD along the horizontal line, equal to the distance of the picture. Draw the perpendiculars $M\mu$, [illegible], $P\pi$, $R\rho$, and the lines MH, NI, PK,

RL, parallel to ED. Draw $C\mu$, $C\nu$, $C\pi$, $C\rho$, and DH, DI, DL, DK, cutting the former in m, n, p, r; the figure $mnpr$ will be the picture required.

It is not necessary that CD be equal to CE, but only that the lines MH, NI, &c., be parallel to DE.

PROBLEM IV.

To put a reticulated square in perspective, having two of its sides parallel to the ground line.



Produce the sides MN, RP, and the intermediate parallels, to meet the ground line in H, K, &c.; also, produce the diagonal MP to meet this line in L. From C, the centre of the picture, draw the lines CH, CK, &c.; and from D, the point of distance, draw the line DL, cutting the former lines in the points m, 1, 2, 3, p. Through these points draw the line mr, np, &c., parallel to HK; then will $mnpr$ be the representation of the reticulated square $MNPR$.

For since the angle MHL is a right angle, and the angle $MLH = MPN =$ half a right ngle; therefore $HL=HM$, $KL=KP$, &c. Hence m is the representation of M, p the representation of P, and so on Also, since MR, NP, &c. are parallel to HK; mr, np, &c., which are also parallel to HK, are the representations of MR, NP, &c.

PROBLEM V.

To put any curve line or irregular figure on the ground plane in perspective.

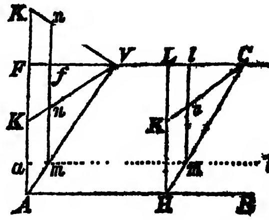
First method.—Find the images of a sufficient number of points by the preceding problems; join these points; and this will be the picture required.

Second method.—Make a true reticulated square, $MNPR$, surrounding the given figure; and put the square in perspective by the last problem. Then observe through which of the points in the reticulated square, $MNPR$, the given figure passes; and through the like points in the representation, $mnpr$, draw a corresponding figure; this will be the perspective representation of the figure required.

Thus, suppose it were required to put a circle in perspective. Describe a reticulated square, $MNPR$, about this circle, and put the square in perspective by the last problem. Observe through which points of the reticulated square the circle passes, and through the corresponding points in the figure $mnpr$ draw the oval, as is shown in the diagram; this will be the representation of the original circle.

PROBLEM VI.

To put a vertical line of a given length in perspective, standing on a given point of the picture.



Let AB be the ground line, FC the horizontal line, and C the centre of the picture. Also, let m be the representation of any point, M, on the ground plane (prob. 1). Through H draw HL perpendicular to AB, and make HK equal to the height of the given line MN. Join CK, and draw mnl parallel to HL, and we have $mn : ml :: HK : HL ::$ height of the given line : height of the eye. But ml is the distance of the point m from the horizontal line, therefore mn is the required picture of the vertical line (prop. 6).

Scholium.—When there are many vertical lines to be drawn, it will be convenient to have a separate piece of paper, and to draw on it a horizontal line, such as VF, and through any point F to draw the vertical line FK, making FA equal to the height of the eye. Through any point, V, of the horizontal line, draw VA; then, if AK be equal to any line in the original, mn will be equal to its representation in the picture when standing on the point m. And all vertical lines standing on the ground plane, which terminate in the parallel amb, must be measured along the line mn on a scale of equal parts.

PROBLEM VII.

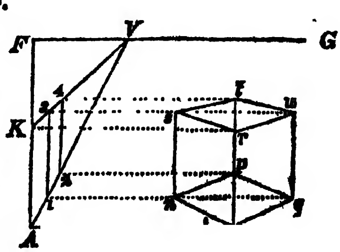
To put any sloping line in perspective.

From the extremities of this line suppose perpendiculars to be drawn to the ground plane, meeting it in two points, which may be called the base points of the sloping line. Put these base points in perspective, and draw, by the last problem, the perpendiculars from the extremities. Join these by a straight line, and it will be the representation required.

PROBLEM VIII.

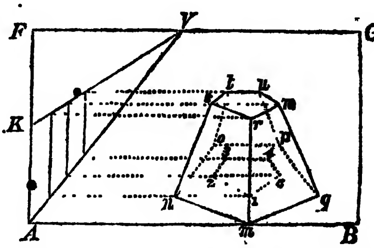
To put any solid in perspective.

From all the angles of the body let fall perpendiculars upon the ground plane, and put their base points in perspective, as in the last problem. From all these points raise perpendiculars of proper lengths, as if taken from the solid; the upper extremities of these lines being joined, will give the solid in perspective.



Ex. 1.—*To put a cube in perspective, as seen from one of its angles.*—Since the base of a cube, standing on the ground plane, and seen from one of its angles, is a square seen from one of its angles, draw first such a perspective square (prob. 3); then raise, from any point of the ground line AB, the perpendicular AK, equal

to the side of the square, and draw to any point V in the horizontal line FG the straight lines VA, VK. From the angles n, p, q draw the lines $n1$, $p2$, &c. parallel to AB; and also the lines 13, 24, parallel to AK. Lastly, draw the perpendiculars $mr = AK$, $ns = 13$; &c. Then, if the points r, s, t, u be joined, the whole cube will be in perspective.



Ex. 2. *To put the frustum of a pyramid in perspective.*—Let the base of the frustum be a pentagon. If from each angle of the upper end a perpendicular be supposed to fall upon the base, these perpendiculars will mark the bounding points of a pentagon, the sides of which will be parallel to the sides of the base of the frustum within which it is inscribed. Join these points, and the interior pentagon will be formed with its sides respectively parallel to the corresponding sides of the base of the frustum. From the ground line AB raise the perpendicular AK, and make it equal to the perpendicular height of the frustum. To any point, V, in the horizontal line, draw the straight lines AV, KV, and by a similar construction to that in the last example, find the different altitudes in perspective at the points 1, 2, 3, 4, 5, due to the height of the frustum: connect the upper points by straight lines, and draw mr, ns, ot, pu, qw; the truncated pyramid will be completed.

PROBLEM IX.

A point being given in the picture, to find its original on the ground plane. (See the next figure.)

Let m be the given point, and C the centre of the picture. Take CD on the horizontal line equal to the distance of the picture from the eye. Join Cm, Dm, and produce them to meet the ground line AB in H, K. From H draw HM perpendicular to AB and equal to HK, then M is the original of the point m. This is sufficiently evident from problem 1, of which this problem is the converse.

Cor.—Hence, if any line in the picture be given, its original on the ground plane may easily be determined, by finding the originals of the extreme points.

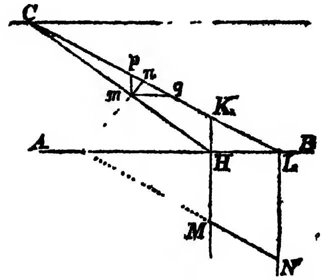
Scholium.—The centre of the picture is determined by finding the intersection of any two lines whose originals are perpendicular to the picture; and the distance of the picture will be known from the proportion $EC : MH :: mC : mH$, when the real distance of any object from the ground line is given.

PROBLEM X.

A vertical line being given in the picture, to find the height of the original on the ground plane.

Let mp be the line given, and C the centre of the picture. Draw the lines CmH, CpL, meeting the ground line in H and L; and

through H draw HK perpendicular to AB, meeting CL in K; then is HK the height of the original object on the ground plane.



This will be sufficiently clear from the demonstration of problem 4.

Cor. 1.—The original of the horizontal line mq parallel to AB is equal to HL.

Cor. 2.—Hence, if the picture of any solid body be given, its original may be found. For all the base lines will be found by problem 8, and all the perpendiculars by this problem; therefore all the surfaces that bound the solid will be known.

Scholium.—The preceding rules contain the fundamental principles of all that is really useful in the art of perspective. Indeed, it may be said that they are all comprised in the 5th and 6th theorems; and the student will find, by carefully considering these propositions, that they comprehend all the various constructions which have been given by different writers on the subject.

Our limits, in a work of this nature, altogether preclude us from giving such a number of examples as might appear necessary to enable the student to apply the rules with confidence and facility. We must therefore refer him to the several practical treatises on the subject, and more particularly to Mr. Fielding's "Synopsis of Lineal and Aërial Perspective."

PROJECTION OF THE SPHERE.

The projection of the sphere is a perspective representation of the surface of the sphere, upon a plane called the ***plane of projection***; and it is variously denominated, according to the different positions of the eye and plane of projection.

There are three principal kinds of projection, the ***stereographic***,* the ***orthographic***,† and the ***gnomonic***.‡ In the stereographic projection, the eye is supposed to be placed on the surface of the sphere; in the orthographic, it is so distant that all the lines drawn from the sphere to the eye are supposed to be perpendicular to the plane of projection; and in the gnomonic projection, the eye is placed at the centre of the sphere. We mean only to notice the two first in this work; the last is chiefly useful in dialling.

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* From στερεὸς, solid, and γράφω, I write, because this projection arises from the intersection of two solids, the sphere and the cone.

† From ὀρθὸς, right, and γράφω, because all the lines are drawn at *right* angles to the plane of projection.

‡ From γνώμων, a dial, because it is applied to dialling.

STEREOGRAPHIC PROJECTION.

DEFINITIONS.

The *stereographic projection* of the sphere is such a representation of the circles on its surface, on the plane of one of its great circles, as would appear to an eye situated in the pole of that circle.

The great circle on whose plane the sphere is projected, is called the *primitive circle.*

The place of the eye is called the *projecting point.*

The *line of measures* of any circle of the sphere, is that diameter of the primitive, produced indefinitely, which passes through the centre of the projected circle.

PROP. I.—THEOREM.

Any circle passing through the projecting point is projected into a straight line. (See the next figure.)

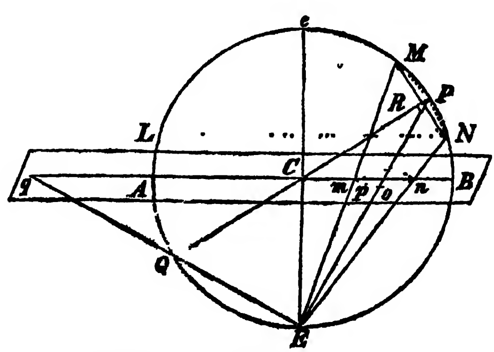
For all lines drawn from the projecting point to any point in the circumference will be in the plane of this circle, and consequently they will meet the primitive in the intersection of this circle and the plane of projection.

Cor. 1.—A great circle passing through the projecting point is projected into a straight line passing through the centre of the primitive; for every great circle passing through E must also pass through e, the opposite pole, and C is the projection of the point e.

Cor. 2.—Every arc eM, reckoned from the opposite pole of the primitive, is measured by Cm, which is the tangent of the angle CEM, or half the arc eM. This is often called the *semitangent* of the arc eM.

PROP. II.—THEOREM.

Every circle of the sphere which does not pass through the projecting point is projected into a circle.



1. If the circle MN be parallel to the primitive; then lines drawn to all points of its circumference from the projecting point E, will form the surface of a cone, which being cut parallel to the base by the plane of projection, the section mn, into which MN is projected, will be a circle. (Conic Sections, art. 225.)

2. But if the circle MRN be not parallel to the plane of projection, let the great circle $EAPB$ pass through the projecting point E, and also through P, the pole of MRN; and let $EAPB$ cut the primitive in the diameter AB, and the circle MRN in the line MN; draw the

diameter PCQ. Because CP passes through the pole of the circle MRN, it is perpendicular to this circle, and will therefore pass through its centre (Trig. arts. 121, 122). Hence MN is a diameter of the circle MRN, and the plane PCE is perpendicular to this circle. Through N draw NL parallel to AB; also, join EM, EN, meeting AB in the points m and n. Then, because LN is parallel to AB, the arc $AL = BN$; also, the quadrantal arc $EA = EB$; therefore the arc $EL = EN$, and the angle ENL or $Enm = EMN$. And because the oblique cone, which has the circle MRN for its base and E for its vertex, is cut through the axis by a plane, MEN, perpendicular to the base; and it is also cut by the primitive perpendicular to EMN, making the angle $Enm = EMN$; it follows that mrn is a subcontrary section (Con. Sec. art. 228). Hence mrn is in this case also a circle, of which mn is a diameter.

Cor. 1.—Bisect the diameter mn in o, then will o be the centre of the circle of projection. Put $Pe = I$, $PM = PN = D$; also, let $Co = d$, $mo = on = r$: then, (Trig. art. 66),

$$d = \tfrac{1}{2}(Cn + Cm) = \tfrac{1}{2}\tan\frac{I+D}{2} + \tfrac{1}{2}\tan\frac{I-D}{2} = \frac{\sin I}{\cos I + \cos D},$$

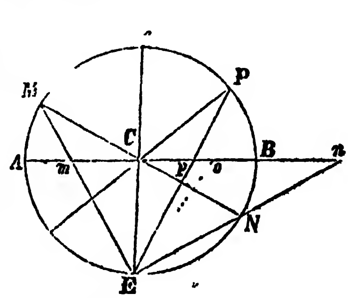
$$r = \tfrac{1}{2}(Cn - Cm) = \tfrac{1}{2}\tan\frac{I+D}{2} - \tfrac{1}{2}\tan\frac{I-D}{2} = \frac{\sin D}{\cos I + \cos D}.$$

Cor. 2.—If p, q be the projections of P and Q, the poles of the circle MN; then

$$Cp = \tan CEp = \tan \tfrac{1}{2}I,$$
$$Cq = \tan CEq = \cot CEp = \cot \tfrac{1}{2}I.$$

PROP. III.—THEOREM.

The distance of the centre of a projected great circle from the centre of the primitive is equal to the tangent of its inclination to the primitive, and its radius is equal to the secant of its inclination.

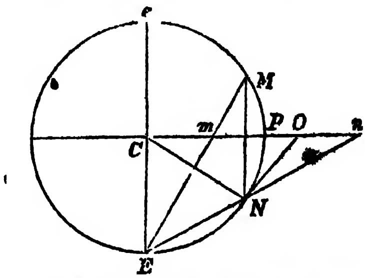


Let E be the projecting point, BEA a great circle passing through the point of projection, and the pole of the circle to be projected. Let MN be the intersection of this plane with the plane of EAB; then, by the last proposition, the circle MN will be projected into a circle of which the diameter is mn. Bisect mn in o, and join Eo. Now, since MN is a diameter, the angle MEN or mEn is a right angle, therefore a circle described from the centre o, with the radius om or on, will pass through E, and consequently the angle $oEn = onE = CEm$; therefore $2oEn$ or $CoE = 2CEm$ or eCM, and CEo, the complement of $CoE = ACM$, the complement of eCM. Hence

$$d = Co = \tan CEo = \tan ACM = \tan I,$$
$$r = om = oE = \sec ACM = \sec I.$$

PROP. IV.—THEOREM.

The centre of a projected small circle, perpendicular to the primitive, is distant from the centre of the primitive the secant of the small circle's distance from its own pole; and its radius is the tangent of that distance.

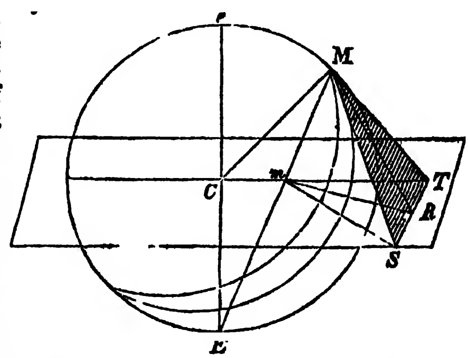


Let E be the projecting point, EMN a great circle passing through E, and also through P, the pole of this small circle; and let the circle EMN cut the small circle in the diameter MN. Then, by prop. 2, the circle MN will be projected into a circle whose diameter is mn. Bisect mn in o, and join CN, oN. Because the angle $EMN = Enm$ (prop. 2); a circle will pass through the four points M, m, N, n; and since mn bisects the chord MN at right angles, it will be a diameter of this circle, and o will be its centre: consequently the angle $CoN = 2CnN$. And because CE, MN are parallel, the angle $CNM = ECN = 2EMN$; hence the angle $CNM = CoN$, and the angle $CNo = CNM + oNM = CoN + oNM =$ a right angle; consequently No is a tangent to the circle EMN at N. But the arc NP is the distance of the small circle MN from its nearest pole; hence No or on is the tangent, and Co the secant of the distance of the small circle from its nearest pole.

Cor.—Hence $d = Co = \sec D$; $r = oN = \tan D$.

PROP. V.—THEOREM.

The angle made by two tangents, MR, MS, *on the surface of the sphere, is equal to the angle,* RmS, *made by their projections in the plane of the primitive.*



Through the points E, M, let a great circle be drawn, which will be perpendicular to the plane of the primitive, because it passes through its pole (Trig. art. 122). Let MT be the intersection of the plane MRS with the plane of the circle EM. Because CM is perpendicular to MR, MS, therefore CM and the circle CME are perpendicular to the plane MRS; hence CM is also perpendicular to MT. And because the circle ME is perpendicular both to the plane MST, and also to the plane of the primitive, it is therefore perpendicular to their common intersection ST, and consequently the angles STM, STm are right angles.

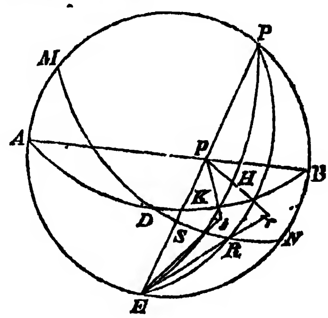
And because the angle $CMT =$ a right angle $= CmE + CEm$,

taking away the equal angles CME, CEM, there remains the angle $mMT = CmE = MmT$; and consequently $TM = Tm$. Hence the two sides of the triangles MTR, mTR, having the two sides of the one equal to the two sides of the other, each to each, and the included angles right angles, the angle $TMR = TmR$. In like manner it may be shown that the angle $TMS = TmS$. Hence, taking equals from equals, the angle on the sphere $RMS =$ the projected angle RmS.

Cor.—An angle contained by any two circles of the sphere is equal to the angle formed by their projections.

PROP. VI.—THEOREM.

Any projected arc of a great circle of the sphere is measured by that arc of the primitive which is cut off by right lines drawn from the projected pole through the two extremities of the given arc.



Let ADB be the primitive circle, E the place of the eye, Dsr the projection of the great circle MDN, and p the projection of its pole, P. Through P, E, describe the great circle PBE; and through P, E, and the points R, S, describe the two small circles PRE, PSE, cutting the primitive in H and K. Join pH, and produce it to meet ER, produced in r; also, produce pK to meet ES, produced in s; then, because pH, pK are in the plane of the primitive, r, s will evidently be the projections of the points R, S, and it is required to prove that the arc rs is measured by the arc of the primitive HK.

For since P, E are the poles of the great circles MDN, ADB, the chord $PR =$ chord EH, and therefore the arc $PR =$ arc EH; hence, taking away the common arc HR, the remaining arcs PH, ER are equal. In like manner it may be proved that the arcs PK, ES are equal And because the arcs PH, PK, and the included angle HPK, are equal to the arcs ER, ES, and the included angle RES, respectively, the figures PHK, ERS are symmetrical, and the arc $HK = RS$.* But RS is the original of rs, and therefore HK is the measure of the projected arc rs.

PROP. VII.—THEOREM.

Any angle formed by the projections of two great circles of the sphere is measured by that arc of the primitive which is cut off by straight lines

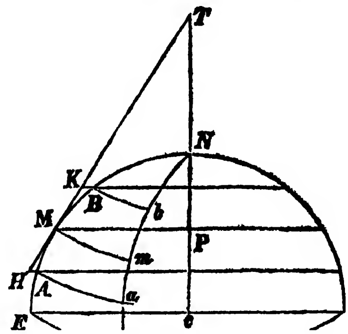
* For if we conceive the chords PH, ER to be drawn and produced to meet in T, the angle PET will evidently $= EPT$, and $PT = ET$. In like manner, if the chords PK, ES be drawn and produced to meet in V, $PV = EV$. Hence the two plane triangles PTV, ETV, having the sides PT, PV equal to ET, EV, respectively, and TV common, the angle $TPV = TEV$. Again, the two plane triangles PHK, ERS have the chords PH, PK equal to ER, ES, respectively, and the included angle HPK has just been proved $= RES$; therefore the chord $HK =$ chord RS, and consequently the arc $HK =$ arc RS.

CONSTRUCTION OF MAPS BY DEVELOPEMENT.

The practical application of the preceding methods of projection to the construction of maps is usually confined to the representation of an entire hemisphere; but for laying down a particular country, or portion of a hemisphere, the method of developement is generally adopted. Since the surface of a cone is described by the motion of a straight line passing continually through a fixed point and the circumference of a given circle, it is evident that if a right cone be rolled round upon a plane, while the vertex continues at the same point, every point of the surface of the cone will come in contact with a corresponding point in the plane surface; so that a sector of a circle will be described, with which the surface of the cone, if expanded, would exactly coincide.

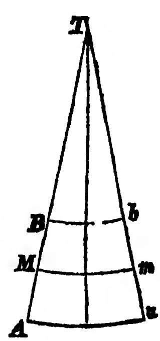
PROP.—THEOREM.

To develope a portion of a sphere, considered as the surface of a cone.



Let ENQ be a section of the sphere made by a plane of the meridian. Let CN be the axis, and EQ a diameter of the equator. Also, let $AabB$ be a portion of the sphere to be developed. Bisect AB in M, and at M draw the tangent MK, meeting CN, produced in T. Then, if the plane figure $TMEQ$ be supposed to revolve about the axis TC, NEC will generate a hemisphere, and TH will generate a conical surface, which will touch the hemisphere. Now it is evident that if AB be an arc of not many degrees, the spherical surface $ABab$, generated by AB, will nearly coincide with the conical surface described by HK.

To develope this surface, take TM (fig. 2) $= TM$ (fig. 1); and take the lines MA, MB also equal to the arcs MA, MB. From the centre T, with the radii TA, TM, TB, describe the arcs Aa, Mm, Bb. Lay off the degrees in Mm (fig. 2) equal in length to the corresponding degrees on the sphere. Then if the line $Tbma$ be drawn, the plane surface $AaBb$ (fig. 2) will bear a near resemblance to the corresponding portion of the sphere (fig. 1); and the resemblance will be more exact, in proportion as the zone comprehended between Aa and Bb is diminished in breadth.



Cor.—It is evident that the degree of the parallel Mm is to a degree on the equator EcQ, as PM to CE, or as $\cos EM$ to rad. Hence, if l be the latitude of M, or the middle latitude of AB,

$$\text{degree of longitude on } Mm = \text{degree of latitude on } AB \times \cos l.$$

It is also manifest that $TM = CM \times \cot l$, or if TM be measured in

degrees or minutes of the meridian, since the radius of a circle $= 57^{\circ}\cdot 29578 = 3437'\cdot 74677$,

$$TM = 57^{\circ}\cdot 3 \cot l = 3438' \cot l.$$

Scholium.—In drawing a map of any country of small extent, such as England, Ireland, &c., it is usual to make all the meridians and parallels of latitude straight lines; and to make the extreme parallels, and the meridian passing through the centre of the map, proportional to their real magnitude. This will be a tolerably correct representation of the whole country.

Another method, similar to this, but more exact, is to make the meridian passing through the centre of the map, and all the parallels of latitude, straight lines, as in the last method. Then all the degrees on each of the parallels are made proportional to their magnitude, and the lines passing through the corresponding points of division on the parallels will represent the meridians. These will be curved lines, and not straight, as in the last method. This is usually called *Flamstead's* projection, as it was first used by that astronomer in constructing his "Celestial Atlas;" and it is extremely useful in geographical maps, for countries lying on both sides of the equator.

A considerable improvement of this method, for countries of some extent, such as Europe, Russia, &c., is to represent all the parallels of latitude by concentric circles, according to the principles of the conical developement; and then to lay off the degrees on each parallel, proportional to their real magnitude. The lines drawn through the corresponding divisions of these parallels will represent the meridians. This delineation, perhaps, will give the different parts of a map of some extent, in as nearly their due proportions as the nature of the case will admit.

MENSURATION.

The term *mensuration* is usually applied to a system of rules and methods by which numerical measures of geometrical quantities are obtained. In the following treatise we shall principally confine ourselves to such practical rules as may be immediately derived from the previous parts of this work; although we may add one or two others, which can be introduced most conveniently in this place.

CHAP. I.—MENSURATION OF GEOMETRICAL FIGURES.

RECTILINEAL PLANE FIGURES.

The measure of the space or surface contained within the boundaries of any plane figure is called its area or superficial content. This is estimated in acres, square yards, square feet, or some other fixed or

determinate measure. Thus, if the adjoining figure represent the top of a rectangular table whose length is 6 feet and breadth 3 feet, then the upper surface will contain 6×3 or 18 square feet (prop. 56); a square foot being the unit or integer by which the area is estimated.

PROB. I.—*To find the area of a square, a rectangle, or any parallelogram.*

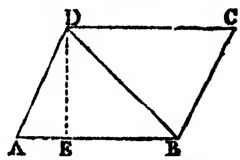
(1.) *When the base and perpendicular height are given.*

Multiply the base by the perpendicular height, and the product will be the area.

This rule is demonstrated in Geometry, prop. 56.

(2.) *When two adjacent sides, and their contained angle, are given.*

Multiply continually together the two sides and the natural sine of the contained angle, and the last product will be the area.



For, by Trigonometry (art. 16),

$$\text{radius or } 1 : \sin A :: AD : DE,$$

$$\therefore \quad DE = AD \times \sin A;$$

hence the area of the parallelogram is equal to

$$AB \times DE = AB \times AD \times \sin A.$$

Note.—If the calculation be made by logarithms, 10 must be rejected in the index of the sum of the logarithms, because the logarithm of the radius, in the table of log. sines, is equal to 10.

Ex. 1.—To find the area of a parallelogram, the length being 12·25, and breadth or height 8·5.

$$\text{The area} = 12{\cdot}25 \times 8{\cdot}5 = 104{\cdot}125.$$

2. Suppose the sides AB and AD are 36 feet and 25·5 feet, and the angle A is 58°; required the area.

The natural sine of 58° to rad 1 is ·848048, therefore

$$\text{the area} = 36 \times 25{\cdot}5 \times {\cdot}848048 = 778{\cdot}5 \text{ square feet.}$$

By Logarithms.

log sin 58°	-	9·928420
log 36	- -	1·556303
log 25·5	- -	1·406540
log area		2·891263

The number corresponding to this logarithm is 778·5, the area required.

3. To find the area of a square whose side is 35·25 chains.

Ans. 124 acres 1 rood 1 perch.

4. To find the area of a rectangular board whose length is $12\frac{1}{2}$ feet, and breadth 9 inches. Ans. $9\frac{3}{8}$ feet.

5. To find the content of a piece of land, in form of a rhombus, its length being 6·20 chains, and perpendicular breadth 5·45.

Ans. 3 acres 1 rood 20 perches.

6. To find the number of square yards of painting in a parallelogram whose length is 37 feet, and height 5 feet 3 inches.

Ans. $21\frac{7}{12}$ square yards.

PROB. II.—*To find the area of a triangle.*

(1.) *When the base and perpendicular are given.*

Multiply the base by the perpendicular height, and take half the product for the area. Or, multiply one of these dimensions by half the other.

Ex.—To find the area of a triangle whose base is 625, and perpendicular height 520 links.

The area $= 625 \times 260 = 162500$ square links $=$ 1 acre 2r. 20p.

(2.) *When two sides and their contained angle are given.*

Multiply together the two sides and the natural sine of the contained angle, and half the last product will be the area of the triangle.

This rule follows immediately from the second rule of the preceding problem; for the triangle *ABD* is half the parallelogram *ABCD*.

Ex.—What is the area of a triangle whose two sides are 30 and 40, and their contained angle 28° 57′?

The natnral sine of 28° 57′ is ·484046, and

$\frac{1}{2} \times 40 \times 30 \times \cdot 484046 = 290\ 4276$, the area required.

By Logarithms.

log sin 28° 57′	-	9·684887
$\frac{1}{2} \times 40 \times 30 = 600$ - log	-	2·778151
log area	-	2·463038

The number corresponding to this logarithm is 290·4, the area required.

(3.) *When the three sides are given.*

Add all the three sides together, and take half that sum. Next, subtract each side severally from the said half sum, thus obtaining three remainders. Lastly, multiply the said half sum and the three remainders together, and extract the square root of the last product, for the area of the triangle.

That is, if a, b, c be the three sides of the triangle, and $p = \frac{1}{2}(a + b + c)$, then the

$$\text{area} = \sqrt{[p(p-a)(p-b)(p-c)]}.$$

This rule is proved in the application of Algebra to Geometry, p. 418.

Ex.—To find the area of the triangle whose three sides are 20, 30, and 40.

	20	45	45	45
	30	20	30	40
	40	25	15	5
	2)90			
half sum	45			

Then $45 \times 25 \times 15 \times 5 = 84375$; and the square root of this number is 290·47, the area required.

By Logarithms.

log 45....	1·653213
log 25....	1·397940
log 15....	1·176091
log 5....	·698970
	2)4·926214
290·47....	2·463107

Examples.

1. How many square yards contains the triangle whose base is 40, and perpendicular 30 feet? Ans. $66\frac{2}{3}$ square yards.

2. To find the number of square yards in a triangle whose base is 49 feet, and height $25\frac{1}{4}$ feet. Ans. 68·7361.

3. To find the area of a triangle whose base is 18 feet 4 inches, and height 11 feet 10 inches. Ans. 108 feet 5 twelfths 8 inches.

4. How many square yards contains the triangle of which one angle is 45°, and its containing sides 25 and $21\frac{1}{4}$ feet? Ans. 20·86947.

5. How many square yards of plastering are in a triangle whose sides are 30, 40, 50 feet?

6. How many acres does the triangle contain whose sides are 2569, 4900, 5025 links? Ans. 61 acres 1 rood 39 perches.

PROB. III.—*To find the area of a trapezoid.*

Add together the two parallel sides; then multiply their sum by the perpendicular breadth, or distance between them; and take half the product for the area.

This rule is demonstrated in Geometry, prop. 54.

Ex. 1.—In a trapezoid, the parallel sides are 750 and 1225 links, and the perpendicular distance between them 1540 links: to find the area.

$\frac{1}{2}\times(750+1225)\times1540=152075$ square links $=15$ acres 33 perches.

Ex. 2.—How many square feet are contained in the plank whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches? Ans. $13\frac{13}{24}$ feet.

PROB. IV.—*To find the area of any trapezium.*

Divide the trapezium into two triangles by a diagonal; then find the reas of these triangles, and add them together.

Or thus:—Let fall two perpendiculars on the diagonal from the other two opposite angles; then add these two perpendiculars together, and multiply the sum by the diagonal, taking half the product for the area of the trapezium.

Ex. 1.—To find the area of the trapezium whose diagonal is 42, and the two perpendiculars on it 16 and 18.

Here $\frac{1}{2}\times(16+18)\times42=714$, the area.

2. How many square yards of paving are in the trapezium whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet? Ans $222\frac{1}{12}$ yards.

3. In the quadrilateral field $ABCD$, the perpendiculars DE, BF could not be measured, on account of obstructions; and therefore the following lines were measured:—The side $BC=265$ yards; $AD=220$; the diagonal $AC=378$; the segment $AE=100$; and $CF=70$ yards. Required the area, in acres. Ans. 17 acres 2 roods 21 perches.

PROB. V.—*To find the area of a regular polygon.*

Rule 1.—Multiply the perimeter of the polygon by the perpendicular, let fall from its centre on one of its sides, and half the product will be the area of the polygon.

To find the perpendicular.—Divide 180° by the number of sides, and then use this proportion—

radius : cotangent of this angle : : $\frac{1}{2}$ side of polygon : perpendicular.

Rule 2.—If a be the length of one of the sides of the polygon, and n the number of sides, then

$$\text{area of the polygon} = \frac{na^2}{4}\cot\frac{180^\circ}{n}.$$

These rules are only in effect resolving the polygon into as many equal triangles as it has sides, by drawing lines from the centre to all the angles, then finding the areas, and adding them all together.

Ex. 1.—To find the area of a regular pentagon, each side being 25 feet, and the perpendicular from the centre, on each side, 17·2047737.

Here $25 \times 5 = 125$ is the perimeter. Therefore the area $= \frac{1}{2} \times 17{\cdot}2047737 \times 125 = 1075{\cdot}298$.

2. To find the area of the hexagon whose side is 20.
Ans. 1039·23048.

3. To find the area of an octagon whose side is 20.
Ans 1931·37084.

4. To find the area of a decagon whose side is 20.
Ans. 3077·68352.

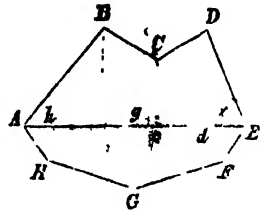
5. To find the area of a dodecagon whose side is 20.

PROB. VI.—*To find the area of an irregular polygon.*

Rule 1.—Draw diagonals dividing the proposed polygon into trapeziums and triangles. Then find the areas of these separately, and add them together, for the content of the whole polygon.

Rule 2.—Draw any convenient line as a base; and from all the angular points of the polygon let fall perpendiculars upon the base; the figure will then be divided into trapezoids and triangles, the sum of which will be the content of the polygon.

Ex. 1.—What is the content of the octangular figure $ABCDEFGH$, the diagonal AE being taken as a base, and the perpendiculars Bb, Cc, &c., drawn from the angular points to AE: when the lengths of the perpendiculars, &c., are as follows?



$Bb=294$; $Cc=142{\cdot}5$; $Dd=224$;
$Ff=121$; $Gg=195{\cdot}5$; $Hh=142$.
$Ah=44.5$; $hb=124{\cdot}25$; $bg=80$;
$gc=41$; $cd=130{\cdot}5$; $df=50$; $fE=52{\cdot}5$.

Ans. 162463·9.

Ex. 2.—Let *ABCDEFGH* be the vertical section of a rampart, and *Bb, Cc, Dd,* ... the perpendiculars on the base *AH*. What is the area of the section when the heights and breadths are as follows?

Heights, *Bb, Co, Dd,* &c....12; 12·5; 13 5; 13·5; 18; 16 feet.
Breadths, *Ab, bc, cd,* &c....16; 18; 2; 3; 2; 12; 10 feet.

Ans. 698·5 feet.

MENSURATION OF THE CIRCLE.

Prob. VII.—*To find the circumference of a circle from the diameter.*

Multiply the diameter by 3·1415926, and the product will be the circumference.

In general it will be sufficient to use four places of decimals, 3·1416. Sometimes also the number $3\frac{1}{7}$ is used, as an approximation. This number is expressed by the Greek letter π.

This rule is deduced from the corollary to prop. 89. Thus, since the circumferences of circles are to one another as their diameters, and π is the circumference of a circle whose diameter is 1; if c be the circumference of a circle whose diameter is d, or radius r, we have

$$1 : \pi :: d : c, \quad \text{therefore} \quad c = \pi d = 2\pi r.$$

As the number π, and other numbers dependent on it, are continually occurring, we shall give their values here, for the sake of easy reference.

π	=	3·14159265	=	3·1416	nearly.
$\frac{1}{4}\pi$	=	·78539816	=	·7854	„
$\frac{1}{6}\pi$	=	·5235988	=	·5236	„
$\frac{\pi}{180}$	=	·0174533	=		
$\frac{1}{\pi}$	=	·3183099	=	·3183	„
$\frac{1}{4\pi}$	=	·0795775	=	·07958	„

Ex. 1.—If the diameter of a well be 3 ft. 9 in., what is its circumference? Ans. 11 ft. 9 in.

2. The diameter of a circular plantation is 100 yards; what did it cost fencing round, at 6*s*. 9*d*. a rood? Ans. 19*l*. 5*s*. $6\frac{1}{2}$*d*.

3. The diameter of the sun is 883320 miles; what is its circumference? Ans. 2,774,724 miles.

4. If the equatorial diameter be 7924·[illegible] miles, what is the length of a degree of the equator? Ans. 69·156 miles.

Prob. VIII.—*To find the diameter from the circumference.*

Multiply the circumference by ·3183099, or (unless great accuracy be required) by ·3183, and the product will be the diameter.

For $d = \frac{c}{\pi}$; and $\frac{1}{\pi} = \cdot 3183\ldots$; therefore $d = \cdot 3183c$.

Ex. 1.—What is the diameter of a stone column whose circumference measures 9 ft. 6 in.? Ans. 3 feet.

2. If the circumference of the earth be 25,000 miles, what is its diameter, supposing it to be a sphere? Ans. 7957 miles.

Prob. IX.—*To find the length of any arc of a circle.*

Multiply the decimal ·01745 by the degrees in the given arc, and that product by the radius of the circle, for the length of the arc; that is, if r be the radius of the circle, and n the number of degrees in the arc,

$$\text{length of the arc} = \cdot 01745nr.$$

The reason of this rule is sufficiently evident. For the semi-circumference, or 180 degrees, being 3·14159265, when the radius is 1, we have

$$180^\circ : 1^\circ :: 3\cdot 14159265 : \cdot 0174533,$$

the arc of 1 degree. Hence the decimal ·01745, multiplied by any number of degrees, n, will give the length of the arc of those degrees, when the radius is 1. And because similar arcs of circles are proportional to their radii, therefore

$$1 : r :: \cdot 01745n : \cdot 01745nr, \text{ length of the arc required.}$$

Ex. 1.—To find the length of an arc of 30 degrees, the radius being 9 feet. Ans. 4·7115.

2. To find the length of an arc of 12° 10′, or $12\frac{1}{6}°$, the radius being 10 feet. Ans. 2·1231.

Prob. X.—*To find the area of a circle.*

Rule 1.—Multiply half the circumference by half the diameter, and the product will be the area.

Rule 2.—Square the diameter, and multiply that square by the decimal ·7854, for the area.

Rule 3.—Square the circumference and multiply that square by the decimal ·07958.

By the first rule, area $= \frac{1}{4}cd$ (Geom. prop. 89).

$$\text{And since } c = \pi d, \quad \therefore \quad \text{area} = \tfrac{1}{4}\pi d^2 = \pi r^2.$$

$$\text{And also,} \quad \text{area} = \frac{c^2}{4\pi} = \cdot 07958c^2.$$

Ex. 1.—To find the area of a circle whose diameter is 10, and its circumference 31·416.

By Rule 1.	*By Rule* 2.	*By Rule* 3.
31·416	·7854	31·416
10	100	31·416
4)314 16	78·54	986·865
78·54		·07958
		78·54

So that the area is 78·54 by all the three rules.

2. To find the area of a circle whose diameter is 7, and circumference 22. Ans. $38\frac{1}{2}$.

3. How many square yards are in a circle whose diameter is $3\frac{1}{2}$ feet? Ans. 1·069.

4. To find the area of a circle whose circumference is 12 feet. Ans. 11·4595.

Prob. XI.—*To find the area of a circular ring, or of the space included between the circumferences of two concentric circles.*

Take the difference between the areas of the two circles, as found by the last problem, for the area of the ring. Or, multiply the sum of the diameters by their difference, and the product by ·7854.

Ex. 1.—The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences. Ans. 50·2656.

2. What is the area of the ring, the two diameters being 10 and 20? Ans. 235·62.

Prob. XII.—*To find the area of the sector of a circle.*

(1.) *When the radius and the length of the arc are given.*

Multiply the radius by half the arc of the sector, for the area.

(2.) *When the radius and the number of degrees in the arc are given.*

Rule 1.—Compute the area of the whole circle; then say, as 360° is to the degrees in the arc of the sector, so is the area of the whole circle to the area of the sector.

Rule 2.—Multiply continually together the square of the radius, the number of degrees in the sector, and the decimal ·0087266, the last product will be the area: that is, if r be the radius of the circle, and n the number of degrees in the sector,

$$\text{area of the sector} = {\cdot}0087266nr^2.$$

$$\text{For, by problem 9, the length of the arc} = {\cdot}0174533nr;$$

$$\therefore \text{ area of the sector} = \tfrac{1}{2}r \times {\cdot}0174533nr = {\cdot}0087266nr^2.$$

Ex. 1.—To find the area of a circular sector whose arc contains 18 degrees, the diameter being 3 feet.

$$\text{Area of the circle, } {\cdot}7854 \times 3^2 = 7{\cdot}0686.$$

$$\text{Then } 360° : 18° :: 7{\cdot}0686 : {\cdot}35343, \text{ the area of the sector.}$$

By the Second Rule.

$$\text{Area of the sector} = {\cdot}0087266 \times 18 \times 1{\cdot}5^2 = {\cdot}35336.$$

2. To find the area of a sector whose radius is 10, and arc 20. Ans. 100.

3. Required the area of a sector whose radius is 25, and the contained angle 149° 29′.

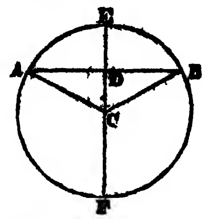
PROB. XIII.—*To find the area of the segment of a circle.*

Find the area of the sector having the same arc with the segment, by the last problem.

Find also the area of the triangle formed by the chord of the segment and the two radii of the sector.

Then the difference of the areas, when the segment is less than a semicircle, or their sum when it is greater, will be the area of the segment.

(1.) *When the radius* AC *and the chord* AB *are given.*



Find the angle ACE from the proportion

$AC : AD ::$ rad of the tables : $\sin ACE$.

Then the radius AC and the angle ACE being known, find the area of the sector by the last problem from this proportion,

As $360° : 2$ angle ACE, or as

$180°$: angle ACE : area of the circle : area of the sector.

Also, area of the triangle $ACB = \frac{1}{2}AC \times CB \times \sin \angle ACB$.

And the area of the segment $ABE =$ sector $ACBE -$ triangle ABC.

(2.) *When the chord* AB *and the height* DE *are given.*

Find DF from the proportion

$$DE : AD :: AD : DF;$$

then the radius CA or $CE = \frac{1}{2}(DE + DF)$, and $CD = CE - ED$. Proceed then to find the angle ACE, and the areas of the sector and the triangle, as in the first case.

Ex.—Given $AB = 12$, $AC = 10$, to find the area of the segment.

From the first proportion above, the angle ACE will be found equal to $36° \; 52'{\cdot}2 = 36°{\cdot}87$. Also, the area of the whole circle $= 400 \times {\cdot}7854 = 314{\cdot}16$. Hence the area of the sector, from the second proportion $= 64{\cdot}3504$. Also, the area of the triangle ACB will be found $= 48$;

$$\therefore \text{ area of the segment } AEB = 16{\cdot}3504.$$

Second method.—From a table of circular segments.

Rule.—Divide the height of the segment by the diameter, and find the area corresponding to the quotient in the following table. Multiply this number by the square of the diameter, and the product will be the area of the segment.

When the segment is greater than a semicircle, find the area of the remaining segment in the Table, and subtract it from ·785398; the remainder will be the tabular area.

Ex.—Taking the same example as before, CD will be found $= 8$, and $DE = CE - CD = 2$. Hence $CD \div EF = 2 \div 20 = 0{\cdot}1$. The number corresponding to 0·1 in the table is ·040875. Hence the area $= {\cdot}040875 \times 20^2 = 16{\cdot}35$.

Ex. 1.—What is the area of the segment whose height is 18, and diameter of the circle 50? Ans. 636·375.

2. Required the area of the segment whose chord is 16, the diameter being 20. Ans. 44·728.

Areas of the segments of a circle.

Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.
·01	·001329	·11	·047005	·21	·119897	·31	·207376	·41	·303187
·02	·003748	·12	·053385	·22	·128113	·32	·216666	·42	·313041
·03	·006865	·13	·059999	·23	·136465	·33	226033	·43	·322928
·04	·010537	·14	066833	·24	·144944	34	·235473	·44	·332843
·05	·014681	·15	·073874	·25	·153546	·35	·244980	·45	·342782
·06	·019239	·16	·081112	·26	·162263	·36	·254550	46	·352742
·07	·024168	·17	088535	·27	·171089	·37	·264178	·47	·362717
·08	·029435	·18	·096134	·28	·180019	38	·273861	·48	·372704
·09	·035011	·19	·103900	·29	·189047	·39	·283592	·49	·382699
·10	·040875	·20	·111823	·30	·198168	·40	·293369	·50	·392699

CURVILINEAL PLANE FIGURES.

Prob. XIV.—*To find the area of an ellipse or oval.*

Multiply the longest diameter, or axis, by the shortest; then multiply the product by the decimal ·7854, for the area.

This rule is proved in Conic Sections, art. 128.

Ex. 1.—Required the area of an ellipse whose two axes are 70 and 50. Ans. 2748·9.

2. To find the area of the oval whose two axes are 24 and 18. Ans. 339 2928.

Prob. XV.—*To find the area of a parabola.*

Multiply the base of the perpendicular height; then take two-thirds of the product for the area.

This rule is proved in Conic Sections, art. 220.

Ex. 1.—To find the area of a parabola; the height being 2, and the base 12.

$$\text{The area} = \tfrac{2}{3} \times 2 \times 12 = 16.$$

2. Required the area of the parabola whose height is 10, and its base 16. Ans. 106⅔.

Prob. XVI —*To find the area of any curvilineal figure.*

Rule 1.—Divide the base into any number of equal parts, and measure the breadths of the figure at the two ends, and at the points of division. Then add together half the sum of the two extreme breadths

and all the intermediate ones; the sum, multiplied by the common distance of these breadths, will be the area, nearly; that is, if a, b, c, ... l be the breadths, and δ the common distance between these breadths,

$$\text{area} = (\tfrac{1}{2}a + b + c + \ldots\ldots + k + \tfrac{1}{2}l) \times \delta.$$

Rule 2.—Divide the base into an *even* number of equal parts, and measure the breadths, as in the first rule. Then add together the two extremes, four times the sum of all the even breadths, and twice the sum of all the odd breadths, excepting the first and last; a third of this sum, multiplied by the common distance of these breadths, will be the area, very nearly; that is,

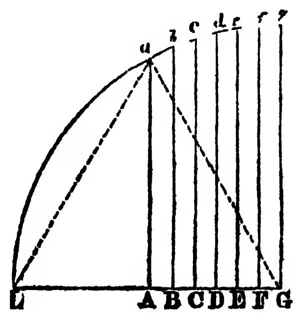
$$\text{area} = (a + 4b + 2c + 4d \ldots\ldots + 2i + 4k + l) \times \tfrac{1}{3}\delta.$$

Both these rules are approximations, but the second rule gives the area much more correctly than the first, and the calculation is nearly as easy.*

Ex. 1.—To find the area of the quadrant of a circle, GLg.

Bisect GL in A, and divide AG into six equal parts, by the perpendiculars Aa, Bb, &c. Also, suppose $GL = Gg = 12$, $GA = 6$, $AB = BC$, &c. $= 1$. We have then

$$Aa = \surd(Ga^2 - GA^2)$$
$$= \surd(144 - 36) = 10{\cdot}3920.$$



In like manner, the other ordinates will be found as below. And because GL is bisected in A, therefore $aL = aG$, and the angle $aGL = 60$ degrees. Hence the angle $aGg = 30°$, and the area of the quadrant $= 3$ times the sector aGg.

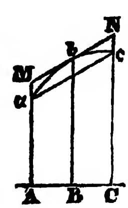
We have then,

* The first rule is founded upon the supposition, that the portions of the curve ab, bc, &c. are so small, that they may be taken for straight lines without any material error. On this hypothesis, the figures Ab, Bc, &c. are trapezoids. Let $Aa = a$, $Bb = b$, &c., and $AB = BC =$ &c. $= \delta$; we have then

$$\text{area } AagG = \tfrac{1}{2}(a + b) \times \delta + \tfrac{1}{2}(b + c) \times \delta \ldots\ldots + \tfrac{1}{2}(f + g) \times \delta$$
$$= (\tfrac{1}{2}a + b + c + d + e + f + \tfrac{1}{2}g) \times \delta.$$

In the second rule, we suppose that the arc abc is a portion of a parabola whose diameter is bB, and the double ordinate ac. If, then, the parallelogram $aMNC$ be completed, the area abc will be equal to $\tfrac{2}{3}$ of the parallelogram aN (Con. Sect. art. 221). We have, therefore,

$$\text{area } AabcC = \text{trapezoid } AacC + \tfrac{2}{3} \text{ parallelogram } aN$$
$$= \tfrac{1}{3} \text{ trapezoid } Ac \times \tfrac{2}{3} \text{ trapezoid } AN$$
$$= \tfrac{1}{2}(Aa + Cc) \times \tfrac{1}{3}AC + \tfrac{2}{3}(AM + BN) \times \tfrac{1}{2}AC$$
$$= \tfrac{1}{3}(a + 1b + c) \times \delta.$$



In like manner, area $CcdeE = \tfrac{1}{3}(c + 4d + e) \times \delta$; and so on.

Hence, adding these together, we have

$$\text{area } AagG = (a + 4b + 2c + 4d + 2e + 4f + g) \times \tfrac{1}{3}\delta.$$

By the First Rule.

Gg....	12
Aa....	10·39230
	2)22·39230
	11·19615
Bb....	10·90871
Cc....	11·31371
Dd....	11·61895
Ee....	11·83216
Ff....	11·95826
area *AagG*....	68·82794
triangle *AGa*...	31·17690
sector *aGg*....	37·65104
	3
quadrant *GLg*..	112·95312

By the Second Rule.

Bb...	10·90871
Dd...	11·61895
Ff...	11·95826
	34·48592
	4
	137·94368
Cc...	11·31371
Ee...	11·83216
	23·14587
	2
	46·29174

Gg....	12
Aa....	10·39230
	137·94368
	46·29174
	3)206·62772
AagG..	68·87591
AaG...	31·17690
sector *aGg*	37·69901
	3
GLg...	113·09703

Since the areas of circles are as the squares of their radii, if we divide each of these results by 144, we shall have the area of the quadrant when the radius is 1. By the 1st rule this will be found $=\cdot 7844$, which fails in the third place of decimals; and by the second rule it will be found $=.78539$, which is true to the last place of decimals.

Ex. 2.—The breadths of an irregular figure, at five equidistant places, are 8·2, 7·4, 9·2, 10·2, 8·6; required the area, by both methods.

3. Find the area of the parabola, from seven ordinates, measured along the axis of the curve, at the distances from the vertex, 0, 1, 2, 3, 4, 5, 6; the ordinates being given from the equation $y^2 = 20x$, where y is the ordinate, and x its distance from the vertex.

4. Find the area of a hyperbola in like circumstances; the ordinates being given from the equation $y^2 = 20x + x^2$.

5. Find the area of a portion of an ellipse in like circumstances, the ordinates being given from the equation $y^2 = \frac{16x}{3} - \frac{4x^2}{9}$; and compare this area with that of a quadrant of a circle whose radius is 6.

MENSURATION OF SUPERFICIES.

PROB. XVII.—*To find the superficies of any solid bounded by plane surfaces.*

Find the areas of several bounding planes by the preceding rules, and their sum will be the superficies required.

Ex. 1.—To find the surface of a cube, the length of each side being 20 feet. Ans. 2400 feet.

2. To find the whole surface of a triangular prism, whose length is 20 feet, and each side of its end or base 18 inches.
Ans. 91·948 feet.

3. What must be paid for lining a rectangular cistern with lead, at 3*d*. a pound weight, the thickness of the lead being such as to weigh

7lb. for each square foot of surface; the inside dimensions of the cistern being as follows, viz., the length 3 feet 2 inches, the breadth 2 feet 8 inches, and depth 2 feet 6 inches? Ans. 3*l.* 5*s.* 9$\frac{3}{4}$*d.*

4. Find the convex surface of a hexagonal pyramid, each side of the base being 2 feet 6 inches, and its perpendicular height 10 feet. Ans. 76·737 square feet.

5. How many square feet are in the surface of the frustum of a square pyramid, each side of the base or greater end being 3 feet 4 inches, and each side of the less end 2 feet 2 inches; also each of the edges of the frustum being 10 feet?

6. Find the whole surface of the frustum of an octagonal pyramid, whose perpendicular height is 6 feet; and each side of the two ends 4 and 5 feet respectively.

Prob. XVIII.—*To find the convex surface of a right cylinder.*

Multiply the circumference of the base by the length of the cylinder, and the product will be the convex superficies.

This is demonstrated in Geometry, prop. 124.

Ex. 1.—To find the convex surface of a cylinder whose length is 20 feet, and the diameter of its base is 2 feet. Ans. 125·664.

2. What is the whole superficies of a cylinder, its length being 10 feet and diameter 3? Ans. 108·38 feet.

Prob. XIX.—*To find the convex surface of a right cone.*

Multiply the circumference of the base by the slant height, and half the product will be the convex surface.

This is proved in Geometry, prop. 126.

Ex. 1.—Required the convex surface of a cone, the slant height being 50 feet, and the diameter of its base 8$\frac{1}{2}$ feet.

2. What is the whole superficies of a cone, the radius of the base being 30 feet, and the perpendicular height 40 feet?

Prob. XX.—*To find the convex surface of the frustum of a right cone.*

Add together the circumferences of the two ends, and multiply their sums by the slant height; then half the product will be the convex superficies.

Ex. 1.—To find the convex surface of the frustum of a cone, the slant height of the frustum being 12$\frac{1}{2}$ feet, and the circumferences of the two ends 6 and 8·4 feet. Ans. 90 feet.

2. What is the whole surface of the frustum of a right cone, the diameters of the two ends being 8 and 12 feet, and its perpendicular height 6 feet?

Prob. XXI.—*To find the surface of a sphere.*

Rule 1.—Multiply the circumference of the sphere by its diameter, and the product will be the whole surface of it.

Rule 2.—Square the diameter, and multiply the square by 3·1416, for the surface.

Rule 3.—Square the circumference, and multiply the square by the decimal ·3183, for the surface.

For the surface of a sphere is equal to the rectangle contained by the diameter and the circumference of the sphere, or is equal to four great circles of the sphere (Geometry, prop. 130, cor. 3); and therefore

$$\text{surface} = cd = \pi d^2 = \frac{c^2}{\pi} = \cdot 3183099\,c^2.$$

Ex. 1.—Required the surface of a sphere whose diameter is 7, and circumference 22. Ans. 154.

2. Required the superficies of a globe whose diameter is 24 inches. Ans. 1809·56.

3. Required the area of the whole surface of the earth, its circumference being 25,000 miles. Ans. 198,943,687 square miles.

PROB. XXII.—*To find the convex surface of a spherical segment or a spherical zone.*

Multiply the circumference of the sphere by the height of the segment or zone, and the product will be the convex superficies.

The truth of this will appear from Geometry, prop. 130.

Ex. 1.—The axis of a sphere being 42 inches, what is the convex superficies of the segment whose height is 9 inches?

Ans. 1187·5248 inches.

2. Required the convex surface of a spherical zone, whose breadth or height is 2 feet, and cut from a sphere of 12½ feet diameter.

Ans. 78·54 feet.

MENSURATION OF SOLIDS.

The magnitude of a solid, its bulk, or its extension, is called its *solidity*, or its *solid content*. The unit of solids is a cube, each of whose edges is the measuring unit of lines, and consequently each of its faces is the measuring unit of surfaces.

PROB. XXIII.—*To find the solid content of a prism or cylinder.*

Multiply the area of the base by the perpendicular height, and the product will be the solid content.

This rule follows immediately from Geometry, prop. 125.

Ex. 1.—To find the solid content of a cube whose side is 24 inches.

Ans. 13824 cubic inches.

2. How many cubic feet are in a block of marble, its length being 3 feet 2 inches, breadth 2 feet 8 inches, and thickness 2 feet 6 inches?

Ans. 21⅑.

3. How many imperial gallons of water will the cistern contain, whose dimensions are the same as in the last example, when 277·3 cubic inches are contained in one gallon? Ans. 131·58.

4. Required the content of a triangular prism whose length is 10 feet, and the three sides of its triangular base are 3, 4, 5 feet.
Ans. 60 cubic feet.

5. Required the content of a round pillar or cylinder, whose length is 20 feet, and circumference 5 feet 6 inches. Ans. 48·1459 feet.

PROB. XXIV.—*To find the content of a pyramid or cone.*

Multiply the area of the base by the perpendicular height, and $\frac{1}{3}$ of the product will be the content.

This rule is demonstrated in Geometry, prop. 128.

Ex. 1.—Required the solidity of a square pyramid, each side of its base being 30, and its perpendicular height 25. Ans. 7500.

2. To find the content of a triangular pyramid, whose perpendicular height is 30, and each side of the base 3. Ans. 38·971143.

3. To find the content of a triangular pyramid, its height being 14 feet 6 inches, and the three sides of its base 5, 6, 7 feet.
Ans. 71·0352.

4. What is the content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet? Ans. 27 5276.

5. What is the content of the hexagonal pyramid, whose height is 6·4 feet, and each side of its base 6 inches? Ans. 1·38564 feet.

6. Required the content of a cone, its height being $10\frac{1}{2}$ feet, and the circumference of its base 9 feet. Ans. 22·56093.

PROB. XXV.—*To find the solidity of the frustum of a cone or pyramid.*

(1.) *When the ends of the frustum are any similar figures.*

Add together the areas of the two ends, and the mean proportional between them; then multiply the sum by the perpendicular height, and one-third of the product will be the content.

(2.) *When the diameters of the two ends of the frustum of a cone are given.*

Add together the squares of the diameters of the two ends and the product of the diameters; then multiply this sum, the perpendicular height, and ·7854 continually together; and one-third of the last product will be the content.

(3.) *When the circumferences of the two ends of the frustum of a cone are given.*

Add together the squares of the circumference of the two ends and the product of the circumferences; then multiply this sum, the perpendicular height, and ·07958 continually together; and one-third of the last product will be the content.

These rules are immediately derived from the 7th problem, p. 422, vol. i.

Ex. 1.—To find the number of solid feet in a piece of timber whose bases are squares, each side of the greater end being 15 inches, and each side of the less end 6 inches; also, the length or the perpendicular altitude 24 feet. Ans. $19\frac{1}{2}$.

2. Required the content of a pentagonal frustum, whose height is 5 feet, each side of the base 18 inches, and each side of the top or less end 6 inches. Ans. 9.31925 feet.

3. To find the content of a conic frustum, the altitude being 18, the greatest diameter 8, and the least diameter 4. Ans. 527·7888.

4. What is the content of the frustum of a cone, the altitude being 25; also, the circumference at the greater end being 20, and at the less end 10? Ans. 464·216.

5. If a cask, which is composed of two equal conic frustums joined together at the bases, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches, how many imperial gallons will it hold? Ans. 65·8668.

PROB. XXVI.—*To find the solidity of a sphere or globe.*

Rule 1.—Take the cube of the diameter, and multiply it by the decimal ·5236, for the content.

Rule 2.—Cube the circumference, and multiply by ·01688, for the content.

These rules are deduced from Geometry, prop. 132. For since a sphere is equivalent to two-thirds of the circumscribing cylinder, we have,

$$\text{content of sphere} = \tfrac{2}{3} \times \pi r^2 \times 2r = \frac{4\pi}{3} r^3$$

$$= \frac{\pi}{6} d^3 = \cdot 5236\, d^3.$$

$$\text{Also the content} = \frac{\pi}{6}\left(\frac{c}{\pi}\right)^3 = \frac{c^3}{6\pi^2} = \cdot 01688\, c^3.$$

Ex. 1.—To find the content of a sphere whose axis is 12. Ans. 904·7808.

2. To find the solid content of the globe of the earth, supposing its circumference to be 25000 miles. Ans. 263750,000000 cubic miles.

PROB. XXVII.—*To find the solid content of a spherical segment.*

(1.) *When the height of the segment and the radius of the sphere are given.*

From 3 times the diameter of the sphere take double the height of the segment; then multiply the remainder by the square of the height, and the product by the decimal ·5236, for the content.

This rule is proved in the application of Algebra to Geometry, page 423, vol. i.

(2.) *When the height of the segment and the radius of its base are given.*

To three times the square of the radius of the segment's base add the square of its height; then multiply the sum by the height, and the product by ·5236, for the content.

Ex. 1.—To find the content of a spherical segment, of 2 feet in height, cut from a sphere of 8 feet diameter. Ans. 41·888.

2. What is the solidity of the segment of a sphere, its height being 9, and the diameter of its base 20? Ans. 1795·4244.

Prob. XXVIII.—*To find the content of any solid.*

Take any principal line in the solid for an axis, and divide it into an *even* number of equal parts.

Find the areas of the transverse sections perpendicular to this axis at the two extremities, and at the intermediate points of division, by the rules given in the mensuration of plane figures.

Then add together the areas of the two ends, four times the sum of the areas at the even points of division, and twice the sum of the areas at the odd points of division, excluding the two ends. One-third of this sum, multiplied by the common distance of these sections, will be the content required, very nearly.

This rule is founded upon the same principles as the second rule in problem 16. (See the note, page 118.)

Ex. 1.—If, in the figure to problem 16, the quadrant Lg be supposed to describe a hemisphere by its revolution about LG; it is required to find the content of the portion of the sphere included between the circles described by the ordinates Aa, Gg.

Let the dimensions be the same as in problem 16. Then the areas of the different sections described by Aa, Bb, &c., will be $\pi.Aa^2$, $\pi.Bb^2$, &c. Hence, according to the rule,

$\pi.Bb^2 = 143\pi$	$\pi.Cc^2 = 140\pi$	$\pi.Aa^2 = 108\pi$
$\pi.Dd^2 = 135\pi$	$\pi.Ee^2 = 128\pi$	$\pi.Gg^2 = 144\pi$
$\pi.Ff^2 = 119\pi$	268π	1588π
397π	2	536π
4	536π	$3)2376\pi$
1588π		792π

Hence the content $= 792\pi \times AB = 2488{\cdot}14$; which is the same result, in this case, as if the content had been found by the rule in problem 27.

Ex. 2.—Required the number of cubic yards which were cut out of a canal, the areas of five transverse sections perpendicular to the axis being

687·6; 822·2; 735·8; 809·5; 509·5

square feet respectively, and the common distance between two transverse sections being 60 feet. Ans. 6811·466 yards.

3. Required the number of cubic yards in an embankment, the tranverse sections perpendicular to the axis being trapezoids whose dimensions were as follows:—

horizontal lines at top...22; 21·9; 21·6; 21·6; 21·8, 21·6; 22 feet.
do. bottom...46; 46·9; 47·6; 50·2; 52·4; 55·2; 62
perpendicular height.... 6; 6·3; 6·6; 7·2; 7·8; 8·4; 10
Also, the common distance between two transverse sections was 50 feet.
Ans. 80842 cubic feet.

4. Required the content of a round bulging haystack, when the different girts or circumferences, taken from the bottom, were as follows:—
127·2; 145·4; 156·5; 168·7; 148·3; 121·8; 68·6 feet,
respectively. The common vertical distance between the sections was 8 feet; and the height of the conical part, above the highest section, 7·4 feet. Ans. 79782·335 cubic feet.

5. Required the number of imperial gallons which a cask will hold, the length being 40 inches, the girts at the two ends 66 inches, at the middle 91·2, and at 10 inches distance from each end, 81·4 and 82 inches respectively; also, the thickness of the cask being ½ an inch.
Ans 69·623 imp. gallons.

CHAP. II.—ARTIFICERS' WORKS; GAUGING; BALLS AND SHELLS.

Artificers compute the contents of their works by several different measures. As,

Glazing and masonry by the foot; painting, plastering, paving, &c, by the yard of 9 square feet; flooring, partitioning, roofing, tiling, &c., by the square of 100 square feet:

And brickwork, either by the yard of 9 square feet, or by the perch, or square rod or pole, containing 272¼ square feet, or 30¼ square yards, being the square of the rod or pole of 16½ feet, or 5½ yards in length.

As it is inconvenient to divide by the number 272¼, the fraction is often omitted in practice, and the content in feet divided by 272 only.

I.—THE CARPENTER'S OR SLIDING RULE.

The carpenter's or sliding rule is an instrument used in measuring timber and artificers' works, both for taking their dimensions and computing their contents.

The instrument consists of two equal pieces, each a foot in length, which are connected together by a folding joint.

One side or face of the rule is divided into inches, and eighths of an inch. On the same face are also several plane scales, divided into twelfth parts by diagonal lines, which are used in planning dimensions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely, each foot into ten equal parts, and each of these into ten parts again: so that, by means of this last scale, dimensions are taken in feet, tenths and hundredths, and multiplied as common decimal numbers, which is the easiest way.

On the one part of the other face are four lines marked *A*, *B*, *C*, *D*;

the two middle ones, B and C, being on a slider, which runs in a groove made in the stock. The same numbers serve for both these two middle lines, the one being above the numbers and the other below.

These four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or lowest line, D, is a single one, proceeding from 4 to 40. It is also called the girt line, from its use in computing the contents of trees and timber; and on it is marked Imp. G. at 18·79 (square root of 277·3 ÷ ·7854), the gauge-point for the imperial gallon, to make this instrument serve the purpose of a gauging rule.

On the other part of this face there is a table of the value of a load, or 50 cubic feet of timber, at all prices from 6 pence to 2 shillings a foot.

When the one at the begining of any line is accounted 1, then the 1 in the midle will be 10, and the 10 at the end 100; but when the 1 at the beginning is counted 10, then the one in the middle is 100, and the 10 at the end 1000; and so on. And the smaller divisions are altered proportionally.

II.—BRICKLAYERS' WORK.

Brickwork is estimated at the rate of a brick and half thick. So that if a wall be more or less than this standard thickness, it must be reduced to it as follows:—

Multiply the superficial content of the wall by the number of half bricks in the thickness, and divide the product by 3.

The dimensions of a building may be taken by measuring half round on the outside, and half round it on the inside; the sum of these two gives the compass of the wall, to be multiplied by the height, for the content of the materials.

Chimneys, on account of the trouble of them, are commonly measured as if they were solid, deducting only the vacuity from the hearth to the mantel. All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed.

The dimensions of a common bare brick are $8\frac{1}{4}$ inches long, 4 inches broad, and $2\frac{1}{2}$ thick; but including the half-inch joint of mortar, when laid in brickwork, every dimension is to be counted half an inch more, making its length 9 inches, its breadth $4\frac{1}{2}$, and thickness 3 inches. So that every 4 four courses of proper brickwork measure just 1 foot, or 12 inches, in height.

Note.—A standard rod of brickwork requires 4500 bricks, including waste; and a square yard of brickwork, one brick thick, requires 100 bricks.

Ex. 1.—How many rods of standard brickwork are there in a wall whose length of compass is 57 feet 3 inches, and height 24 feet 6 inches; the wall being $2\frac{1}{2}$ bricks or 5 half bricks thick?

Ans. 8 rods $17\frac{1}{2}$ yards.

2. Required the content of a wall 62 feet 6 inches long, 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick. Ans. 169·753 yards.

3. A triangular gable is raised $17\frac{1}{2}$ feet high, on an end wall whose length is 24 feet 9 inches, the thickness being two bricks; required the reduced content. Ans. 32·0833 yards.

4. The end wall of a house is 28 feet 10 inches long, and 55 feet 8 inches high, to the eaves; 20 feet high is $2\frac{1}{2}$ bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is $1\frac{1}{2}$ brick thick; above which is a triangular gable of one brick thick, which rises 42 courses of bricks, of which every four courses make a foot. What is the whole content, in standard measure? Ans. 253·626 yards.

III.—MASONS' WORK.

To masonry belong all sorts of stone work; and the measure made use of is a foot, either superficial or solid.

Walls, columns, blocks of stone or marble, &c., are measured by the cubic foot; and pavements, slabs, chimney-pieces, &c., by the superficial or square foot.

Cubic or solid measure is used for the materials, and square measure for the workmanship.

In the solid measure, the true length, breadth, and thickness are taken, and multiplied continually together. In the superficial, there must be taken the length and breadth of every part of the projection, which is seen without the general upright face of the building.

Ex. 1.—Required the solid content of a wall 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick. Ans. $1310\frac{3}{4}$ feet.

2. What is the solid content of a wall, the length being 24 feet 3 inches, height 10 feet 9 inches, and thickness 2 feet?

Ans. 521·375 feet.

3. Required the value of a marble slab, at 8*s*. per foot; the length being 5 feet 7 inches, and the breadth 1 foot 10 inches.

Ans. 4*l*. 1*s*. $10\frac{1}{2}$*d*.

4. In a chimney-piece, suppose the length of the mantel and slab each 4 feet 6 inches; breadth of both together, 3 feet 2 inches; length of each jamb, 4 feet 4 inches; breadth of both together, 1 foot 9 inches. Required the superficial content. Ans. 21 feet 10 inches.

IV.—CARPENTERS' AND JOINERS' WORK.

To this branch belongs all the wood work of a house, such as flooring, partitioning, roofing, &c.

Large and plain articles are usually measured by the square foot or yard, &c; but enriched mouldings, and some other articles, are often estimated by running or lineal measure; and some things are rated by the piece.

In measuring of joists, take the dimensions of one joist, and multiply its contents by the number of them, considering that each end is let into the wall about $\frac{2}{3}$ of the thickness, as it ought to be.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

The measure of centering for cellars is found by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length; but in groin centering, it is usual to allow double measure, on account of the extraordinary trouble.

In roofing, the dimensions, as to length, breadth, and depth, are taken as in flooring joists, and the contents computed the same way.

In floor-boarding, take the length of the room for one dimension, and the breadth for the other, and multiply them together for the content.

For staircases, take the breadth of all the steps, by making a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step, for the whole area. By the length of a step is meant the length of the front and the returns at the two ends; and by the breadth is to be understood the girts of its two outer surfaces, or the tread and riser.

For the balustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel-post, for the one dimension; and twice the length of the baluster on the landing, with the girt of the hand-rail, for the other dimension.

For wainscoting, take the compass of the room for the one dimension, and the height from the floor to the ceiling, making the string ply close into all the mouldings, for the other.

For doors, take the height and the breadth, and multiply them together, for the area. If the door be panelled on both sides, take double its measure for the workmanship; but if one side only be panelled, take the area and its half for the workmanship.—*For the surrounding architrave*, girt it about the uttermost part for its length; and measure over it, as far as can be seen when the door is open, for the breadth.

Window-shutters, bases, &c., are measured in like manner.

In measuring of joiners' work, the string is made to ply close into all mouldings, and to every part of the work over which it passes.

Ex. 1.—Required the content of a floor 48 feet 6 inches long, and 24 feet 3 inches broad. Ans. 11 sq. $76\frac{1}{8}$ feet.

2. How many squares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning? Ans. 18·3973 squares.

3. What cost the roofing of a house at 10*s*. 6*d*. a square, the length within the walls being 52 feet 8 inches, and the breadth 30 feet 6 inches, reckoning the roof $\frac{3}{2}$ of the flat? Ans. 12*l*. 12*s*. $11\frac{3}{4}$*d*.

4. To how much, at 6*s*. per square yard, amounts the wainscoting of a room, the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compass 83 feet 8 inches; also the three window-shutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the doors and shutters, being worked on both sides, are reckoned work and half work? Ans. 36*l*. 12*s*. $2\frac{3}{8}$*d*.

V.—SLATERS' AND TILERS' WORK.

In these articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of slates at the bottom, or for how much one row of slates or tiles are laid over another.

When the roof is of a true pitch, that is, forming a right angle at top, then the breadth of the building, with its half added, is the girt over both sides, nearly.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip.

Deductions are made for chimney-shafts or window holes.

Ex. 1.—Required the content of a slated roof, the length being 45 feet 9 inches, and the whole girt 34 feet 3 inches.

Ans. 174·104 yards.

2. To how much amounts the tiling of a house, at 25*s*. 6*d*. per square, the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches; also the eaves projecting 16 inches horizontally on each side, and the roof being of a true pitch?

Ans. 25*l*. 4*s*. 4½*d*.

VI.—PLASTERERS' WORK.

Plasterers' work is of two kinds; namely, ceiling, which is plastering on laths; and rendering, which is plastering upon walls. These kinds are measured separately.

The contents are estimated either by the foot or the yard, or the square of 100 feet. Enriched mouldings, &c. are rated by running or lineal measure.

Deductions are made for chimneys, doors, windows, &c.

Ex. 1.—How many yards contains the ceiling which is 43 feet 3 inches long, and 25 feet 6 inches broad?

Ans. 122¼.

2. To how much amounts the ceiling of a room, at 10*d*. per yard, the length being 21 feet 8 inches, and the breadth 14 feet 10 inches?

Ans. 1*l*. 9*s*. 8¾*d*.

3. The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering, the former at 8*d*. and the latter at 3*d*. per yard; allowing for the door of 7 feet by 3 feet 8, and a fire-place of 5 feet square?

Ans. 1*l*. 13*s*. 3½*d*.

VII.—PAINTERS' WORK.

Painters' work is computed in square yards. Every part is measured where the colour lies; and the measuring line is forced into all the mouldings and corners.

Windows are done at so much a piece. And it is usual to allow double measure for carved mouldings, &c.

Ex. 1.—How many yards of painting contains the room which is 65 feet 6 inches in compass, and 12 feet 4 inches high?

Ans. 89·759 yards.

2. The length of a room being 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows, each 6 feet by 3 feet 2 inches?

Ans. 73·074 yards.

VIII.—GLAZIERS' WORK.

Glaziers take their dimensions either in feet, inches, and parts, or feet, tenths, and hundredths. And they compute their work in square feet.

In taking the length and breadth of a window, the cross bars between the squares are included. Also, windows of round or oval forms are considered as rectangles, and measured to their greatest length and breadth, on account of the waste in cutting the glass.

Ex. 1.—How many square feet contains the window which is 4·25 feet long, and 2·75 feet broad? Ans. 11·6875.

2. What will the glazing a triangular sky-light come to, at 10*d.* per foot; the base being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches? Ans. 1*l.* 15*s.* 1$\frac{3}{4}$*d.*

3. There is a house with three tiers of windows, three windows in each tier, their common breadth 3 feet 11 inches: now, the height of the first tier is 7 feet 10 inches; of the second, 6 feet 8 inches; of the third, 5 feet 4 inches. Required the expense of glazing, at 14*d.* per foot. Ans. 13*l.* 11*s.* 10$\frac{1}{2}$*d.*

IX.—PAVERS' WORK.

Pavers' work is done by the square yard; and the true area is taken for the content.

Ex. 1.—What cost the paving a footpath, at 3*s.* 4*d.* a yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches?

Ans. 5*l.* 7*s.* 11$\frac{1}{2}$*d.*

2. What cost the paving a court, at 3*s.* 2*d.* per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches?

Ans. 7*l.* 4*s.* 5$\frac{1}{4}$*d.*

3. What will be the expense of paving a rectangular court yard, whose length is 63 feet, and breadth 45 feet; in which there is laid a footpath of 5 feet 3 inches broad, running the whole length, with broad stones, at 3*s.* a yard; the rest being paved with pebbles at 2*s.* 6*d.* a yard? Ans. 40*l.* 5*s.* 10$\frac{1}{2}$*d.*

X.—PLUMBERS' WORK.

Plumbers' work is rated at so much a pound, or else by the hundred weight of 112 pounds.

Sheet lead, used in roofing, guttering, &c., is from 6 to 10lb. to the square foot. And a pipe of an inch bore is commonly 13 or 14lb. to the yard in length.

Ex. 1.—How much weighs the lead which is 39 feet 6 inches long, and 3 feet 3 inches broad, at 8$\frac{1}{2}$lb. to the square foot?

Ans. 1091$\frac{3}{16}$lb.

2. What cost the covering and guttering a roof with lead, at 18*s.* the cwt.; the length of the roof being 43 feet, and breadth or girt over it 32 feet; the guttering 57 feet long, and 2 feet wide; the former weighing 9·831lb., and the latter 7·373lb. the square foot?

Ans. 115*l.* 9*s.* 1$\frac{1}{4}$*d.*

XI.—TIMBER MEASURING.

PROB. I.—*To find the area, or superficial content, of a board or plank.*

Multiply the length by the mean breadth.

By the Sliding Rule.—Set 12 on *B* to the breadth in inches on *A*; then against the length in feet on *B* is the content on *A*, in feet and fractional parts.

Note.—When the board is tapering, add the breadths of the two ends together, and take half the sum for the mean breadth. Or else take the mean breadth in the middle.

Ex. 1.—What is the value of a plank, at $1\frac{1}{2}d.$ per foot, whose length is 12 feet 6 inches, and mean breadth 11 inches? Ans. 1*s.* 5*d.*

2. Required the content of a board whose length is 11 feet 2 inches, and breadth 1 foot 10 inches. Ans. 20 ft. 5 twelfths 8 inches.

What is the value of a plank which is 12 feet 9 inches long, and 1 foot 3 inches broad, at $2\frac{1}{2}d.$ a foot? Ans. 3*s.* $3\frac{1}{2}d.$

4. Required the value of 5 oaken planks at 3*d.* per foot, each of them being $17\frac{1}{2}$ feet long, and their several breadths as follows, namely, two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones each 18 inches at the broader end, and $11\frac{1}{4}$ at the narrower. Ans. 1*l.* 5*s.* $9\frac{3}{4}d.$

Prob. II.—*To find the solid content of squared or four-sided timber.*

Multiply the mean breadth by the mean thickness, and the product again by the length, for the content, nearly.

By the Sliding Rule.—As the length in feet on *C* is to 12 on *C* when the quarter girt is in inches, or to 10 on *D*, when it is in tenths of feet; so is the quarter girt on *D* to the content on *C*.

Note.—If the tree taper regularly from one end to the other, either take the mean breadth and thickness in the middle, or take the dimensions at the two ends, and half their sum will be the mean dimensions, which, multiplied as above, will give the content nearly.

If the piece do not taper regularly, but be unequally thick in some parts and small in others; take several different dimensions, add them all together, and divide the sum by the number of them, for the mean dimensions.

Ex. 1.—The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and less ends 1 foot 6 inches and 1 foot 3 inches, and the thickness at the greater and less ends 1 foot 3 inches and 1 foot. Required the solid content. Ans. 28 feet $7\frac{7}{12}$ twelfths.

2. What is the content of the piece of timber whose length is $24\frac{1}{2}$ feet, and the mean breadth and thickness each 1·04 feet? Ans. $26\frac{1}{2}$ feet.

3. Required the content of a piece of timber whose length is 20·38 feet, and its ends unequal squares, the sides of the greater being $19\frac{1}{8}$ inches, and the side of the less $9\frac{7}{8}$ inches. Ans. 29·7562 feet.

4. Required the content of the piece of timber whose length is 27·36 feet; at the greater end the breadth is 1·78, and thickness 1·23; and at the less end the breadth is 1·04, and thickness 0·91 feet. Ans. 41·278 feet.

Prob. III.—*To find the solidity of round or unsquared timber.*

Multiply the square of the quarter girt, or of $\frac{1}{4}$ of the mean circumference, by the length, for the content.

By the Sliding Rule.—As the length upon C is to 12 or 10 upon D, so is the quarter girt, in 12ths or 10ths, on D to the content on C.

Note.—When the tree is tapering, take the mean dimensions, as in former problems, either by girting it in the middle, for the mean girt, or at the two ends, and taking half the sum of the two; or by girting it in several places, then adding all the girts together, and dividing the sum by the number of them, for the mean girt. But when the tree is very irregular, divide it into several lengths, and find the contents of each part separately.

This rule, which is commonly used, gives the answer about $\frac{1}{4}$ less than the true quantity in the tree, or nearly what the quantity would be, after the tree is hewed square in the usual way; so that it seems intended to make an allowance for the squaring of the tree.

Ex. 1.—A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches, what is the content? Ans. 116·375 ft.

2. The length of a tree is 24 feet, its girt at the thicker end 14 feet, and at the smaller end 2 feet; required the content. Ans. 96 feet.

3. What is the content of a tree whose mean girt is 3·15 feet, and length 14 feet 6 inches? Ans. 8·9922 feet.

4. Required the content of a tree whose length is $17\frac{1}{4}$ feet, which girts in five different places as follows, namely, in the first place, 9·43 feet; in the second, 7·92; in the third, 6·15; in the fourth, 4·74; and in the fifth, 3·16. Ans. 42·519525.

GAUGING.

Gauging is the art of finding the contents of all kinds of vessels used by maltsters, brewers, distillers, &c.; which come under the cognizance of the Excise.

PROB. I.—*To gauge a round guile tun according to the method used by the Excise.*

Guile tuns are vessels in which wort is fermented, in order to convert it into ale or beer.

Measure two diameters at right angles to each other, in the middle of every ten inches from the bottom upwards; that is, measure two diameters at five inches from the bottom, at fifteen inches, twenty-five inches, and so on: and take half the sum of each two for a mean diameter (d). Then the content of each ten inches of the vessel will be equal to

$$\frac{\cdot 7854d^2 \times \text{depth}}{277\cdot 3} = \cdot 002833d^2 \times \text{depth, nearly};$$

and these added together will give the whole content in imperial gallons.

Ex. 1.—Suppose the perpendicular depth of a guile tun to be 30 inches, and the three mean diameters to be 42·2, 45·3, and 48·6 inches respectively; required its content in imperial gallons.

Ans. 175·46 gallons.

2. If the depth of the liquor in the preceding guile tun be 18·4 inch how many gallons does it contain? Ans. 99·26 gallons.

Prob. II.—*To gauge a back or cooler.*

Backs or coolers are vessels which receive the wort, when let out of the copper in order to be cooled. They are generally of a rectangular form, and seldom exceed nine or ten inches in depth.

Rule.—Multiply together the mean length, the mean breadth, and the mean depth; the last product, divided by 277·3, will give the quantity of wort in imperial gallons.

Note.—It has been found, by experiments, that ten gallons of hot wort will only measure nine gallons when the wort is cold. When, therefore, the wort has been gauged in a hot state, the quantity will be reduced to a cold state by multiplying the number of gallons by ·9.

Ex. 1.—The length of a cooler is 125·6 inches, the breadth 73·4 inches, and the mean depth of the wort, taken in ten different places, 4·5; how many imperial gallons are contained in the cooler?

Ans. 149·58 gallons.

2. If the mean depth of warm wort in the preceding cooler be 6·4 inches; for how many gallons must the duty be charged when cold?

Ans. 191·4624 imperial gallons.

Prob. III.—*To gauge a cistern, couch, or floor of malt.*

Multiply the mean length by the mean breadth, and divide the product by 2218·2; the quotient will be the content for one inch in depth. This, multiplied by the mean depth, will give the content in bushels.

Ex. 1.—The mean length of a cistern is 96, the mean breadth 64, and the mean depth 32 inches: required the content in imperial bushels.

Ans. 88·608 bushels.

2. The mean length of a floor of malt is 115, the mean breadth 112, and the mean depth 4·6 inches: what is its content in bushels?

Ans. 26·7076 bushels.

Prob. IV.—*To gauge a cask.*

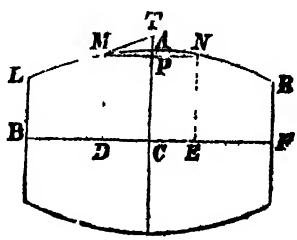
Casks have been usually divided into four forms or varieties. (1). The middle frustum of a spheroid. (2). The middle frustum of a parabolic spindle. (3). The two equal frustums of a paraboloid. (4). The two equal frustums of a cone. But as few casks contain so much as the first variety, or so little as the third or fourth; and none of them agree altogether with the general forms of casks; we shall not notice them further, but give another rule, suited to casks of all forms, which has been simplified from a rule of Dr. Hutton's, and adapted to imperial measure.

This rule is founded upon the supposition that the staves of all casks, for about a third of their length at each end, are nearly straight; and that in the middle part they are curved, so as to form the bulge or middle of the cask. And as all curves, for a short distance near the vertex, differ insensibly from a parabola, Dr. Hutton supposed the middle part of a cask to be formed by the revolution of a parabola round its axis, and the ends to be two equal frustums of a cone: and this agrees well, he observes, with the real contents of casks, as he

proved from several casks, which he actually filled with a true gallon measure after their contents were computed by this method.

(1). *To compute the content of a cask from three dimensions—viz. the length, and the bung and the head diameters.*

Rule.—Add together the square of the bung diameter, half this square, the square of the head diameter, and the product of the diameters; multiply the sum by the length, and the product by ·000812; the last product will give the content in imperial gallons, very nearly.*



* Let $BD = DE = EF = \frac{1}{3}l$; the bung diameter $= a$, the head diameter $= b$; and the diameters at M and $N = c$. By the supposition BM, ER may be considered as the frustums of a cone; and MAN as the arc of a parabola, by whose revolution round the axis, BF, the middle portion of the cask, DN, is generated.

Now it is easily shown, from the Integral Calculus, that the content of the parabolic spindle, DN, is equal to

$$\frac{\pi l}{180}(8a^2 + 4ac + 3c^2),$$

and it is proved, vol. i., page 422, that the content of the frustum of the cone BM

$$= \frac{\pi l}{36}(b^2 + bc + c^2) = \frac{\pi l}{180}(5b^2 + 5bc + 5c^2).$$

Produce LM and CA to meet in T. Then, since LM and MA have the same directions at M, MT will be a tangent to the parabola MA, and therefore $PA = \frac{1}{2}PT$. Now it is evident, from similar triangles, that

$$CT - DM : DM - BL :: 2(a - c) : c - b :: 1 : 2.$$

Hence $c = \frac{1}{5}(4a + b)$. Substituting this value in the two last equations, we get, for the sum of the parabolic spindle and the two conic frustums,

$$\text{Content of the cask} = \frac{\pi l}{180 \times 25}(488a^2 + 324ab + 313b^2)$$

$$= \frac{\pi l}{360}(39{\cdot}04a^2 + 25{\cdot}92ab + 25{\cdot}04b^2).$$

And putting $\frac{\pi}{360 \times 277{\cdot}3} = {\cdot}000031473 = M,$

$$\text{Content in gallons} = Ml(39a^2 + 26ab + 25b^2) \text{ nearly}.$$

This is Dr. Hutton's rule, adapted to imperial measure.

To simplify this expression, put

$$M(39a^2 + 26ab + 25b^2) = M'(39a^2 + 26ab + 26b^2).$$

And if $M' = M - \mu$, then

$$\frac{\mu}{M} = \frac{b^2}{39a^2 + 26ab + 26b^2}.$$

And as this is a small fraction, we may take for b, in all cases, its average value, $\frac{4}{5}a$, without any sensible error. Hence

$$\mu = \frac{M}{130}; \text{ and } M' = M - \mu = {\cdot}0000312309.$$

$$\therefore \quad \text{content} = M'l(39a^2 + 26ab + 26b^2) = 26M'l(\tfrac{3}{2}a^2 + ab + b^2),$$

where $26M' = {\cdot}0008120034 = {\cdot}000812$, very nearly.

Ex. 1.—The length of a cask is 30, the bung diameter 24, and the head diameter 18 inches; what is the content in imperial gallons?

24^2............	576	1620
Half do.........	288	30
24 × 18........	432	48600
18^2............	324	·000812
	1620	·39·4632 Imperial gallons.

2. The length of a cask is 48, the bung diameter 36, and the head diameter 27 inches; what is its content in imperial gallons?
Ans. 133·1883 gallons.

3. The bung diameter of a cask is 45, the head diameter 34·2, and the length 56·4 inches; what is its content in imperial gallons?
Ans. 263·155 gallons.

(2). *To compute the content from four dimensions—viz., the lengthh the bung and head diameters, and the diameter taken at equal distances, from the bung and head.*

Rule.—Add together the square of the bung diameter, the square of the head diameter, and the square of double the middle diameter; multiply the sum by the length of the cask; and the product, multiplied by ·000472, will give the content in imperial gallons.

This rule is founded upon the 2nd rule in prob. 16, and is extremely correct for casks of all kinds. "So that more perfect than this, both with respect to truth and expedition, nothing can be expected, or indeed wished for, in gauging."

To find the diameter at equal distances from the bung and head: place a straight graduated rule at the bung, parallel to the axis of the cask, and measure each way from the bung on the rule, a fourth part of the length of the cask. At these points measure the distances between the rule and the cask; the sum of these distances, taken from the bung diameter, will give the middle diameter.

Ex. 1.—The length of a cask is 40 inches, the bung diameter 32 the head diameter 24, and the middle diameter 28·8 inches; to find the content in imperial gallons.

2. The length of a cask is 50 inches; the head diameter is 34·3, the middle diameter 37·7, and the bung diameter 40·5; to find the content in imperial gallons.

(3). *To find the content of a cask by the diagonal or gauging rod.*

This rod has two lines of numbers graduated upon it. On one of them is a scale of inches for measuring the diagonal, a line from the bung to the intersection of the head with the stave opposite to it; and on the other is the content of a cask in imperial gallons, corresponding to the casks diagonal in inches and tenths.

The diagonal rod is much used in gauging, on account of the facility with which the contents of casks may be found by it. Its construction is founded upon the theorem, that similar solids are to one another as

the cubes of their like sides. The same result may be obtained by calculation from the following rule, which is derived entirely from experience.

Rule.—Measure the diagonal of the cask in inches and tenths, and multiply the cube of this diagonal by ·002266; the product will be the content in imperial gallons.

Ex. 1.—The diagonal of a cask is 20 inches; what is its content in imperial gallons? Ans. 18·128 gallons.

The length of a cask is 30, the bung diameter 24, and the head diameter 18 inches; required the diagonal and content.

Ans. Diagonal = 25·8 inches; content = 38·936 gallons.

PROB. V.—*To find the ullage of a cask.*

The quantity of liquor contained in a cask when it is not full, is called the *wet ullage;* and the content of the part not filled is termed the *dry ullage.*

(1). *To ullage a standing cask.*

Rule.—Add together the square of the diameter at the surface, the square of the diameter of the nearest end, and the square of double the diameter taken in the middle between the other two; multiply the sum by the length between the surface and nearest end, and this again by ·000472; the last product will be the content of the less part of the cask in imperial gallons, whether empty or filled.

This rule is the same as that given in prob. 128, page 124.

Ex.—Required the ullage of a standing cask, whose diameters at the end, middle, and surface, are 24, 27, and 29 inches; when the number of wet inches is ten.

(2). *To ullage a lying cask.*

Rule.—Divide the wet inches by the bung diameter; and find the area corresponding to this decimal in the table of circular segments (page 117): multiply this segment by the whole content of the cask, and add a fourth part to it; the sum will be the ullage nearly.

This rule is founded on the supposition that the ullage is proportional to the segment of the bung circle, cut off by the surface of the liquor. Hence

$$\text{ullage} : \text{content of cask} :: \text{segmental area} \times d^2 : \cdot7854d^2;$$

and because this proportion gives rather too much for the ullage, we have, for an approximation,

$$\text{ullage} = \text{content} \times \frac{\text{segmental area}}{\cdot 8} = \text{content} \times \text{segmental area} \times 1\tfrac{1}{4}.$$

Ex.—Required the ullage of a lying cask, whose length is 40 inches, bung diameter 32, and head diameter 24 inches; supposing 8 to be the number of wet inches.

WEIGHT OF BALLS, SHELLS, AND POWDER.

The weight and dimensions of balls and shells may easily be found from those of a given size, by considering that the weights must be

proportional to their magnitudes, and that the magnitudes of spheres are proportional to the cubes of their diameters.

PROB. I.—*To find the weight of an iron ball, from its diameter.*

An iron ball of 4 inches diameter weighs 9lb., and therefore as 64 (the cube of 4) is to 9, its weight, so is the cube of the diameter of any other ball to its weight.

Or take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, and add the two together, for the weight.

Ex. 1.—The diameter of an iron shot being 6·7 inches, required its weight. Ans. 42·294lb.

2. What is the weight of an iron ball whose diameter is 5·54 inches? Ans. 24lb. nearly.

PROB. II.—*To find the weight of a leaden ball.*

A leaden ball of 1 inch diameter weighs $\frac{3}{14}$ of a lb.; therefore $\frac{3}{14}$ of the cube of the diameter will be the weight in pounds, nearly.

Ex. 1.—Required the weight of a leaden ball of 6·6 inches diameter. Ans. 61·606lb.

2. What is the weight of a leaden ball of 5·30 inches diameter? Ans. 32lb. nearly.

PROB. III.—*To find the diameter of an iron ball.*

Multiply the weight by $7\frac{1}{9}$, and the cube root of the product will be the diameter.

Ex. 1.—Required the diameter of a 42lb. iron ball. Ans. 6·685 inches.

2. What is the diameter of a 24lb. iron ball? Ans. 5·54 inches.

PROB. IV.—*To find the diameter of a leaden ball.*

Multiply the weight by 14, and divide the product by 3; the cube root of the quotient will be the diameter.

Ex. 1.—Required the diameter of a 64lb. leaden ball. Ans. 6·684 inches.

2. What is the diameter of an 8lb. leaden ball? Ans. 3·342 inches.

PROB. V.—*To find the weight of an iron shell.*

Take $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter for the weight of the shell.

That is, from the cube of the external diameter take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

Ex. 1.—The outside diameter of an iron shell being 12·8, and the inside diameter 9·1 inches; required its weight. Ans. 188·941lb.

2. What is the weight of an iron shell whose external and internal diameters are 9·8 and 7 inches? Ans. 84·12lb.

Prob. VI.—*To find how much powder will fill a rectangular box.*

Find the content of the box in inches, by multiplying the length, breadth, and depth all together. Then divide by 30 for the pounds of powder.

This and the following rules are only approximative rules, founded upon the supposition that, at a medium, 30 cubic inches weigh a pound. Of 18 different kinds of gunpowder used in the royal laboratory, Woolwich, the specific gravities varied from ·929 to ·727. The specific gravity of French gunpowder usually lies between ·944 and ·897.

Ex. 1.—Required what quantity of powder will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches.

2. How much powder will fill a cubical box whose side is 12 inches?

Prob. VII.—*To find how much powder will fill a cylinder.*

Multiply the square of the diameter by the length, and this again by ·7854. The last product divided by 30 will give the pounds of gunpowder.

Ex. 1.—How much powder will the cylinder hold whose diameter is 10 inches, and length 20 inches?

2. How much powder can be contained in the cylinder whose diameter is 4 inches, and length 12 inches?

Prob. VIII.—*To find how much powder will fill a shell.*

Multiply the cube of the internal diameter, in inches, by ·5236, and divide this product by 30, for the lbs. of powder.

Ex. 1.—How much powder will fill the shell whose internal diameter is 9·1 inches?

2. How much powder will fill a shell whose internal diameter is 7 inches?

Prob. IX.—*To find the size of a cubical box, to contain a given weight of powder.*

Multiply the weight in pounds by 30, and the cube root of the product will be the side of the box, in inches.

Ex. 1.—Required the side of a cubical box, to hold 50lbs. of gunpowder.

2. Required the side of a cubical box, to hold 400lbs. of gunpowder.

Prob. X.—*To find what length of a cylinder will be filled by a given weight of gunpowder.*

Multiply the weight in pounds by 30, and divide the product by the area of the base, in inches; the quotient will be the length in inches.

Ex. 1.—What length of a 36-pounder gun, of $6\frac{3}{8}$ inches diameter, will be filled with 12lbs. of gunpowder?

2. What length of a cylinder, of 8 inches diameter, may be filled with 20lbs. of gunpowder?

PILING OF BALLS AND SHELLS.

PROB. I.—*To find the number of balls in a triangular pile.*

Multiply continually together the number of balls in one side of the bottom row, and that number increased by one, also the same number increased by two; then $\frac{1}{6}$ of the last product will be the answer: that is,

$$\text{the number} = \tfrac{1}{6}n(n+1)(n+2);$$

where n is the number in each side of the bottom row.

Ex. 1.—Required the number of balls in a triangular pile, each side of the base containing 30 balls. Ans. 4960.

2. How many balls are in the triangular pile, each side of the base containing 20? Ans. 1540.

PROB. II.—*To find the number of balls in a square pile.*

Multiply continually together the number in one side of the bottom course, that number increased by one, and double the same number increased by one; then $\frac{1}{6}$ of the last product will be the answer: that is,

$$\text{the number} = \tfrac{1}{6}n(n+1)(2n+1).$$

Ex. 1.—How many balls are in a square pile of 30 rows?

2. How many balls are in a square pile of 20 rows?

PROB. III—*To find the number of balls in a rectangular pile.*

From three times the number in the length of the base row, subtract one less than the breadth of the same, multiply the remainder by the same breadth, and the product by one more than the same, and divide by 6 for the answer: that is,

$$\text{the number} = \tfrac{1}{6}n(n+1)\,[3l-(n-1)];$$

where l is the length, and n the breadth of the lowest course.

Note.—In triangular and square piles, the number of horizontal rows or courses is always equal to the number of balls in one side of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom row. Also, the number in the top row or edge is one more than the difference between the length and breadth of the bottom row.

Ex. 1.—Required the number of balls in a rectangular pile, the length and breadth of the base row being 46 and 15. Ans. 4960.

2. How many shot are in a rectangular complete pile, the length of the bottom course being 59, and its breadth 20? Ans. 11060.

PROB. IV.—*To find the number of balls in an incomplete pile.*

From the number in the whole pile, considered as complete, subtract the number in the upper pile which is wanting at the top, and the remainder will be the number in the frustum or incomplete pile.

Ex. 1.—To find the number of shot in the incomplete triangular pile, one side of the bottom course being 40, and the top course 20. Ans. 10150.

2. How many shot are in the incomplete rectangular pile of 12 courses, the length and breadth of the base being 40 and 20? Ans. 6146.

MISCELLANEOUS QUESTIONS IN MENSURATION.

1. The length of a rectangular field is 13 chains 64 links, and its breadth 11 chains 9 links; required the content, in acres.
Ans. 15 acres 10 perches.

2. The sides of a triangular field are 174, 161, and 145 yards; required the content, in acres. Ans. 2·2527 acres.

3. If the side of a rhombus is $29\frac{1}{2}$ feet, and one of its angles 62°; what is its content in square yards? Ans. 85·38 yards.

4. The top and bottom of a ditch are horizontal, and its breadth at the top is 72 feet, and at the bottom $38\frac{3}{8}$; also, the sloping sides are $26\frac{5}{8}$ and 20 feet. Required the area of the transverse sections.
Ans. $885\frac{5}{8}$ feet.

5. A ceiling contains 114 yards 6 feet of plastering, and the room is 28 feet broad; what is the length of it? Ans. $36\frac{4}{7}$ feet.

6. A wooden cistern cost 3*s.* 2*d.* painting within, at 6*d.* per yard; the length of it was 102 inches and the depth 21 inches, what was the width? Ans. $27\frac{1}{4}$ inches.

7. If my court-yard be 47 feet 9 inches square, and I have laid a footpath with Purbeck stone, of 4 feet wide, along one side of it; what will paving the rest with flint stones come to, at 6*d.* per square yard?
Ans. 5*l.* 16*s.* $0\frac{1}{2}$*d.*

8. The paving of a triangular court, at 18*d.* per foot, came to 100*l.*; the longest of the three sides was 88 feet; required the sum of the other two equal sides. Ans. 106 85 feet.

9. The perambulator, or surveying wheel, is so contrived as to turn just twice in the length of 1 pole, or $16\frac{1}{2}$ feet; required the diameter.
Ans. 2·626 feet.

10. What is the side of that equilateral triangle whose area cost as much paving, at 8*d.* a foot, as the palisading the three sides did at a guinea a yard? Ans. 72·746 feet.

11. In the trapezium $ABCD$ are given, $AB = 13$, $BC = 31{\cdot}2$, $CD = 24$, and $DA = 18$, also B, a right angle; required the area.
Ans. 410·122.

12. A roof which is 24 feet 8 inches by 14 feet 6 inches is to be covered with lead, at 8*d.* per square foot; what will it come to at 18*s.* per cwt.? Ans. 22*l.* 19*s.* $10\frac{1}{4}$*d.*

13. Given two sides of an obtuse-angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land. Ans. 58·876, or 23·099.

14. The end wall of a house is 24 feet 6 inches in breadth, and 40 feet to the eaves; one-third of which is 2 bricks thick, one-third more is $1\frac{1}{2}$ brick thick, and the rest 1 brick thick. The triangular gable rises 38 courses of bricks, 4 of which make a foot in depth, and this is half a brick thick; what will this piece of work come to, at 5*l.* 10*s.* per statute rod? Ans. 20*l.* 11*s* $7\frac{1}{2}$*d.*

15. How many bricks will it take to build a wall 10 feet high and 500 feet long, of a brick and a half thick; reckoning the brick 9 inches long, and 4 courses to the foot in height? Ans. 80000.

16. If from a right-angled triangle, whose base is 12, and perpendicular 16 feet, a line be drawn parallel to the perpendicular, cutting off a triangle whose area is 24 feet square: required the sides of this triangle. Ans. 6, 8, and 10.

17. If a round pillar 7 inches over, have four feet of stone in it; of what diameter is the column, of equal length, that contains 10 times as much? Ans. 22·136 inches.

18. A circular fish-pond is to be made in a garden, that shall take up just half an acre; what must be the length of the cord that strikes the circle? Ans. 27¾ yards.

19. When a roof is of a true pitch, or making a right angle at the ridge, the rafters are nearly ¾ of the breadth of the building; now supposing the eaves-boards to project 10 inches on a side horizontally, what will the new ripping a house cost, that measures 32 feet 9 inches long, by 22 feet 9 inches broad on the flat, at 15*s.* per square?

Ans. 8*l.* 19*s.* 11*d.*

20. If the side of an equilateral triangle be 10, what are the radii of the inscribed and circumscribed circles?

Ans. 2·8868 and 5·7736.

21. If the radius of a circle be 10, what are the sides of a regular inscribed pentagon, hexagon, octagon, and decagon?

Ans. 11·756; 10; 7·654; 6·18.

22. How many degrees are contained in that arc of a circle whose length is equal to the radius? Ans. $57^{\circ}{\cdot}295779$.

23. If the centre of a circle whose diameter is 20, is in the circumference of another circle whose diameter is 40; what are the areas of the three included spaces?

Ans. 173·852; 140·308; and 1116·332.

24. What quantity of canvass is necessary for a conical tent whose perpendicular height is 8 feet, and the diameter at the bottom 18 feet?

Ans. 210¼ square feet.

25. Suppose a sack, when laid flat, to be 2 feet broad and 5 feet long; how many imperial gallons will it contain, if it has a circular bottom, and 9 inches is left for tying the top? Ans. 33·7247.

26. Supposing that the meridian of the earth be considered as a circle, and that its mean radius is 20,888761 feet; required the length of a French metre in English inches, and a kilométre in furlongs: the arc of the meridian, from the pole to the equator, being divided into ten million métres.

27. Required the value of a French *are* in English inches, and of a French litre in imperial pints: an are being equal to 100 square métres, and a litre equal to a cubic decimétre, or a cube whose side is a tenth of a métre.

28. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semicircle, is equal to 100; what is the diameter of the semicircle? Ans. 26 32148.

29 The base of the great pyramid of Egypt is a square, each side of which is 763·4 feet; and the length of each of the edges is 714·47 feet. Required the area of the ground on which it stands, the convex surface of the pyramid, and the number of cubic yards of stone contained in it.

30. A carpenter is to put an oaken curb to a round well, at 8*d.* per foot square: the breadth of the curb is to be $7\frac{1}{2}$ inches, and the diameter within $3\frac{1}{2}$ feet; what will be the expense? Ans. 5*s.* $2\frac{1}{4}$*d.*

31. A gentleman has a garden 100 feet long and 80 feet broad, and a gravel walk is to be made of an equal width half round it; what must the breadth of the walk be, to take up just half the ground?

Ans. 25·968 feet.

32. The top of a maypole being broken off by a blast of wind, struck the ground at 10 feet distance from the foot of the pole; what was the height of the whole maypole, supposing the length of the broken piece to be 26 feet? Ans. 50 feet.

33. Seven men bought a grinding-stone of 60 inches diameter, each paying $\frac{1}{7}$ part of the expense; what part of the diameter did the first and last grind down? Ans. The 1st, 4·4508; 7th, 22·6778 inches.

34. If 20 feet of iron railing weigh half a ton, when the bars are an inch and a quarter square; what will 50 feet come to at $3\frac{1}{2}$*d.* per lb., the bars being $\frac{5}{8}$ of an inch square? Ans. 20*l.* 0*s.* 2*d.*

35. The value of diamonds is proportional to the square of their weight. What, then, is the value of the famous diamond called *Koh-i-noor*, or the *mountain of light;* its weight, according to Sir A. Burnes, being $3\frac{1}{2}$ rupees; also the weight of a sicca rupee being 7 dwt. 12 gr., and the value of a wrought diamond of one carat ($3\frac{1}{10}$ grains) being 8*l.*? Ans. 330405*l.* 16*s.*

36. What will the painting of a conical spire come to, at 8*d.* per yard; supposing the height to be 118 feet, and the circumference of the base 64 feet? Ans. 14*l.* 0*s.* $8\frac{2}{3}$*d.*

37. Suppose the ball on the top of St. Paul's church is 6 feet in diameter, what did the gilding of it cost, at $3\frac{1}{2}$*d.* per square inch?

Ans. 237*l.* 10*s.* 1*d.*

38. What will the frustum of a marble cone come to, at 12s. per solid foot; the diameter of the greater end being 4 feet, that of the less end $1\frac{1}{2}$, and the length of the slant side 8 feet? Ans. 30*l.* 1*s.* $10\frac{1}{4}$*d.*

39. To divide a cone into three equal parts by sections parallel to the base, and to find the altitudes of the three parts, the height of the whole cone being 20 inches.

Ans. The upper part 13·867; the middle part, 3·604; the lower part, 2·528.

40. A gentleman has a bowling-green, 300 feet long and 200 feet broad, which he would raise 1 foot higher, by means of the earth to be dug out of a ditch that goes round it; to what depth must the ditch be dug, supposing its breadth to be everywhere 8 feet?

Ans. 7·2674 feet.

41. How high above the earth must a person be raised, that he may see one-tenth of its surface?

Ans. To the height of one-fourth of the earth's radius.

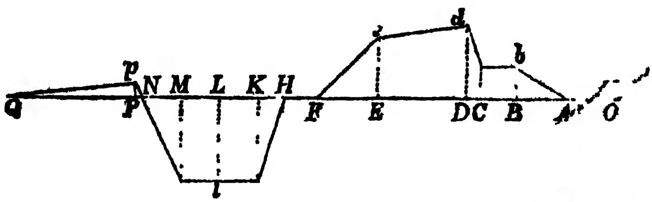
42. A cubic foot of brass is to be drawn into wire of $\frac{1}{40}$ of an inch in diameter; what will the length of the wire be, allowing no loss in the metal? Ans. 97784·797 yards, or 55 miles 984·797 yards.

43. Let AdF be the profile or transverse section of the parapet of a fort, HlN that of the ditch, and NpQ that of the glacis: Also, suppose $AB = 5$, $BC = 4$, $CD = 1$, $DE = 9$, $EF = 6$,

$$FH = 3, \quad HK = MN = 4, \quad NP = 1\tfrac{1}{2}, \quad PQ = 15:$$

and also $Bb = Cc = 3$, $Dc = 7\tfrac{1}{2}$, $Ee = 6$, $Ll = 8$, $Pp = 1\tfrac{1}{2}$:

to find the breadth of the ditch at top and bottom, when the earth thrown out, is sufficient to make the parapet and glacis, supposing that, after being excavated, its bulk is increased in the proportion of 10 to 9.



Ans. 17·036, and 9·036.

44. Supposing the diameter of an iron 9lb. ball to be 4 inches, it is required to find the diameters of the several balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 32, 36, and 42lb., and the calibre of their guns, allowing $\frac{1}{50}$ of the calibre, or $\frac{1}{49}$ of the ball's diameter, for windage. Answer.

Weight of Ball.	Diameter of Ball.	Calibre of Gun.	Weight of Ball.	Diameter of Ball.	Calibre of Gun.
1	1·9230	1·9622	12	4·4026	4·4924
2	2·4228	2·4723	18	5·0397	5·1425
3	2·7734	2·8301	24	5·5469	5·6601
4	3·0526	3·1149	32	6·1051	6·2297
6	3·4943	3·5656	36	6·3496	6·4792
9	4·0000	4·0816	42	6·6844	6·8208

45. Supposing the windage of all mortars to be $\frac{1}{80}$ of the calibre, and the diameter of the hollow part of the shell to be $\frac{7}{10}$ of the calibre of the mortar; it is required to determine the diameter and weight of the shell, and the quantity or weight of powder requisite to fill it, for each of the several sorts of mortars, namely, the 13, 10, 8, 5·8, and 4·6 inch mortars. Answer.

Calibre of Mortar.	Diameter of Shell.	Weight of Shell empty.	Weight of Powder.	Weight of Shell filled.
4·6	4·523	8·320	0·583	8·903
5·8	5·703	16·677	1·168	17·845
8	7·867	43·764	3·065	46·829
10	9·833	85·476	5·986	91·462
13	12·783	187·791	13·151	200·942

MECHANICS.

DEFINITIONS.

1. MECHANICS is the science which treats of the laws of rest and motion of bodies, whether solid or fluid.

This science is usually divided into four parts:—

I. *Statics*, which considers the laws of equilibrium of solid bodies.
II. *Dynamics*, which treats of the laws of motion of solid bodies.
III. *Hydrostatics*, which relates to the laws of equilibrium of fluid bodies; and
IV. *Hydrodynamics*,* which treats of the laws of motion of fluid bodies.

2. *Motion* is the passage from place to place; *rest* is a permanency in the same place.

3. By *matter* we understand any substance that affects our senses. *Bodies* are certain portions of matter limited in every direction. The *mass* of a body is the quantity of matter of which it is composed. A *material* or *elementary particle* is a body indefinitely small in every direction. The space occupied by any body is called its *volume*, or *solid content*.

Density is the proportional quantity of matter contained in a body of a given magnitude; and it is said to be uniform when equal quantities of matter are always contained in equal magnitudes.

4. By *force* we understand any cause which tends to impress or destroy motion. As we have no means of estimating force except by its effects, it is differently measured in statics and dynamics.

In statics, force is measured by the *pressure* which it causes a body, when at rest, to exert against another with which it is in contact. Thus, when a heavy body is supported by the hand, it exerts a pressure downwards on the hand, and is sustained by the pressure of the hand upwards. The former pressure is called the *action*, and the latter the *reaction*; and they are evidently equal to each other. The pressures exerted by strings pulled by any forces are called *tensions*.

In dynamics, force is measured by the velocity uniformly generated in a given time. Thus, if the heavy body be not supported by the hand, it will fall towards the earth; and the velocity which it acquires in a given time is taken as the measure of the force of attraction.

5. *Gravity* is the tendency which all bodies have to the centre of the earth. We are convinced of the existence of this tendency by observing

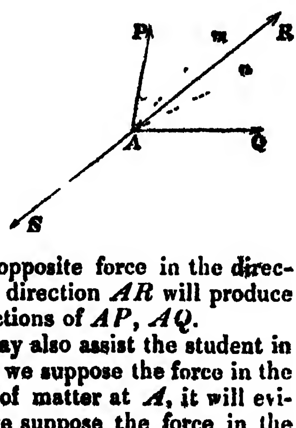
* Mechanics, from *μηχανή*, a machine; statics, from *στατική*, the science of weights; dynamics, from *δύναμις*, force; hydrostatics, from *ὕδωρ*, water, &c.

that whenever a body is sustained, its pressure is exerted in a direction perpendicular to the horizon; and that when every impediment is removed, it always descends in that direction.

6. The *weight* of a body is its tendency to the earth, compared with the like tendency of some other body which is considered a standard. Thus, if a body which can bend a steel spring into a certain position be called one pound, any other body which, by its gravity, will produce the same effect, is also called a pound; and these two together two pounds; and so on.

PART I.—STATICS.

CHAP. I.—COMPOSITION AND RESOLUTION OF FORCES.



7. When two forces act at any point, or upon the same particle of matter, they will produce the same effect as a single force acting in some intermediate direction. Thus, if we suppose two forces to act at the point *A* in the directions of *AP*, *AQ*, it is evident that a third force may be found of such magnitude, and in such a direction, *AS*, as to balance the two forces in the directions *AP*, *AQ*. But the force *AS* will counteract an equal and opposite force in the direction *AR*, and consequently the force in the direction *AR* will produce the same effect as the two forces in the directions of *AP*, *AQ*.

The following dynamical considerations may also assist the student in forming a clear idea of this proposition:—If we suppose the force in the direction *AP* to act alone upon a particle of matter at *A*, it will evidently move in the direction *AP*; and if we suppose the force in the direction *AQ* to act alone upon this particle of matter, it will move in the direction *AQ*: and when both of these forces act together, it will neither move in the direction *AP* nor *AQ*, but will move in some intermediate direction such as *AR*. Now we may conceive this motion in the direction *AR* to have been produced by a *single* force; which single force, therefore, produces the same effect as the two forces in the directions *AP* and *AQ* acting together.

8. The force in the direction *AR* is called the *resultant* of the two forces in the directions *AP* and *AQ*; and the forces in the directions *AP* and *AQ*, are called the *component* parts of the force *AR*.

9. Axiom. I.—*When two forces act in the same direction, the resultant is equal to the sum of the forces, and acts in the same direction; and if*

two forces act in opposite directions, the resultant is equal to the difference of the forces, and acts in the direction of the greater.

Thus, if a weight of 3 pounds and another of 5 pounds be suspended from the same point, the pressure on this point will be equivalent to a weight of 8 pounds. And if a point be acted upon by a force of 8 pounds in one direction, and 5 pounds in the opposite, it will be drawn with a force of 3 pounds in the first direction. For the force of 8 may be considered as composed of 5 and 3, of which the 5 will be destroyed by the opposite force, and the 3 will remain effective.

10. AXIOM. II.—*The effect of any force is the same at every point of the line in which it acts.*

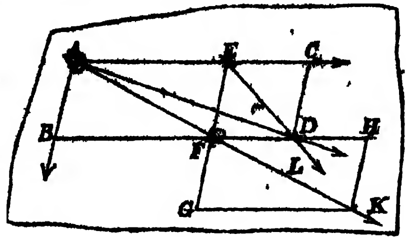
This is a matter of universal experience. Thus, the weight of a body is the same, whatever be the length of the string by which it is suspended; and the *tension* of the string, or the force by which it is pulled, is manifestly everywhere the same, if the weight of the string is neglected.

11. AXIOM III.—*If two equal forces act upon the same point, the force which is equivalent to the two will bisect the angle formed by the direction of these forces.*

(1). The resultant will be in the same plane with the directions of these forces. For whatever reason may be given that the resultant be without the plane PAQ, on one side of it, the same reason may be assigned that it shall be similarly situated on the other side of the plane. Hence it is evident that the resultant can be on neither side of PAQ, but in the plane itself.

(2). In the same manner it may be shown that the resultant will bisect the angle PAQ. For if AR bisect the angle PAQ, and Am be the resultant of two equal forces AP, AQ; there will be another line, An, similarly situated on the other side of AR; and any reason which can be given to prove that Am is the resultant, will in like manner prove that An is the resultant; from whence it follows that there are two resultants; which is manifestly impossible. Hence the resultant can be only in the direction AR.

12. PROP. I.—*If the resultant of two forces,* p, q, *acting at a point, be in the direction of the diagonal of the parallelogram, of which the sides represent the magnitude and direction of the component forces; and if this be true with respect to two other forces,* p, r, *then will this proposition also be true for the forces* p *and* q + r, *the two forces in each case being inclined to each other at the same angle.*



Let A be the point on which the forces p and $q+r$ act; AB, AC, their direction: take AB, AE, EC, in the proportion of p, q, r; and complete the parallelograms $ABDC$, $ABFE$. The force AC is equal to the two forces AE, EC, acting in the same direction (art. 9); therefore the two forces AB,

AC are equivalent to the three forces AB, AE, EC, acting at the point A. But the force EC, acting at A, has the same effect as if it acted at E in the direction EC (art. 10). Also, the two forces AB, AE are, by hypothesis, equivalent to a single force, which we will call R, acting in the direction of the diagonal AFK; and this force has the same effect as if it acted at F in the direction FK. Hence the two forces AB, AC acting at A are equivalent to r acting at E in the direction EC, and R acting at F in the direction FK.

Again, if we take two forces, FG, FH (equal to AB, AE), acting at F in the directions EF, BF, produced, these will manifestly have a resultant R equal to the resultant of AB, AE, because they are equal to these forces, and are inclined to each other at the same angle, and therefore the force R acting at F in the direction FK, is equivalent to the two forces p, q acting at F in the directions FG, FH; or is equivalent to the force p acting at E in the direction EF, and q acting at D in the direction DH. Hence it appears that the two forces AB, AC acting at A, are equivalent to the forces p, r acting at E in the directions EF, EC, and the force q acting at D in the direction DH.

But the two forces EF, EC have, by hypothesis, a resultant in the direction EDL, which will have the same effect, if it act at D in the direction DL; therefore the three forces EC, EF, FH, are equivalent to two forces, both of which pass through D, and therefore the resultant of these three forces will also pass through D. Thus the two forces AB, AC, acting at A, are equivalent to a single force which passes through D; but the resultant of AB, AC, also passes through A, and therefore AD must be the direction of this resultant.

13. Prop. II.—*If two forces acting upon the same point be to each other as any two integral numbers* m *and* n, *the force which is equivalent to the two will be in the* direction *of the diagonal of the parallelogram described on the two sides which represent the magnitude and direction of these forces.*

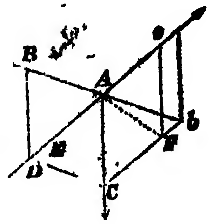
When the forces are equal, it appears, from the third axiom, that the resultant bisects the angle formed by the directions of these two forces, and therefore, in this case, it is evidently in the direction of the diagonal of the parallelogram described on these two lines. Hence, if $p=q=r$ in the last article, the proposition is true for p and q, and also for p and r, and therefore it was proved by that article to be true for p and $q+r$, or for p and $2p$. Again, since the proposition is true for p and $2p$, and also for p and p, it is true for p and $2p+p$, or $3p$. In like manner, it is true for p and $3p+p$, or $4p$; and also for p and $4p+p$, or $5p$; and generally for p and mp, m being any whole number.

Again, since the proposition is true for mp and p, it is also true for mp and $p+p$, or $2p$. Also, since it is true for mp and $2p$, and also for mp and p, it is true for mp and $2p+p$, or $3p$; and generally it is true for mp and np, m and n being any integral numbers.

14. *Cor.*—If the numbers m and n are not *commensurable*, still it is evident, from the scholium (vol. i. p. 168), that the resultant of two forces P and Q will be in the direction of the diagonal of the parallelogram constructed on the lines representing these forces.*

* Let AB, AC represent the two forces P, Q; then, if the resultant do not act along the diagonal AD, let it act in the direction AO. Divide AB into a number of equal

15. Prop. III.—*If any two forces act at the same point, the force which is equivalent to the two is expressed in magnitude by the diagonal of the parallelogram, of which the sides represent the magnitude and direction of the component forces.*

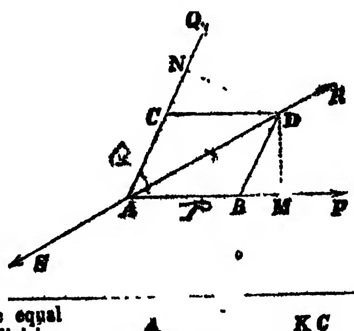


Let AB, AC represent the component forces; complete the parallelogram $ABCD$. From the last article it appears that the two forces AB, AC are equivalent to a single force which acts in the direction AD. Let AE represent this force, and produce it to e, so that $Ae = AE$. Now, since the two forces AB, AC are equivalent to AE, and the force AE is kept at rest by an equal force Ae, acting in the opposite direction, the two forces AB, AC will be kept at rest by the force Ae; and the three forces AB, AC, Ae, will keep each other in equilibrium. Complete the parallelogram $ACFe$ and produce BA, CF to meet in b. The figure $ADCb$ is manifestly a parallelogram, therefore $Ab = CD = AB$. Hence the two forces AC, Ae, which are kept at rest by the force AB, are equivalent to the equal force Ab acting in the opposite direction. But, by the last proposition, the resultant of the two forces AC, Ae is in the direction, AF, of the parallelogram $ACFe$. Therefore the forces AC, Ae are equivalent to the force Ab, and also to a force acting in the direction AF, which is manifestly impossible, unless the lines AF and Ab coincide, or unless $CF = Cb$, or $AE = AD$. Hence the resultant is represented in magnitude by AD, the diagonal of the parallelogram $ABDC$.

16. Cor 1.—Let $AB = P$, $AC = Q$, and the resultant $AD = R$; also, let the angle $BAC = \alpha$, then, (Trig. art. 109),

$$AD^2 = AC^2 + CD^2 - 2AC \times CD \cos ACD$$
$$= AC^2 + AB^2 + 2AC \times AB \cos BAC.$$

$$\therefore R^2 = P^2 + Q^2 + 2PQ \cos \alpha.$$



Also, to determine the angles BAD, CAD, we have

$$R : Q :: \sin \alpha : \sin BAD,$$
$$R : P :: \sin \alpha : \sin CAD.$$

17, Cor. 2—If from any point, D, in the direction of the resultant, DM, DN be drawn perpendicular to the directions of the forces AB, AC, then

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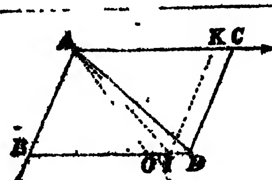


Parts, each less than OD, and apply these equal parts to the line BD; one of the points of division will fall between O and D. Let I be the point; draw IK parallel to AB. Now, since the two forces AB, AK are commensurable, the resultant will be in the direction AI: call this force S. But the two forces AB, AC are equivalent to AB, AK, and KC; that is, a force S acting in the direction AI, and the force KC acting in the direction AC. And the resultant of these two last forces S, KC will evidently be within the angle CAI, and cannot therefore be in the direction AO. Hence it is impossible that the resultant of the two forces AB, AC can be in any other direction than the diagonal AD

$$P : Q :: \sin CAD : \sin BAD$$
$$:: DN : DM;$$

that is, the forces P, Q are inversely proportional to the perpendiculars on their directions from any point in the direction of the resultant.

18. *Cor.* 3,—If three forces, P, Q, S, acting on a point keep it at rest, each of these forces is proportional to the sine of the angle made by the other two. Let the forces P and Q be equivalent to the force R, then, since P and Q balance the force S, the force R will also balance S, and therefore R is equal and opposite to S. But, by the last corollary,

$$R : P :: \sin \alpha : \sin QAR \text{ or } \sin QAS,$$
$$R : Q :: \sin \alpha : \sin PAR \text{ or } \sin PAS;$$
$$\therefore \quad S : P : Q :: \sin \alpha : \sin QAS : \sin PAS.$$

19. *Cor.* 4.—If the three sides of any triangle be parallel to three forces which act on a point and keep it at rest, these forces will be proportional to the sides of the triangle. For this triangle will manifestly be similar to the triangle ABD, and the sides of this triangle are proportional to the three forces P, Q, S, which keep a particle at rest.

20. *Scholium.*—The proposition which we have just demonstrated is generally known by the name of the *parallelogram of forces*, and is the foundation of the whole doctrine of equilibrium. Many proofs of this important theorem have been given by some of the most eminent philosophers, suc has D. Bernoulli, Dalembert, &c., and in later times by Laplace and Poisson; but they are generally of too abstruse a nature to be introduced in a work like the present. Among others of a more elementary nature, none appear to be so direct, or founded on such self-evident principles, as that which we have introduced above, and which was first given by Duchayla in the correspondance de l'ecole polytechnique.

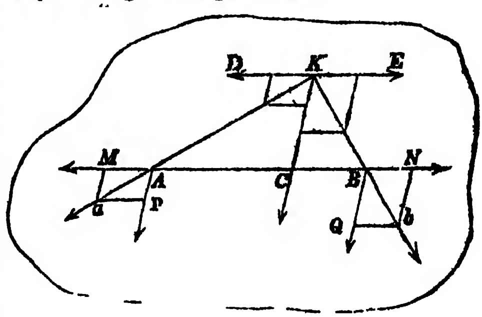
There is one proof of this theorem, however, frequently given by English writers on mechanics, which not improbably led to its discovery, and which we think it right to notice in this place. Although the demonstration is founded on the principles of dynamics, which are foreign to the subject of equilibrium; and although the axioms on which it rests may themselves stand in need of demonstration, yet from its extreme simplicity we consider it well adapted to those who are only commencing the study of mechanics.

Let AB, AC (see the last figure) represent two forces acting on a point A, then these forces will be proportional to the velocities communicated to the particle A in their directions, and consequently to the spaces which it would uniformly describe in a given time. Complete the parallelogram AD, then the motion in the direction AC can neither accelerate nor retard the approach of the body to the line BD, which is parallel to AC; hence the body will arrive at BD in the same time that it would have done had no motion been communicated to it in the direction AC. In like manner, the motion in the direction AB can neither make the body approach to nor recede from CD; therefore, in consequence of the motion in the direction AC, it will arrive in the same time that it would have done had no motion been communicated in the direction AB. Hence it follows that, in consequence of the two

motions, the body will be found both in BD and CD at the end of this time, and will therefore be found in D, the point of their intersection. And since AB, AC, AD, represent the spaces uniformly described by the particle A in the same time, they are proportional to the forces acting in those directions; that is, the forces AB, AC, acting at the same time, produce a force which is represented in magnitude and direction by AD.

RESULTANT OF TWO PARALLEL FORCES.

21. PROP. IV.—*To find the resultant of two parallel forces* AP, BQ, *acting in the same direction upon a rigid straight line* AB.



Let AP, BQ represent the two forces P, Q; and suppose that two equal and opposite forces S, T, represented by AM, BN, be applied to the points A, B, in the direction of AB, produced. Then it is evident, since the two additional forces S, T counteract each other, that the resultant of the two forces P, Q is the same as that of the four forces P, Q, S, T. If we now complete the parallelograms $APaM$, $BQbN$, the resultant of the two forces AM, AP is Aa, and the resultant of the two forces BQ, BN is Bb; and consequently the two forces P, Q are equivalent to the two forces Aa, Bb. Produce Aa, Bb until they meet in K, and through K draw DE parallel to AB, and KC parallel to AP or BQ. Now the force Aa will have the same effect on the system as an equal force at K acting in the direction KA (art. 10); and this force may evidently be resolved into two others: one in the direction KD, parallel and equal to AM, and the other in the direction KC, parallel and equal to AP. In like manner the force Bb may be supposed to be transferred to K, and to be resolved into two forces, one in the direction KE, parallel and equal to BN; and the other in the direction KC, parallel and equal to BQ. Hence the four original forces P, Q, S, T are equivalent to the same four forces all acting at the point K; of which the two forces S, T, being equal and opposite, will counteract each other, and therefore will have no effect on the system; and the two others, P, Q, acting in the same direction KC, will produce a resultant equal to their sum $P+Q$. Also,

$$P : S :: AP : AM \text{ or } Pa :: KC : AC,$$

$$T : Q :: BN \text{ or } Qb : BQ :: BC : KC;$$

$$\therefore \textit{ ex æquali, } \quad P : Q :: BC : AC.$$

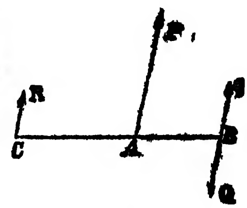
22. COR.—If R be the resultant of the two forces P and Q, then $R = P + Q$; and since $P : P + Q :: BC : AB$, we have

$$P : Q : R :: BC : AC : AB,$$

that is, each of the three forces, P, Q, R, is proportional to the part of the line AB, intercepted between the directions of the other two forces.

23. PROP. V.—*To find the resultant of two parallel forces* P, Q, *acting in opposite directions upon a rigid body.*

Suppose P to be the greater of the two forces P, Q. Let P be resolved into two parallel forces acting in the same direction, the one, q, equal and opposite to Q, and the other a force, R, acting at an unknown point C. Then, since P is the resultant of q and R, we have, by the last proposition,



$P = q + R$; and $R : q :: AB : AC$.

Also, since P is equivalent to q and R, the two forces P, Q are equivalent to the three forces Q, q, R. But the forces q, Q counteract each other, consequently the two forces P, Q are equivalent to R.

24. *Cor.*—Because $R : Q :: AB : AC$, we have

$$AC = \frac{Q}{R} AB = \frac{Q}{P - Q} AB.$$

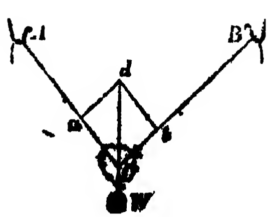
When, $P = Q$, the resultant $R = 0$, and the distance AC becomes infinitely great. In this case no single force could produce the effect of the two forces P, Q. Their tendency is to turn the system round in the plane in which they are, without producing any motion except a rotatory one.

The following problems will serve to illustrate the preceding propositions.

PROBLEMS.

25. PROBLEM I.—A, B *are two fixed points, and* W *is a weight suspended at the knot* C *in the string* ACB ; *it is required to find the forces exerted by the strings* CA, CB.

The point C is in this case kept at rest by three forces; the weight W acting by the string CW, and the tensions or forces of the strings CA, CB acting in the directions of the strings. From any point, d, in the line WC, produced, draw db, da parallel to CA, CB. In order to support the weight W, the resultant of the forces of the strings CA, CB must be in the direction Cd, and must be equal to the weight W. The forces will therefore be as Ca, Cb, and their resultant as Cd (art. 15). Hence, if P, Q represent the forces of the strings CA, CB, we have

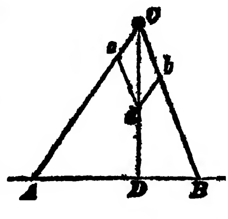


$P : Q : W :: Ca : Cb : Cd :: \sin dCB : \sin dCA : \sin ACB$,

whence P and Q are known.

Cor.—The forces of the strings measure their *tensions*, and these are measured by the pressure exerted on the immovable points A, B.

26. PROBLEM II.—*A given weight* W *is supported by two props* AC, BC, *situated upon a horizontal plane* AB; *to find the pressure upon each.*



It is evident that the pressures must be in the direction of the props, for a force which is perpendicular to either prop will only tend to destroy the equilibrium, unless it be counteracted. In the vertical line CD, take Cd to represent the weight W, and draw db, da parallel to CA, CB; then we shall have the weight W, and the re-actions of the two props CA, CB, proportional to Cd, aC, bC. Hence

$$P : Q : W :: Ca : Cb : Cd :: \sin DCB : \sin DCA : \sin ACB$$
$$:: \cos B : \cos A : \sin C.$$

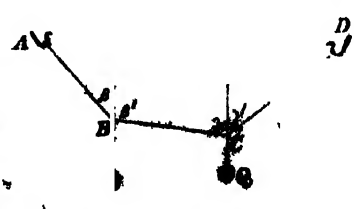
27. *Cor.* 1.—This result might have been obtained by resolving the pressures into two forces, one horizontal and the other vertical. These forces, from what we have proved, will be $P \cos A$, $Q \cos B$, in a horizontal direction, and $P \sin A$, $Q \sin B$, in a vertical direction; and since the horizontal parts counteract each other, and the vertical parts are evidently equivalent to W, we have

$$P \cos A = Q \cos B; \quad P \sin A + Q \sin B = W,$$

from whence the same result may be obtained as before.

28. *Cor.* 2.—The pressures upon the plane at A and B, are evidently equal to the pressures exerted on the upper extremities of the props; and if these be resolved as before, we shall have the vertical parts equal to $P \sin A$, $Q \sin B$, which are counteracted by the re-action of the plane. Also, the horizontal parts, equal to $P \cos A$, $Q \cos B$, thrust the props outwards, and will make them slide in opposite directions, unless they are counteracted either by immovable obstacles or the friction of the plane.

29. PROBLEM III.—*A string* ABCD, *of which the extremities* A, D, *are fixed, is kept in a given position by the weights* P, Q, *suspended at the knots* B, C.



Let the sides of the polygon make, with the vertical lines at the points B, C, the angles β, β'; γ, γ'. Let T be the force or *tension* of the string AB, which will manifestly be the same both at A and B, acting in opposite directions; let T' be the tension of BC, which will be the same both at B and C, and T'' the tension of CD. The point B is kept at rest by three forces, the weight P, the tension T in the direction BA, and the tension T' in the direction BC, and therefore (art. 17)

$$P : T' :: \sin ABC : \sin ABP :: \sin(\beta + \beta') : \sin \beta.$$

Hence $P = T' \dfrac{\sin(\beta + \beta')}{\sin \beta} = T' \sin \beta' \dfrac{\sin \beta \cos \beta' + \cos \beta \sin \beta'}{\sin \beta \sin \beta'}$

$$P = T' \sin \beta' (\cot \beta' + \cot \beta).$$

Also, the point C is kept at rest by three forces, the weight Q, and the tensions T'', T''', in the directions CB, CD, therefore, as before,

$$Q = T'' \sin\gamma(\cot\gamma + \cot\gamma');$$

and since γ is the supplement of β', $\sin\gamma = \sin\beta'$, therefore

$$P : Q :: \cot\beta + \cot\beta' : \cot\gamma + \cot\gamma'.$$

And if there were more angles, we should find each of the weights to be proportional to the sum of the cotangents of the angles which the supporting strings make with a vertical line.

A cord kept at rest in this manner is called a *funicular polygon.*

Problems for Practice.

1. Two forces, represented by 12lbs. and 15lbs., are inclined to each other at an angle of 60°; required the magnitude of the resultant and its inclination to the greater.

Ans. Magnitude, 23·43; inclination to the greater force, 26° 20′.

2. Three forces, which are to each other as 3, 4, 5, act upon a point and keep it at rest; required the angles at which these forces are inclined to each other. Ans. 126° 52′; 143° 8′; and 90°.

3. A boat, fastened to a fixed point, P, by a rope, is acted on at the same time by the wind and the current. Suppose that the wind was SE., the direction of the current S., and the direction of the boat from the fixed point S. 20° W., and also that the pressure on P was 150lbs: it is required to find the forces of the wind and the current.

Ans. Force of the wind, 72½lbs.; current 192lbs.

4. In pulling a weight along the ground by a rope inclined to the horizon at an angle of 45°, I exerted a power of 40lbs.; required the force with which I dragged the body horizontally.

Ans. 28¼lbs. nearly.

5. The resultant of two forces is 50lbs., and the angles which it makes with their directions is 20° and 30°; find the component forces.

Ans. 32·6 and 22·3lbs.

6. If two forces be inclined to each other at an angle of 135°; find the ratio between them when the resultant is equal to the less.

Ans. $\sqrt{2} : 1$.

7. Two forces, which are to each other as 2 to $\sqrt{3}$, act upon a point, and produce a force equivalent to half the greater; find the angle at which they are inclined to each other. Ans. 150°.

8. If three forces keep a body at rest, and three lines be drawn at right angles to the directions of these forces; the sides of the triangle formed by these perpendiculars will represent the three forces respectively.

9. A weight of 60lbs., suspended freely from a fixed point A, is drawn by the hand in a horizontal direction through an angle of 30°; required the pressure at A, and the force exerted by the hand.

Ans. 69·28 and 34·64lbs.

10. Suppose the ends of a thread 12 feet long to be fastened at two fixed points, A, B, in the same horizontal line, at a distance of 8 feet

from each other; what must be the proportion between two weights placed at 4 and 6 feet from the two ends of the string, so that, when suspended, they shall be in the same horizontal line?

11. Two parallel forces acting upon a rigid line in opposite directions, at a distance of 12 inches, are to each other as 3 to 5; required the magnitude and position of a third force which will keep the other two in equilibrium. Ans. A force of 2 acting at 18 in. from the greater.

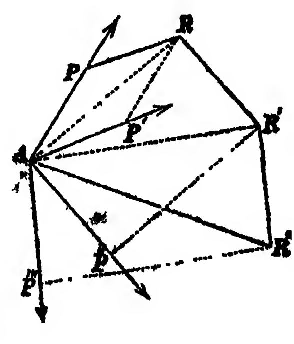
12. Three forces, acting upon a parallelogram, $ABCD$, are represented in magnitude and direction by the lines AB, DC, BD. Find the magnitude and direction of a third force which will keep the parallelogram at rest. Ans. The diagonal AC.

Note.—The student may pass over the next chapter, on a first perusal of this part of the Course.

CHAP. II.—GENERAL FORMULÆ ON THE EQUILIBRIUM OF FORCES.

FORCES ACTING AT A POINT.

30. PROP. I.—*If any number of forces,* P, P′, P″, &c., *in the same plane, act upon a point, it is required to find a single force which will be equivalent to them all.*



Let the lines AP, AP', AP'', &c. represent any number of forces acting at the point A. Describe a parallelogram $APRP'$ on two of the lines AP, AP'; and the diagonal AR will be the resultant of these two forces. Again, with the two sides AR, AP'', describe a new parallelogram $ARR'P''$, and the diagonal AR' will be the resultant of the two forces AR, AP''; or will be equivalent to the three forces AP, AP', AP''. Continuing in this manner, the last diagonal will be the resultant of all the forces AP, AP', AP'', &c. Hence the following construction is evident:

Describe a polygon, $APRR'$...., whose sides, AP, PR, RR'.... shall be, respectively, equal and parallel to the lines which represent the forces AP, AP', AP''.... ; then will

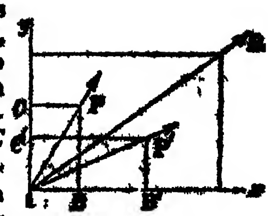
AR be the resultant of P and P';

AR' the resultant of P, P', P'';

and the remaining side of the polygon

AR'' the resultant of all the forces P, P' P'', P'''.

The following method, however, is better adapted to analytical calculation.



Let A be the point at which all the forces act. Through A draw any two lines Ax, Ay, in the plane of the forces, at right angles to each other. Let AP represent the force P in magnitude and direction, draw PB, PC perpendicular to Ax, Ay, then the figure $ABPC$ is a parallelogram, and the force AP is equivalent to the two forces AB, AC, acting in the directions Ax, Ay. In the same manner we may resolve all the other forces, P', P'', &c., into two others in the directions Ax, Ay. Now, if we call the angles PAx, $P'Ax$, &c., α, α', &c., we shall have $AB = P\cos\alpha$, $AC = BP = P\sin\alpha$; and in like manner the component parts of P', P'', &c., acting in the direction Ax, are $P'\cos\alpha'$, $P''\cos\alpha''$, &c.; and the component parts in the direction Ay, are $P'\sin\alpha'$, $P''\sin\alpha''$, &c. Hence, if we put X for the sum of all the forces in the direction Ax, and Y for the sum of all the forces in the direction Ay, we have

$$\left.\begin{aligned} X &= P\cos\alpha + P'\cos\alpha' + P''\cos\alpha'' + \text{\&c.} \\ Y &= P\sin\alpha + P'\sin\alpha' + P''\sin\alpha'' + \text{\&c.} \end{aligned}\right\} \ldots\ldots\ldots (1).$$

Let R be the resultant of all these forces, and a the angle which it makes with Ax, then

$$\text{also,}\quad \left.\begin{aligned} R &= \sqrt{(X^2 + Y^2)}; \quad \tan a = \frac{Y}{X}; \\ X &= R\cos a; \quad Y = R\sin a. \end{aligned}\right\} \qquad (2)$$

31. *Cor.* 1.—If any of the component forces AB, AB' &c., AC, AC', &c., be measured in the opposite direction from A, they must be considered negative. (Vol. i. p. 417).

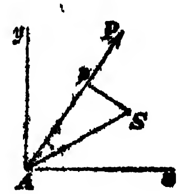
32. *Cor.* 2.—When there is an equilibrium, the resultant of all the forces $= 0$, and therefore $R = \sqrt{X^2 + Y^2} = 0$. Hence it is manifest that $X = 0$, and $Y = 0$, or,

$$P\cos\alpha + P'\cos\alpha' + P''\cos\alpha'' + \text{\&c.} = 0,$$
$$P\sin\alpha + P'\sin\alpha' + P''\sin\alpha'' + \text{\&c.} = 0.$$

33. Def. 1.—The product of a force and the perpendicular distance of a given point from its direction, is called the *moment of the force with respect to that point.*

Def. 2.—If through this point an axis be drawn at right angles to the plane, passing through the point and the direction of the force, the product is called the *moment of the force with respect to the axis.*

34. Prop. II.—*The moment of the resultant of any number of forces acting at a point, with respect to a point S in the same plane, is equal to the sum of the moments of the component forces.*



Let S be any point in the plane in which the forces P, P', &c. act; let $AS = s$, and the angle $SAx = \theta$. If, then, we multiply the values of X and Y, taken from equations (1) and (2), by $s\sin\theta$, and $s\cos\theta$, respectively, we have

$$Rs\cos a\sin\theta = Ps\cos\alpha\sin\theta + P's\cos\alpha'\sin\theta + \text{\&c.}$$
$$Rs\sin a\cos\theta = Ps\sin\alpha\cos\theta + P's\sin\alpha'\cos\theta + \text{\&c.}$$

Subtracting the first of these equations from the second, and substituting $\sin(\alpha - \theta)$ for $\sin\alpha\cos\theta - \cos\alpha\sin\theta$; &c. we get

$$Rs\sin(\alpha - \theta) = Ps\sin(\alpha - \theta) + P's\sin(\alpha' - \theta) + \&c.$$

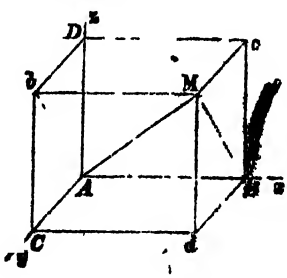
But $s\sin(\alpha - \theta) = s\sin SAP =$ the perpendicular Sp; $s\sin(\alpha' - \theta) = Sp'$, and so on. Putting these perpendiculars $= p, p', p''$, &c.; and the perpendicular upon the resultant $= r$, we obtain

$$Rr = Pp + P'p' + P''p'' + \&c.$$

Cor.—These moments may be either positive or negative; they are positive when they tend to make the system turn round the point S in the same direction as the resultant; and they are negative when they tend to make it turn round in the opposite direction.

35. Prop. III.—*To find the resultant of any number of forces acting in any direction at a point.*

Let AM represent any force acting at the point A. From A draw the three rectangular co-ordinates Ax, Ay, Az, and complete the rectangular parallelopiped AM. Join MB, Ab. Because AB, Mb are equal and parallel (Geom. prop, 107), Bb is a parallelogram; hence the force AM, acting on the point A, may be resolved into two forces represented by AB, Ab, acting at A. Also, since CD is a parallelogram, the force Ab may be resolved into the two forces AC, AD, acting at A. Hence the force AM is equivalent to the three forces AB, AC, AD, acting at A in the directions of the three axes.



Let the force AM be represented by P, and let α, β, γ be the angles which AM makes with Ax, Ay, Az. Then, because AB is perpendicular to the plane cd, it is perpendicular to MB, and therefore

$$AB = AM\cos MAx = P\cos\alpha.$$

In like manner $AC = P\cos\beta$, and $AD = P\cos\gamma$.

Let P', P'', &c. be any other forces acting at A, and making with Ax angles α', α'', &c.; with Ay angles β', β'', &c.; and with Az angles γ', γ'', &c. Then, if we put

$$X = P\cos\alpha + P'\cos\alpha' + P''\cos\alpha'' + \&c.$$
$$Y = P\cos\beta + P'\cos\beta' + P''\cos\beta'' + \&c.$$
$$Z = P\cos\gamma + P'\cos\gamma' + P''\cos\gamma'' + \&c.$$

X, Y, Z will be equal to the sum of the resolved forces acting in the directions of the three axes Ax, Ay, Az, respectively.

Let R be the resultant of the three forces X, Y, Z, and a, b, c the angles which it makes with Ax, Ay, Az, then will $R\cos a$, $R\cos b$, $R\cos c$ be the component forces in the direction of the three axes, and therefore

$$R\cos a = X;\quad R\cos b = Y;\quad R\cos c = Z.$$

Adding the squares of these equations together,

$$R^2(\cos^2 a + \cos^2 b + \cos^2 c) = X^2 + Y^2 + Z^2.$$

But, (Curve Lines, art. 280), $\cos^2 a + \cos^2 b + \cos^2 c = 1$,

$$\therefore \quad R = \sqrt{(X^2 + Y^2 + Z^2)},$$

and also, $\cos a = \frac{X}{R}$; $\cos b = \frac{Y}{R}$; $\cos c = \frac{Z}{R}$.

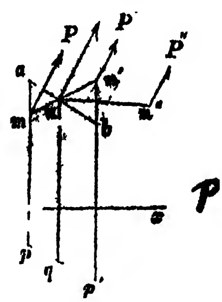
36. *Cor.* 1.—If any of the component forces AB, AC, AD, &c. be measured in the opposite direction from A, they must be considered negative (vol. i. p. 417).

37. *Cor.* 2.—When there is an equilibrium, the resultant of all the forces $= 0$, and therefore $R = \sqrt{(X^2 + Y^2 + Z^2)} = 0$. Hence it is evident that $X = 0$, $Y = 0$, $Z = 0$.

PARALLEL FORCES.

38. Prop. IV.—*To find the resultant of any number of parallel forces acting on a rigid body.*

Let any number of parallel forces, P, P', P'', &c., act upon a rigid body at the points m, m', m'', &c. Let any three axes, Ax, Ay, Az, be drawn at right angles to each other; and let x, y, z be the three co-ordinates of the point m; x', y', z' those of the point m', and so on. From the points m, m', m'', &c. draw mp, $m'p'$, &c. perpendicular to the plane xAy, we have then $z = mp$, $z' = m'p'$, $z'' = m''p''$, &c. If now we join mm', and divide this line in n, so that



$$P : P' :: m'n : mn,$$

the resultant of the two forces P, P' will pass through n, and will be parallel to them, and equal to $P + P'$. Through n draw nq parallel to mp, and ab parallel to pp', we have then

$$P : P' :: m'n : mn :: m'b : ma :: z' - nq : nq - z,$$

Hence $P \times (nq - z) = P' \times (z' - nq)$, and therefore

$$(P + P') \times nq = Pz + P'z'.$$

Again, let us take the two parallel forces $P + P'$ and P'', acting at the points n and m''; then, if nm'' be joined, and n' be the point of application of the resultant in the line nm'', and $n'q'$ its distance from the plane of xy, we have, as before,

$$(P + P' + P'')\, n'q' = (P + P')\, nq + P''z'' = Pz + P'z' + P''z''$$

Proceeding in the same manner, if R be the resultant of all the forces, and z_1 the distance of its point of application from the plane of xy, then

$$R = P + P' + P'' + \&c.; \quad Rz_1 = Pz + P'z' + P''z'' + \&c.$$

Similarly, if x_1, y_1 be the distances of the point of application of R from the planes of yz, xz,

$$Rx_1 = Px + P'x' + P''x'' + \&c.; \quad Ry_1 = Py + P'y' + P''y'' + \&c.$$

39 *Cor.* 1.—When any of the forces P, P', &c. act in the opposite direction, they must be considered negative. Also, if any of the co-ordinates x, y, z; &c., be measured in the direction opposite to that which has been assumed as positive, they must be considered negative.

40. *Cor.* 2.—The expression Px is called the *moment of the force* P, *with respect to the plane* yAz. Hence it appears that the sum of the moments of any number of parallel forces, with respect to any plane, is equal to the moment of their resultant.

41. *Cor.* 3.—The values of the co-ordinates x_1, y_1, z_1 are independent of the angle which the directions of the forces make with the axes. If, therefore, these directions be turned about the points of application of the forces, and at the same time continue parallel to each other, the point of the application of the resultant will not move. Hence this point is called the *centre of parallel forces*, and also the *centre of gravity*. (See art. 66.)

42. Prop. V.—*To find the conditions of equilibrium of parallel forces acting upon a rigid body.*

In order that there may be an equilibrium, one of the forces, as P, must be equal and directly opposed to the resultant of all the others. Let R' be the resultant of P', P'', &c., we shall then have

$$R' = P' + P'' + \&c., \text{ and also } R' = -P.$$

Hence, substituting and transposing,

$$P + P' + P'' + \&c. = 0.$$

Also, since the position of the axes is arbitrary, we may suppose Az to be parallel to the common direction of the forces, and, therefore, the plane xAy will be perpendicular to this direction. And since the force P and the resultant R' are in the same straight line, they will meet the plane xAy in the same point, and therefore x, y, the co-ordinates of P, will be the co-ordinates also of R'. Hence

$$R'x = P'x' + P''x'' + \&c.; \quad R'y = P'y' + P''y'' + \&c.,$$

and substituting for R' its value $-P$, and transposing

$$Px + P'x' + P''x'' + \&c = 0; \quad Py + P'y' + P''y'' + \&c. = 0.$$

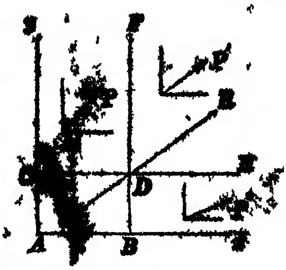
43. *Cor.*—If the rigid body have one point fixed, it is manifest that the equilibrium will subsist if the resultant pass through this point, for it will be counteracted by the resistance of the fixed point. Hence, if we suppose this point to be the origin of the co-ordinates, we shall have, in article 40, $x_1 = 0$, $y_1 = 0$; therefore, by that article,

$$Px + P'x' + P''x'' + \&c. = 0; \quad Py + P'y' + P''y'' + \&c. = 0.$$

FORCES ACTING IN ANY DIRECTIONS UPON A RIGID BODY.

44. Prop. VI.—*To find the resultant of any number of forces acting in the same plane upon a rigid body.*

Let P, P', P'', &c. be the forces acting upon the body in the plane xAy. Let the two lines Ax, Ay be drawn in this plane, at right angles to each other; let x, y be the co-ordinates of the point M; x', y' those of the point M'; and so on. Also, let α, α', α'', &c. be the angles at which the forces P, P', P'', &c. are inclined to the axis Ax. Resolve



each of the forces P, [illegible] two others, the one parallel to [illegible] and the other parallel to the [illegible] the two component forces of P [illegible], $P\sin\alpha$; those of P' will [illegible] α', $P'\sin\alpha'$; and so on. [illegible] are resolved into two sets [illegible], acting at the points M, M', [illegible] parallel to Ax, Ay, respectively.

(1). Let us first suppose [illegible] parallel forces may be [illegible] force; and let X be the [illegible] forces parallel to Ax, and Y the resultant of those [illegible] Ay. Then we have (art. [illegible])

$$X = P [\text{illegible}] + P''\cos\alpha'' + \&c.$$

$$Y = P\sin [\text{illegible}] \alpha' + P''\sin\alpha'' + \&c.$$

Also, let y_1 be the distance of the force X from the axis Ax, and x_1 the distance of the force Y from the axis Ay; then

$$Xy_1 = Py\cos\alpha + P'y'\cos\alpha' + P''y''\cos\alpha'' + \&c.$$

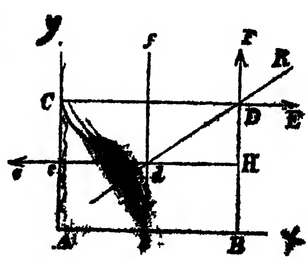
$$Yx_1 = Px\sin\alpha + P'x'\sin\alpha' + P''x''\sin\alpha'' + \&c.$$

The values of x_1 and y_1 being determined from the last equations, take $AB = x_1$, $AC = y_1$, and draw CE, BF parallel to the axes Ax, Ay, meeting in D. The forces X, Y acting in the directions CE, BF, may be supposed to act at the point D in the directions DE, DF. Let R be the resultant of X and Y, and a the angle which this direction makes with Ax, we shall then have

$$R = \sqrt{(X^2 + Y^2)}; \qquad \tan a = \frac{Y}{X}.$$

Hence the magnitude and position of the resultant are known.

(2). Suppose now that one of the sets of parallel forces is reducible to a single force, but the other set, as that parallel to AX, can only be reduced to two equal and parallel forces acting in contrary directions, but not directly opposed. Let CE, ce be the directions of the two equal and opposite forces, X', $-X'$, parallel to Ax; BF the direction of the force Y; and D the point of intersection of the lines CE, BF. The two forces X', Y may be supposed to act at the point D, and will have a single resultant R. Let DR be the direction of this resultant, and let R be produced until it meet ce in d; then R may be supposed to act at d in the direction dR, and it may be resolved again into the two component forces X', Y, in the directions dH, df. The force X' will be counteracted by an equal and an opposite force $-X'$, and therefore there will remain the single force Y acting in the direction df. Thus the force Y, acting in in the direction df, is equivalent to the three forces X', $-X'$, Y acting in the directions CE, ce, and BF.



(3). Lastly: suppose that neither of the sets of parallel forces can be reduced to a single force. Let X', $-X'$ be the two equal forces parallel to Ax, [illegible]ing in contrary directions, but not directly opposed: and let Y', $-$[illegible] be the two equal forces acting in the same manner, parallel to the axis Ay. Let the resultant of the two forces X', Y' be R'; then the resultant of the two forces $-X'$, $-Y'$ will evidently be $-R'$, equal to th[illegible]rmer, but acting in a contrary direction. If, then, R', $-R'$ d[illegible]not act in the same line, these forces are not reducible to a sin[illegible]force. [illegible] R', $-R'$ are directly opposed, they will keep the bod[illegible]n equili[illegible].

45. *Cor.* 1.—Since the resultant R, in the first case, passes through the point D, whose co-ordinates are x_1, y_1; if t and u be the co-ordinates of any other point in the direction of this force, the equation to the line is (Theory of Curve Lines, art. 21)

$$u - y_1 = \tan a (t - x_1) = \frac{Y}{X}(t - x_1);$$

$$\therefore \quad Xu - Yt = Xy_1 - Yx_1 = L, \quad \text{by substitution.}$$

46. *Cor.* 2.—If the forces parallel to Ax are not reducible to a single force, then $X = 0$; but Xy_1 is finite. We have, therefore in this case,

$$-Yt = Xy_1 - Yx_1 = L; \quad \text{and,} \; t = -\frac{L}{Y}.$$

The same result might be obtained independently from the last figure.

47. Prop. VII.—*To find the conditions of equilibrium of any number of forces acting in the same plane upon a rigid body.*

In order that there may be an equilibrium, one of the forces, as P, must be equal and directly opposed to the resultant of all the others. Let R' be the resultant of P', P'', &c., and let a' be the angle which it makes with the axis Ax. We have then (art. 44)

$$R' \cos a' = P' \cos \alpha' + P'' \cos \alpha'' + \text{\&c.}$$

$$R' \sin a' = P' \sin \alpha' + P'' \sin \alpha'' + \text{\&c.}$$

Let $X' = R' \cos a'$, $Y' = R' \sin a'$; also, let t', u' be the variable co-ordinates of the line which is in the direction of R', and $L' = P'(y' \cos \alpha' - x' \sin \alpha') + \text{\&c.}$: we have then, from the same article.

$$X'u' - Y't' = L' \ldots\ldots\ldots (a).$$

Now, since the two forces P and R' are equal and opposite,

$$P \cos \alpha = -R' \cos a', \qquad P \sin \alpha = -R' \sin a',$$

from which two equations we get

$$P \cos \alpha + P' \cos \alpha' + \text{\&c.} = 0; \quad P \sin \alpha + P' \sin \alpha' + \text{\&c.} = 0 \ldots (1).$$

And, in order that they may be directly opposed, the point M, at which P acts, shall be situated in the line in which R' acts; or x, y, the coordinates of M, when substituted for t', u', shall satisfy equation (a).

Hence $\quad X'y - Y'x = -Py \cos \alpha + Px \sin \alpha = L'.$

Substituting for L' its value, and transposing, we get

$$P(y \cos \alpha - x \sin \alpha) + P'(y' \cos \alpha' - x' \sin \alpha') + \text{\&c.} = 0 \ldots (2).$$

These equations denoted (1) and (2) are the equations of condition for the equilibrium of the forces.

48. *Cor.*—If a point in the rigid body in the given plane be fixed, the equilibrium will subsist if the resultant of the forces pass through this point, for it will be counteracted by the resistance of this point. Hence, in this case, equation (2) is the only condition of equilibrium.

49. Prop. VIII.—*The moment of the resultant of any number of forces in the same plane, with respect to a given point, is equal to the sum of the moments of the component forces.*

Suppose this point to be the origin of co-ordinates; and let p, p', p'', &c. be the perpendiculars drawn from the origin A to the directions of the forces P, P', P'', &c. Then, since P is the resultant of the two component forces $P\cos\alpha$, $P\sin\alpha$,

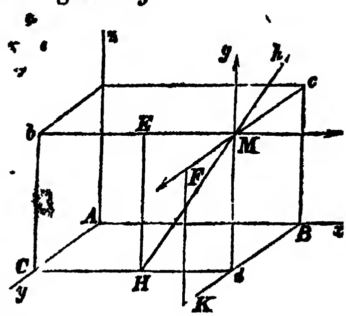
$$\text{the moment } Pp = Py\cos\alpha - Px\sin\alpha \quad \text{(art. 34).}$$

In like manner $P'p' = P'y'\cos\alpha' - P'x'\sin\alpha'$; $P''p'' =$ &c., and $Rr = Ry_1\cos a - Rx_1\sin a$. Hence

$$Rr = Pp + P'p' + P''p'' + \text{\&c.},$$

observing that these moments must be considered negative when they tend to make the system turn round in the opposite direction.

50. Prop. IX.—*To find the conditions of equilibrium of any number of forces acting in any directions upon a rigid body.*



Let P, P', P'', &c. be the forces acting upon a body at the points M, M', M'', &c. Let Ax, Ay, Az be three axes drawn at right angles to each other; and let x, y, z be the co-ordinates of the point M; x', y', z' those of the point M'; and so on. Also, let α, β, γ; α', β' γ', &c. be the angles which the directions of the forces P, P', &c. make with lines parallel to Ax, Ay, Az, respectively. Resolve the force P into three others, $P\cos\alpha$, $P\cos\beta$, $P\cos\gamma$ parallel to the three axes; in like manner resolve the force P' into the three forces $P'\cos\alpha'$, $P'\cos\beta'$, $P'\cos\gamma'$; and so on. Then the forces are resolved into three sets of forces acting at the points M, M', &c., and parallel to Ax, Ay, Az, respectively.

At the point M suppose two equal and opposite forces g, $-g$, to be added to the system parallel to the axis Az; these will counteract each other, and the effect of all the forces will be the same as before. The forces, then, which act at M may be arranged thus:

$$P\cos\alpha, \text{ and } g\,; \qquad P\cos\beta, \text{ and } -g\,; \qquad P\cos\gamma.$$

Let the two forces $P\cos\alpha$, g have a resultant Mh; Mh will be in the plane $MbCd$. Let Mh produced meet Cd in H; then the force Mh at M will have the same effect on the system as an equal force acting at H; and the force Mh acting at H is equivalent to the two forces

$P\cos\alpha$ and g in the directions Hd, HE; and consequently the three forces $P\cos\alpha$, g, and Mh, are proportional to Hd, HE, and HM. Hence

$$Hd : HE :: P\cos\alpha : g;$$

$$\therefore \quad Hd = \frac{Pz\cos\alpha}{g}, \quad \text{and} \quad CH = x - \frac{Pz\cos\alpha}{g}.$$

Hence $P\cos\alpha$ and g acting at M are equivalent to $P\cos\alpha$ parallel to Ax, and g parallel to Az, both acting at a point H, of which the co-ordinates, parallel to Ax and Ay, are $x - \dfrac{Pz\cos\alpha}{g}$ and y.

In the same manner $P\cos\beta$ and $-g$ are equivalent to a force acting in the direction MK in the plane MB; and this produces the same effect as if it acted at K. Resolve this force again at K into the two forces $P\cos\beta$ and g, parallel to Ay and Az, respectively. Then, as before,

$$Kd : KF :: P\cos\beta : g;$$

$$\therefore \quad Kd = \frac{Pz\cos\beta}{g}, \quad \text{and} \quad BK = y + \frac{Pz\cos\beta}{g}.$$

Hence $P\cos\beta$ and $-g$ at M are equivalent to $P\cos\beta$ parallel to Ay, and $-g$ parallel to Az, both acting at a point K, of which the co-ordinates, parallel to Ax and Ay, are x and $y + \dfrac{Pz\cos\beta}{g}$.

The force $P\cos\gamma$ is parallel to Az, and produces the same effect as if it acted at d, of which the co-ordinates are x and y.

Hence it appears that the force P acting at M is equivalent to the five forces.

$P\cos\alpha$ parallel to Ax, acting at H;
$P\cos\beta$ parallel to Ay, acting at K;
$P\cos\gamma$ parallel to Az, acting at d;
$+g$ parallel to Az, acting at H; and
$-g$ parallel to Az, acting at K.

Let the forces P' P'', &c. be resolved in the same manner, we shall then have two sets of forces; one set being in the plane xAy, and the other perpendicular to it. And it is evident that the body will be at rest if each of these sets of forces be in equilibrium separately.

Now we have for the equilibrium of the forces parallel to Az (art. 42),

$$P\cos\gamma + P'\cos\gamma' + P''\cos\gamma'' + \&c. = 0,$$

$$Px\cos\gamma + g\left(x - \frac{Pz\cos\alpha}{g}\right) - gx + P'x'\cos\gamma' + \&c. = 0;$$

$$Py\cos\gamma + gy - g\left(y + \frac{Pz\cos\beta}{g}\right) + P'y'\cos\gamma' + \&c. = 0;$$

and reducing these equations, they become

$$P\cos\gamma + P'\cos\gamma' + P''\cos\gamma'' + \&c. = 0,$$
$$P(x\cos\gamma - z\cos\alpha) + P'(x'\cos\gamma' - z'\cos\alpha') + \&c. = 0,$$
$$P(y\cos\gamma - z\cos\beta) + P'(y'\cos\gamma' - z'\cos\beta') + \&c. = 0$$

Also, we have, for the equilibrium of the forces in the plane xAy (art. 47), the three equations,

$$P\cos\alpha + P'\cos\alpha' + P''\cos\alpha'' + \&c. = 0,$$
$$P\cos\beta + P'\cos\beta' + P''\cos\beta'' + \&c. = 0,$$
$$P(y\cos\alpha - x\cos\beta) + P'(y'\cos\alpha' - x'\cos\beta') + \&c. = 0.$$

And these six equations express the conditions of equilibrium.

51. *Cor.* 1.—These six equations are *sufficient* for the equilibrium of the forces P, P', &c.; they are also *necessary*; for except both sets of forces be in equilibrium separately, the system cannot be at rest. If possible, let the system be at rest when the forces parallel to Az are not separately in equilibrium. The equilibrium will still subsist if we suppose any line in the plane xAy to be fixed. But in that case all the forces in the plane xAy will be counteracted by the resistance of this line. And the forces parallel to Az will turn the system about this line in some of its positions. Hence the equilibrium will not subsist. And since the forces parallel to Az are in equilibrium separately, the other forces must also be in equilibrium separately.

52. *Cor.* 2.—If the rigid body have a fixed point, we may make this point the origin of co-ordinates. Then the body will be in equilibrium, if the resultant of the forces in the plane xAy passes through A, and also the resultant of the forces parallel to Az. Hence it appears, from articles 43, 48, that the 2d, 3d, and 6th of the last six equations in article 50 must be satisfied.

53. Prop. X.—*To find the condition requisite that a system of forces, acting in any manner upon a rigid body, may have a single resultant.*

Proceeding in the same manner as in article 50, we may reduce the forces to two sets, one in the plane xAy, and the other perpendicular to it. If each of these sets of forces have a single resultant, and these resultants intersect each other, they may be compounded into a single force, which will be the resultant of the whole. But if the two resultants do not intersect each other, this will be impossible. Let

$$X = P\cos\alpha + P'\cos\alpha' + \&c.; \quad Y = P\cos\beta + P'\cos\beta' + \&c.$$
$$L = P(y\cos\alpha - x\cos\beta) + P'(y'\cos\alpha' - x'\cos\beta') + \&c.$$

Then the equation of the resultant of these forces in the plane of xy will be (art. 45)

$$Xu - Yt = L,$$

t and u being the co-ordinates of any point of this line. Let, also,

$$Z = P\cos\gamma \quad + P'\cos\ ' + P''\cos\gamma'' + \&c.$$
$$M = P(x\cos\gamma - z\cos\alpha) + P'(x'\cos\gamma' - z'\cos\alpha') + \&c.$$
$$N = P(z\cos\beta - y\cos\gamma) + P'(z'\cos\beta' - y'\cos\gamma') + \&c.;$$

and let x_1, y_1 be the co-ordinates of the point where the resultant Z meets the plane of xy, we shall have then (art. 38)

$$Zx_1 = M, \qquad Zy_1 = -N.$$

And, in order that the resultant R may pass through the point (x_1, y_1), these values of x_1, y_1, when substituted for t and u, must satisfy the

equation $Xu - Yt = L$. Hence, making this substitution, we have, for the equation of condition,

$$LZ + MY + NX = 0 \ldots\ldots\ldots (C).$$

54. *Cor.* 1.—If $X = 0$, $Y = 0$, $Z = 0$, and L, M, N are finite, the equation of condition is satisfied; and yet the forces P, P', &c. have not a single resultant; for the equations from which (C) was deduced did not apply to this case. In this case the forces may be reduced to two that are equal and parallel, but not directly opposed.

55. *Cor.* 2.—In all other cases when this equation is satisfied, the forces P, P', &c. will have a single resultant. Thus, if $X = 0$, $Y = 0$, and Z is finite, we must have $L = 0$, to satisfy equation (C). In this case the forces in the plane yAx are in equilibrium, and the forces parallel to Az have a single resultant. The same remarks may be made if $X = 0$, $Z = 0$; or if $Y = 0$, $Z = 0$; for it is manifest that we may reduce the forces to two sets, one in the plane xAz or zAy, and the other perpendicular to it; and the equations will be exactly similar to those which we had when the forces were reduced to two sets, one in the plane yAx, and the other perpendicular to it.

56. Prop. XI.—*When the equation of condition is satisfied, it is required to find the resultant of any number of forces acting in any directions on a rigid body.*

The force in the plane xAy is composed of X and Y, and the force parallel to Az is Z, and the directions of these two forces meet each other in the point (x_1, y_1). Hence, if R be the resultant, and a, b, c the angles which it makes with lines parallel to the three axes, we shall have

$$R^2 = X^2 + Y^2 + Z^2,$$

$$R \cos a = X; \qquad R \cos b = Y; \qquad R \cos c = Z.$$

And the point where it cuts the plane yAx is known from the equations

$$Zx_1 = M, \qquad Zy_1 = -N.$$

57. Prop. XII.—*When a number of forces are not reducible to one force, they may always be reduced to two.*

For if we add to the system two new forces, S, $-S$, acting at the origin A in opposite directions, the same effect will be produced as before.

Let X', Y', Z' be the component parts of the force S; and let these be united with the components of the forces P, P' &c. Since S passes through A, the values of L, M, N will not be affected by it; therefore the first member of equation (C) will become

$$L(Z + Z') + M(Y + Y') + N(X + X').$$

And since X', Y', Z' are independent of each other, we may give any arbitrary value to two of these quantities, and the third will then be determined by putting the above expression equal to zero. Let R' be the resultant of these forces, then the given forces are reduced to R', and the force $-S$ acting at the point A.

63. *Cor.*—It is evident from this proposition that the two resultants R', $-S$, are not determined either in magnitude or direction.

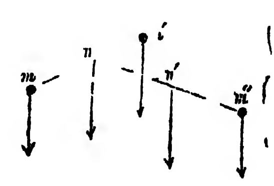
CHAP. III.—THE CENTRE OF GRAVITY.

58. *Gravity* is a term used to denote that force by which every material particle is urged towards the surface of the earth, as soon as it is left unsupported. A line drawn in the direction of this force, is called a *vertical* line; and any plane passing through this line is called a *vertical* plane. A plane perpendicular to this plane is called a *horizontal* plane.

59. From numerous experiments which have been made, it appears that the intensity of the force of gravity is constant at the same place, and, therefore, we may consider a homogeneous body as a collection of material points, to each of which is applied an equal force. Since these forces may all be considered parallel, and exerted in the same direction; and also the resultant of any two parallel forces is equal to their sum (art 21), it follows that the weight is proportional to the mass. In heterogeneous bodies, the weight is not proportional to the magnitude, we therefore suppose that bodies of different substances have a different number of material points of the same gravity contained in the same volume.

60. Def.—The *centre of gravity* of any body, or system of bodies, is a point upon which the body or system, acted upon only by the force of gravity, will balance itself in all positions. We shall, in the following proposition, prove that every body has such a point.

61. Prop. I.—*Every body, or system of bodies, has a centre of gravity.*

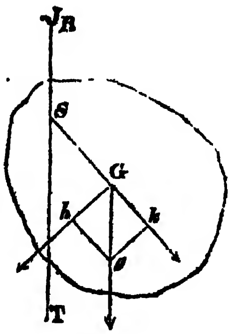


Let m, m', m'', &c, be a system of particles, of different weights, acted on by the force of gravity, and let the weights of these particles be represented by m, m', m'' &c. Divide mm' in n, so that $m : m' :: m'n : mn$; then, since the two particles m, m' are acted on by two parallel forces, which are proportional to their weights, their resultant will be equal to their sum $m + m'$, and pass through the point n (art. 21). Suppose, now, the position of the system to be altered in any way, the resultant will still pass through the point n, and be equal to $m + m'$. Hence, if the point n be supported, the two particles m, m' will be supported in all positions; and the pressure at n will be equal to $m + m'$. Again, if we take a third particle, m'', and join nm''; and divide the line nm'' in n', so that $m + m' : m'' :: m''n' : nn'$; the resultant of the two parallel forces, proportional to the weights $m + m'$ and m'', will pass through n', and will be equal to $m + m' + m''$. If, likewise, the position of the system is altered, the resultant will still pass through n'; and, therefore, if the point n' be supported, the three particles m, m', m'' will be supported in all positions: therefore n' is the centre of gravity of the three particles m, m', m''. And in like manner it may be proved, whatever be the number of forces.

62. *Cor.*—Hence it follows, that the weight of any body may be

supposed to be collected at its centre of gravity, and the effect of the system will be the same as before.

63. Prop. II.—*If a body be suspended by any point, it will not remain at rest, until the centre of gravity be in a vertical line passing through that point.*



Let S be the point of suspension of the body ABC, G its centre of gravity. Then the effort of gravity to put the body in motion is the same as if all the particles were collected at G (art. 61). Join SG and produce it; through S and G draw ST, Gg, in the direction of gravity. Take Gg to represent the force in that direction, and draw gk perpendicular to SG produced, and complete the parallelogram $Ghgk$. Then the force Gg is equivalent to the two forces Gh, Gk, of which Gk is sustained by the re-action at the point S, and Gh ($=Gg \times \sin Ggh = Gg \times \sin GST$) is entirely effective in moving the centre of gravity in a direction perpendicular to SG; therefore the centre of gravity cannot remain at rest until Gh vanishes, that is, until the angle GST vanishes, or SG coincides with ST.

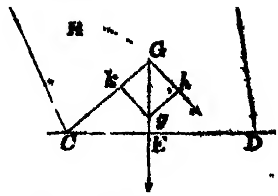
64. *Cor.* 1.—When the point G is in the line ST, *below* S, the weight of the body will be entirely effective in *pulling* the point S, which will resist it, and no motion will ensue. But if G be in the line ST, *above* S, the weight will then produce a *pressure* on S, which will be resisted by this point. There is, however, an important difference in the two cases; for if the body be made to deviate from the position of equilibrium in the first case, it has a tendency to return to it; but in the latter case, if the position of the body be changed in the smallest degree, it has a tendency to move further from the position of equilibrium until it assume some new position. The first, therefore, is called *stable* equilibrium, and the second *unstable* equilibrium.

65. *Cor.* 2.—If the body be suspended by a thread RS, it may be proved in the same manner that when the body is in equilibrium the line RS will be vertical, and will pass, when produced, through the centre of gravity of the body.

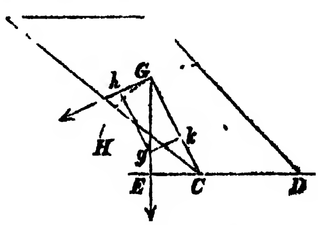
66. *Cor.* 3—Hence the centre of gravity of any body may easily be found *experimentally*. For if the body be suspended by a thread, and be in equilibrium, the centre of gravity will be somewhere in the direction of the vertical line, passing through the point of suspension, or in the line of the thread produced. Again, if the body be suspended from some other point, and a vertical line be drawn through this point, the centre of gravity will be also in this new line; and, therefore, it will be in the intersection of these two lines.

67. Prop. III.—*If a body be placed upon a horizontal plane, it will be sustained or not, according as the perpendicular to the horizon, drawn through its centre of gravity, falls within or without the base.*

Let G be the centre of gravity of the body P; join CG, and with the centre C and radius CG describe the arc GH, then the body cannot



fall over at C, unless G describes the arc GH. Draw the vertical line GE, and suppose the whole force of gravity to be applied at G, and let Gg represent it; draw gk perpendicular to CG, and complete the parallelogram $Ghgk$. Then the force Gg may be resolved into the two, Gk, Gh, of which Gk being in the direction GC, cannot move the body in the direction GH or Gh; and when E falls within the base, the force Gh opposes the communication of motion in the direction GH, and, therefore, the body will be supported. In like manner, if GD be joined, it may be proved that the body cannot fall over at D.



But if GE falls without the base, Gh acts in the direction of the arc GH at the point G; and since there is no force to counteract this, the point G will move in that direction and the body will fall.

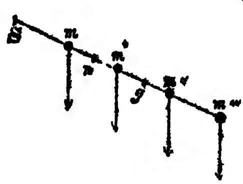
68. *Cor.* 1.—In the same manner it may be shown, that if a body be placed upon an inclined plane, and be prevented from sliding, the body will be sustained or not according as the vertical line GK falls within or without the base.

69. *Cor.* 2.—From similar triangles $Gg : Gh :: CG : CE$, and if CE and the weight of the body Gg be given, the force Gh, which tends to support the body, varies inversely as CG. Hence, as CG increases, or as GE increases, the force which keeps the body steady decreases, or the more easily will the body be overturned.

TO FIND THE CENTRE OF GRAVITY OF ANY SYSTEM OF BODIES.

The centre of gravity of any system of bodies may be found immediately from the general proposition on parallel forces (art. 38); but, as we wish to determine its position independently of the formulæ in the second chapter, we will give a separate demonstration of the two following propositions.

70. Prop. IV.—*To find the centre of gravity of any number of bodies considered as points in the same straight line.*



Let m, m', m'', &c. be any number of bodies in the same straight line. Take any point S in this line; and put $Sm = x$, $Sm' = x'$; and so on. Divide mm' in n, so that $m : m' :: m'n : mn$; n will be the centre of gravity of m and m'. Hence

$$m \times mn = m' \times m'n; \text{ or,}$$

$$m \times (Sn - Sm) = m' \times (Sm' - Sn);$$

$$\therefore (m + m') \times Sn = m \times Sm + m' \times Sm' = mx + m'x'.$$

Again, if we suppose the weights m, m' to be collected at n, and n' to be the centre of gravity of $m + m'$ and m'', we have, as before,

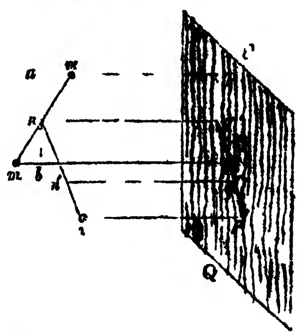
$$(m+m'+m'') \times Sn' = (m+m') \times Sn + m'' \times Sm'' = mx + m'x' + m''x''.$$

Proceeding in this manner, if G be the centre of gravity of all the bodies, $SG = x_1$, and $M = m + m' + m'' + \&c.$, we have.

$$Mx_1 = mx + m'x' + m''x'' + \&c.$$

the distances x, x', &c. being considered negative, if they are measured on the other side of S.

71. Prop. V.—*To find the distance of the centre of gravity of any system of bodies, considered as points, from a given plane.*



Let m, m', m'' be any number of bodies supposed to be connected with each other by rigid lines; PQ the given plane. Draw mp, $m'p'$, $m''p''$, &c. perpendicular to this plane, and put $mp = x$, $m'p' = x'$; and so on. Join mm', and divide it in n, so that $m : m' :: m'n : mn$; then n is the centre of gravity of m and m'. Draw nq perpendicular to the plane PQ, then mp, nq, $m'p'$ are in the same plane, and parallel to each other, also, through n, draw anb perpendicular to mp or $m'p'$; we have then

$$m : m' :: m'n : mn :: m'b : ma$$
$$:: x' - nq : nq - x,$$
$$\therefore \quad m \times (nq - x) = m' \times (x' - nq).$$

Hence $\quad (m + m') \times nq = mx + m'x'.$

Again, join nm'', and divide it in n', so that $m + m' : m'' :: m''n' : nn'$; then n' is the centre of gravity of the three bodies, m, m', m''. And if $n'q'$ be drawn perpendicular to PQ, it may be shown, as before, that

$$(m+m'+m'') \times n'q' = (m+m') \times nq + m''x'' = mx + m'x' + m''x''.$$

And in like manner, if G be the centre of gravity of any number of bodies, and its perpendicular distance from $PQ = x_1$, and $M = m + m' + m'' + \&c.$, we shall have

$$Mx_1 = mx + m'x' + m''x'' + \&c.;$$

$$\therefore \quad x_1 = \frac{mx + m'x' + m''x'' + \&c.}{M};$$

the distances being considered negative, if any of the bodies lie on the other side of the plane.

72. Cor. 1.—If a plane be drawn parallel to PQ, at the distance x_1, the centre of gravity of the system lies somewhere in this plane. In the same manner, two other planes may be found, in each of which the centre of gravity lies, and the point where the three planes cut each other is the centre of gravity of the system. Hence it is evident that every system of bodies can only have one centre of gravity.

73. Cor. 2.—If the plane pass through the centre of gravity, $x_1 = 0$, hence

$$mx + m'x' + m''x'' + \&c. = 0;$$

that is, the sum of the moments on one side of the plane is equal to the sum of the moments on the other side.

74. *Cor.* 3.—When all the bodies are in the same plane, draw any two straight lines at right angles to each other in this plane; then, if x, x', x'', &c. be the lengths of the perpendiculars from the bodies on one of the lines, and y, y', y'', &c. the lengths of the perpendiculars on the other line; and also x_1, y_1 the distances of the centre of gravity from these two lines, we shall have

$$Mx_1 = mx + m'x' + m''x'' + \&c.$$
$$My_1 = my + m'y' + m''y'' + \&c.$$

TO FIND THE CENTRE OF GRAVITY OF CERTAIN FIGURES GEOMETRICALLY.

In finding the centre of gravity of lines and planes, they are supposed to be made up of particles of matter uniformly diffused over them.

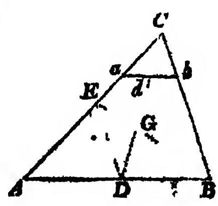
75. PROBLEM I.—*To find the centre of gravity of a material straight line of uniform thickness and density.*

The centre of gravity of a straight line composed of particles of matter which are equal to each other, and uniformly dispersed, is its middle point. For there are equal weights on each side, equally distant from the middle point, which will balance each other in all positions upon that point.

76. PROBLEM II.—*To find the centre of gravity of a triangle.*

Bisect AB, AC in D, E; join CD, BE, cutting each other in G: this point is the centre of gravity of the triangle.

For the triangle may be conceived to be made up of lines parallel to AB, such as ab; then we have, by similar triangles,



$$AD : ad :: CD : Cd :: BD : bd,$$

and $AD = BD$, therefore $ad = bd$; and consequently the line ab will balance itself in all positions upon CD. For the same reason every other line parallel to AB will balance itself in all positions upon CD, and therefore the whole triangle will balance itself upon that line; hence the centre of gravity of the triangle is in the line CD. In the same manner it may be proved that the centre of gravity of the triangle is in the line BE; therefore it is in G, the intersection of the two lines CD, BE.

Join DE; then, since $AD = DB$, and $AE = EC$, DE is parallel to BC, therefore

$$BC : DE :: AB : AD :: 2 : 1.$$

Also the triangles BGC, DGE, are similar, therefore

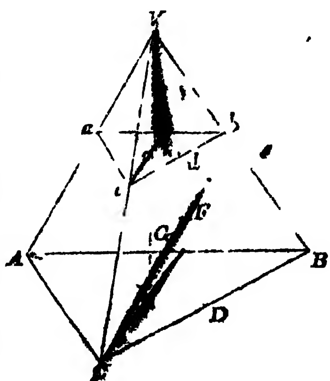
$$\therefore CG : GD :: BC : DE :: 2 : 1.$$

Hence $CG = 2GD$, and $CD = 3GD$.

77. *Cor.* 1.—In the same manner the centre of gravity of a parallelogram will be found by bisecting all its four sides, and joining the points of bisection in the opposite sides; the intersection of these two lines will be the centre of gravity of the parallelogram.

78. *Cor.* 2.—To find the centre of gravity of any rectilineal figure; divide it into triangles, and supposing each of these collected in its centre of gravity, find the centre of gravity of the whole; this will be the centre of gravity of the figure.

79. Problem III.—*To find the centre of gravity of a triangular pyramid.*



Let ABC be the base, and V the vertex of the pyramid. Bisect BC in D; join AD, VD. Take $DE = \frac{1}{3}AD$, then E is the centre of gravity of the triangle ABC. Join VE, then, if any section abc be drawn parallel to the base, cutting VE in e, e will be the centre of gravity of the triangle abc. For since the plane abc is parallel to ABC, bc is parallel to BC, and ad to AD. And because $BD = DC$ and $DE = \frac{1}{3}AD$, therefore $bd = dc$ and $de = \frac{1}{3}ad$; consequently e is the centre of gravity of the triangle abc. Hence if we suppose the pyramid to be composed of an infinite number of indefinitely thin triangular figures parallel to the base, each of these has its centre of gravity in the line VE, and, therefore, the centre of gravity of the pyramid will be in VE.

Again, take $DF = \frac{1}{3}VD$, and join AF, cutting VE in G. Then, as before, the centre of gravity of the pyramid must be in AF; but it is in VE; hence G, the intersection of these lines, is the centre of gravity.

Join EF; then, because $AD = 3DE$, and $VD = 3DF$, therefore EF is parallel to VA. Hence we have, by similar triangles,

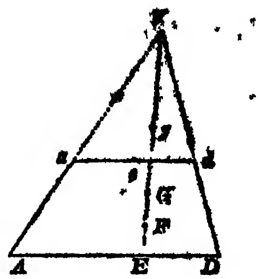
$$VG : GE :: VA : EF :: AD : DE :: 3 : 1;$$

$$\therefore \quad VG = 3GE \text{ and } VE = 4GE.$$

80. *Cor.*—If a line VE be drawn from the vertex of any pyramid or cone to the centre of gravity of its base, and VG be taken equal to $\frac{3}{4}$ths of VE, the point G will be the centre of gravity of the pyramid or cone.

81. Problem IV.—*To find the centre of gravity of the frustum of a pyramid, cut off by a plane parallel to the base.*

Let VAD be a plane passing through V, the vertex of the pyramid, and E the centre of gravity of the base. Let G, g be the centres of gravity of the pyramids VAD, Vad, and F the centre of gravity of the frustum $AabB$; then it is evident, from the last problem, that G, g and F are in the straight line VE. Let P be the weight of the pyramid



VAD, p the weight of Vad, and f the weight of the frustum; then the centre of gravity of f and p will be in the same point as if these bodies were collected in F and g. But the centre of gravity of f and p is the same as that of P, that is, the centre of gravity of F and p is in G.

Hence, if we put $AD = a$, $ad = b$, $Ee = h$, we have (art. 7[illegible])

$$P \times EG = f \times EF + p \times Eg.$$

And since $P : p : f :: a^3 : b^3 : a^3 - b^3$;

$$\therefore \quad EF = \frac{P}{f} EG - \frac{p}{f} Eg = \frac{a^3}{a^3 - b^3} EG - \frac{b^3}{a^3 - b^3} Eg.$$

Also, $\quad a : b :: VE : Ve; \quad \therefore \quad a : a - b :: VE : h;$

$$\text{hence} \qquad EG = \frac{VE}{4} = \frac{h}{4} \frac{a}{a-b}.$$

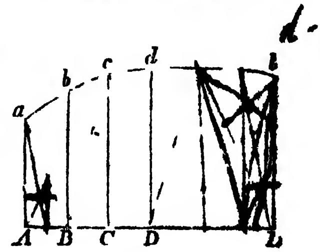
$$\text{Similarly} \quad eg = \frac{h}{4} \frac{b}{a-b}; \quad \text{and} \quad Eg = h + \frac{h}{4} \frac{b}{a-b}.$$

Substituting these values above, we get

$$EF = \frac{h}{4} \frac{a^4 - b^4}{(a^3 - b^3)(a - b)} - h \frac{b^3}{a^3 - b^3}; \text{ and, by reduction,}$$

$$EF = \frac{h}{4} \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

82. Problem V.—*To find the centre of gravity of any curvilinear plane figure.*



Divide the base AL into n equal parts, and draw the ordinates Aa, Bb, Cc, &c. at right angles to AL. Then, if the number of ordinates be sufficiently great, the portions of the curve ab, bc, &c. may be considered as straight lines without any material error. Draw the diagonals Ba, Bc, Dc, &c., and put $Aa = a$, $Bb = b$, &c., and $AB = BC =$ &c. $= \delta$. Then area $AaB = \frac{1}{2}a\delta$; area of trapezium $Babc = b\delta$; area $BcD = c\delta$; and so on.

Also, the centre of gravity of the trapezium $Babc$ is evidently in the line Bb, because it may be divided into two triangles by the diagonal ac; and ac is bisected by Bb, and may be considered as the common base of the triangles. The centre of gravity of the triangle BcD is in Cc; and so on. Hence the figure Al is divided into the following areas:

$$\tfrac{1}{2}a\delta, \quad b\delta, \quad c\delta, \ldots\ldots\ldots \quad k\delta, \quad \tfrac{1}{2}l\delta$$

and the distances of their centres of gravity from the line Aa are, respectively,

$$\tfrac{1}{3}\delta, \quad \delta, \quad 2\delta, \ldots\ldots \quad (n-1)\delta, \quad n\delta - \tfrac{1}{3}\delta.$$

Multiplying each of these areas by the distance of its centre of gravity from Aa, the sum of the moments,

$$\tfrac{1}{6}a\delta^2 + b\delta^2 + 2c\delta^2 \ldots\ldots + (n-1)k\delta^2 + \tfrac{1}{2}nl\delta^2 - \tfrac{1}{6}l\delta^2$$
$$= \delta^2[b + 2c + 3d \ldots\ldots + (n-1)k + \tfrac{1}{2}nl] + \tfrac{1}{6}\delta^2(a - l).$$

Let M = area of the figure Al, and x_1 = distance of the centre of gravity from Aa, then the sum of the moments found above $= Mx_1$, and (Mens. p. 433) $M = \delta[b + c + d \ldots + \frac{1}{2}(a + l)]$.

Hence $x_1 = \delta \dfrac{b + 2c + 3d \ldots\ldots + (n-1)k + \frac{1}{2}nl + \frac{1}{6}(a - l)}{b + c + d \ldots\ldots + \frac{1}{2}(a + l)}$.

83. *Cor.*—In like manner, if the figure $AalL$ be a solid, and a, b, c, &c. represent the areas of the transverse sections, we shall have the same expression for the distance of the centre of gravity from the plane Aa.

Scholium.

84. The preceding method of finding the centre of gravity of any area or solid is extremely simple, and sufficiently accurate for all practical purposes. If, however, we suppose the arcs abc, cde, &c. to be portions of parabolas, as in the Mensuration (see note, p. 433), we shall obtain a more correct result, without much additional labour of calculation. In this case, the base AL must be divided into an *even* number of equal parts; if, then, ac be joined, we shall have

area of the rectilineal triangle $Bac = \frac{1}{2}\delta\,(a + c)$.

Area formed by the parabolic arc abc and the chord ac

$= \frac{2}{3}$ of the circumscribing parallelogram

$= \frac{2}{3}[b - \frac{1}{2}(a + c)] \times 2\delta = \frac{2}{3}\delta\,(2b - a - c)$.

Hence the area $BabcB = \frac{1}{2}\delta(a + c) + \frac{2}{3}\delta\,(2b - a - c)$

$= \frac{1}{6}\delta\,(8b - a - c)$.

It is manifest, also, that the centre of gravity of this area is in the line Bb, since the centre of gravity of the triangle Bac is in this line, and all the double ordinates of the parabola parallel to ac are bisected by Bb. Hence we have the areas AaB, $BabC$, $BcD \ldots\ldots KlL$, respectively, equal to

$\frac{1}{2}a\delta$, $\frac{1}{6}\delta\,(8b - a - c)$, $c\delta$, $\frac{1}{6}\delta\,(8d - c - e) \ldots\ldots\ldots \frac{1}{2}l$;

and the distances of their centres of gravity from the line Aa are, respectively,

$\frac{1}{3}\delta$, δ, 2δ, 3δ, $\ldots\ldots\ldots\ldots$ $n\delta - \frac{1}{3}\delta$.

Taking the moments of these areas as above, we get

$\frac{1}{6}a\delta^2$, $\frac{1}{6}\delta^2(8b-a-c)$, $2c\delta^2$, $\frac{1}{6}\times 3\delta^2(8d-c-e)$, $\ldots\ldots$ $\frac{1}{2}nl\delta^2 - \frac{1}{6}l\delta^2$,

and the sum of these moments will be found equal to

$\frac{\delta^2}{3}(0 \times a + 1 \times 4b + 2 \times 2c + 3 \times 4d \ldots + (n - 1) \times 4k + nl)$.

Hence we shall find, for the distance of the centre of gravity from the line Aa

$$x_1 = \delta \frac{0 \times a + 1 \times 4b + 2 \times 2c + 3 \times 4d \ldots\ldots + (n-1) \times 4k + nl}{a + 4b + 2c + 4d \ldots\ldots + 4k + l}.$$

Problems for Practice.

1. If three equal bodies, considered as points, be placed in the three angles of a triangle, the centre of gravity of these bodies is the same as that of the triangle.

2. If G be the centre of gravity of the triangle ABC, and GA, GB, GC be joined, then $3(GA^2+GB^2+GC^2)=AB^2+AC^2+BC^2$.

3. In the last problem, three forces which are proportional to GA, GB, GC, will keep the point G in equilibrium.

4. If a and b be the two parallel sides of a trapezoid, and h the line which bisects these sides, then the centre of gravity of the trapezoid is in this line, and its distance from a in this line is $\frac{h}{3}\cdot\frac{a+2b}{a+b}$.

5. Four bodies, considered as points, whose weights are 3, 4, 5, and 6lbs., are placed at the successive angles of a square whose side is 12 inches; required the distance of the centre of gravity from the least body, both by construction and calculation.

6. If two spheres touch one another internally; to find the centre of gravity of the solid included between the two surfaces.

7. If two given cones have the same base; to find the centre of gravity of the solid included between their surfaces; first, when the two cones are on contrary sides of the base; secondly, when they are on the same side of it.

8. Seven equal bodies, considered as points, are placed in seven of the angles of a cube; required the distance of their common centre of gravity from the remaining angle.

9. If one of the sides of an isosceles right-angled triangle rest on a horizontal plane, and the other side is vertical, to find the greatest isosceles triangle which can be described on the hypothenuse as a base, so that the whole figure shall not fall.

10. If in a system consisting of any number of particles a point be taken, and if each particle be multiplied into the square of its distance from the point, the sum of these products will be the least when this point is the centre of gravity.

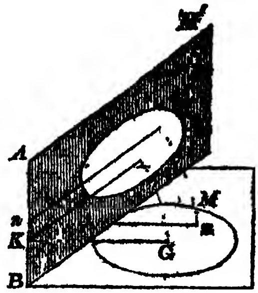
GULDINUS'S PROPERTIES.

85. Prop. VI.—*If any plane figure revolve about an axis in its own plane, the content of the solid generated by this figure in its revolution through any angle is equal to a prism whose base is the revolving figure, and its height the length of the path described by the centre of gravity of the plane figure.*

Also, the area of the superficies generated by the perimeter of this figure is equal to a rectangle whose base is the perimeter, and its height the length of the path described by the centre of gravity of the perimeter.

(1). Let M be the given area, G its centre of gravity, and AB the axis of revolution in the plane of the figure. Let the whole of the

figure M be on one side of the axis, and let it revolve through an angle θ into the position M'. Draw GK perpendicular to AB, and put $GK = k$, then the arc GG', described by the point G, is $= k\theta$. Now, if we suppose the area M to be composed of an indefinite number of elementary portions $m, m' m''$, &c., and x, x', x'', &c. to be the distances of these portions from the axis AB, then the arcs described by these elements will be $x\theta, x'\theta, x''\theta$, &c.; therefore the solids described by m, m', m'', &c. are evidently $mx\theta, m'x'\theta, m''x''\theta$, &c. But these make up the whole solid generated by the area M. Hence this solid



$$= mx\theta + m'x'\theta + m''x''\theta + \&c. = \theta(mx + m'x' + m''x'' + \&c.)$$
$$= \theta . M . k = M \times GG';$$

and, therefore, is equal to the prism whose base is the figure M and altitude GG'.

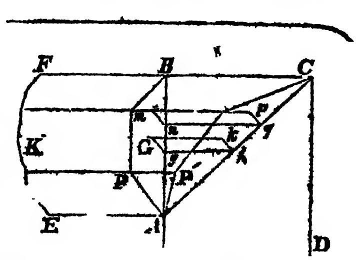
(2). In like manner, if m, m' represent indefinitely small portions of the perimeter M, we shall have the surface generated by each particle equal to $mx\theta, m'x'\theta$, &c., and, therefore, it may be shown, as before, that the whole surface = a rectangle, whose base is M and altitude GG'.

86. *Ex.*—Let the figure be a circle which revolving round an axis without it generates a solid resembling the ring of an anchor. The centre of the circle is the centre of gravity, both of the area and the perimeter. Hence the solid content of the ring = a cylinder whose base is the revolving circle, and altitude the circumference described by the centre of the circle. And the surface of the ring is equal to the surface of this cylinder.

The following proposition is analogous to Guldinus's properties, and will be found useful in estimating the contents of uniform embankments, ramparts, &c., when they vary their direction.

87. Prop. VII.—*The solid content of any rampart is equal to a prism whose base is the transverse section of the rampart, and height the length of the path described by the centre of gravity of the transverse section. (The path being supposed to continue parallel to the ditch round the angles.)*

Let $EFCD$ represent a portion of the rampart, C of one the salient angles; and let CA bisect the angle C. Let APB be the profile or transverse section perpendicular to AE or BF, and $AP'C$ a vertical section passing through AC. Now the proposition is obviously true for the portion $AEFB$, which is a prism whose base is the transverse section and height GK. And the solid $APBCP'A$ is, from the construction, a portion of the same prism, cut obliquely by the section $AP'C$. Let m be any indefinitely small portion of the area APB; through m draw



the vertical plane $mnqp$ perpendicular to AB, and intersecting the horizontal plane ABC in the line nq; draw mp parallel to nq, then $mnqp$ is evidently a parallelogram, and mp perpendicular to the plane APB. The solid which has m for its base and altitude $mp = m \times mp = m \times nq = m \times An \tan\theta = mx \tan\theta$. Now, if we suppose the area APB to be composed of the elementary portions m, m', m'', &c., and each portion to be multiplied by its altitude, these will make up the whole solid $AP'CBP$, and therefore this solid

$$= mx \tan\theta + m'x' \tan\theta + \&c. = \tan\theta\,(mx + m'x' + \&c.)$$
$$= \tan\theta \,.\, M \,.\, Ag = M \,.\, gh = M \,.\, Gk.$$

But the solid $AEFB = M \,.\, GK$, therefore the whole solid $AEFC = M \,.\, Kk$.

88. *Ex.*—Let the dimensions of the transverse section of the rampart of a square fort be the same as those given in question 43, p. 143; and let $OA = 50$ feet, O being the centre of the fort; to find the breadth of the ditch at top and bottom, so that the earth thrown out of the ditch may be just sufficient to make the parapet and glacis, when its bulk, after being excavated, is increased in the ratio of 10 to 9,

The area of the section of the parapet will be found		$=$	103·5 sq. ft.
do.	of the glacis	$=$	12·375 ,,
do.	of the ditch	$=$	$16x-32$,,

Also the distance of

G,	the centre of gravity of the parapet from O		$=$	63.6 feet.
L	do.	glacis	$=$	$84+2x$.
R	do.	ditch	$=$	$78+x$.

Now, the perimeters described by the points G, L, and R are evidently, in this case, $8 \times OG$, $8 \times OL$, and $8 \times OR$. Hence, dividing by 8, we have, by the proposition,

$$63{\cdot}6 \times 103{\cdot}5 + (84 + 2x) \times 12{\cdot}375 = \tfrac{10}{9} \times (78 + x)(16x - 32);$$
$$\text{or,}\qquad 128x^2 + 9550x = 74867{\cdot}5,$$

from whence x will be found $=$ 7·1 feet.

CHAP. IV.—THE MECHANICAL POWERS.

89. The *mechanical powers* are the most simple instruments used for the purpose of supporting weights, or communicating motion to bodies.

These powers are generally reckoned six in number,—the *lever*; the *wheel and axle*, including the *toothed wheel*; the *pulley*; the *inclined plane*; the *wedge*; and the *screw*.

The three first, in the state of equilibrium, may be reduced to the lever, and the three last may be reduced to the inclined plane; so that, strictly speaking, we cannot consider that there are more than two simple machines.

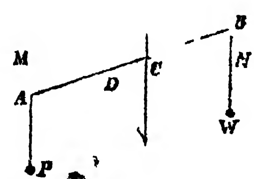
90. When two forces act upon each other by means of any machine, one of them is, for the sake of distinction, called the *power*, and the other the *weight*. The weight is the resistance to be overcome, or the effect to be produced; the power is the force, of whatever nature, which is employed to overcome that resistance, or to produce the required effect.

I. THE LEVER.

91. A *lever* is an inflexible rod moveable in one plane, about a point which is called the *fulcrum* or centre of motion. The portions of the lever into which the fulcrum divides it are called the *arms* of the lever. When the arms are in the same straight line, it is called a *straight* lever; in other cases a *bent* lever.

92. Levers are usually divided into three kinds. In levers of the first kind the fulcrum is situated between the power and the weight, as in steelyards, scissors, pincers, &c. Levers of the second kind have the weight between the power and the fulcrum, as in a pair of nutcrackers, and in the oars of a boat, where the water is considered as the fulcrum. In levers of the third kind, the power is between the weight and the fulcrum, as in tongs, sheep-shears, &c. The bones of animals are generally considered as levers of the third kind, the joint being the fulcrum, the muscle fixed near the joint being the power, and the force exerted by the limb at a greater distance from the joint being the weight.

93. Prop. I.—*To find the conditions of equilibrium, when two forces act in the same plane upon a lever.*



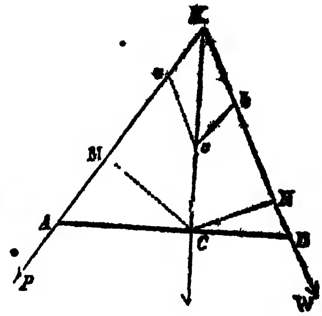
1. Let ACB be a lever, moving about the fulcrum C in the plane $PABW$, and let P, W be two weights or forces acting upon the arms of the lever in this plane, in the directions AP, BW; and, first, let us suppose that these directions are parallel. Now it is evident that, if the resultant of these two forces passes through the fulcrum C, there will be an equilibrium, since the point C is fixed; but that if it passes through any other point in AB, as D, the single force at D will be unsupported, and will make the lever move round C in the direction of this force. Hence it appears, from art. 22, that the line AB must be divided in C, so that

$$P : W :: CB : CA :: CN : CM,$$

the line MCN being drawn through C perpendicular to the directions of the forces.

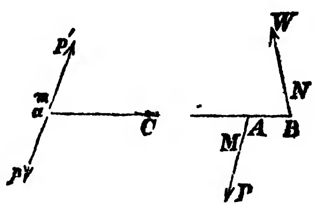
2. But if the forces P and W are inclined to each other, let the directions of these forces meet each other in the point K. Suppose this point to be rigidly connected with AB, then we may conceive the forces to be applied at the point K, instead of the points A and B. Hence the resultant of these two forces will pass through K. But when there is an equilibrium it must also pass through the point C, as in the former case, and therefore KC will be the direction of the resultant.

If, therefore, Kc be taken to represent the pressure on the fulcrum, and the parallelogram $Kacb$ be completed, Ka, Kb will represent the two forces P and W. Draw CM, CN, perpendicular to AK, BK, then we have



$$P : W :: Ka : Kb \text{ or } ac$$
$$:: \sin Kca \text{ or } \sin bKc : \sin aKc$$
$$:: CN : CM.$$

3. Let the forces P, W act on the same side of the fulcrum to turn the lever in opposite directions. Produce AC to a, so that $Ca = CA$, and let two forces p, p', equal and parallel to the force P, be applied at the point a in opposite directions, then no change will be made in the conditions of equilibrium. But it is evident that the forces P and p will balance each other, because they are equal and parallel, and at equal distances from the fulcrum. Also, the forces p' and W will balance each other, when



$$p' : W :: CN : Cm; \text{ or,}$$
$$P : W :: CN : CM.$$

But if the four forces P, W, p, p' be in equilibrium, and p, p' counteract each other, the two forces P, W must balance each other. Hence also, in this case, when P and W are in equilibrium,

$$P : W :: CN : CM.$$

94. *Cor.* 1.—If $CA = a$; $CB = b$, the angle $CAP = \alpha$, $CBW = \beta$, there will be an equilibrium, when

$$Pa \sin \alpha = Wb \sin \beta.$$

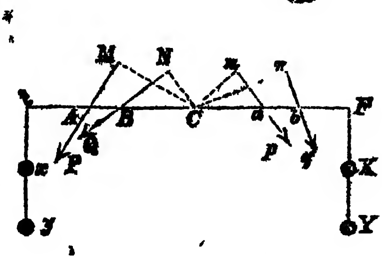
95. *Cor.* 2.—[illegible]his proposition is equally true in bent levers of any form; for we m[illegible]ose a rigid li[illegible] passing through the fulcrum to meet the direc[illegible]e forces, and t[illegible]rces to be transferred to the extremities of this line.

96. Prop. II.—*If any number of forces*, P, Q, &c.; p, q, &c., *acting upon the arms of a lever to turn it in opposite ways, be such that*

$$P \times CM + Q \times CN + \&c. = p \times Cm + q \times Cn + \&c.$$

there will be an equilibrium.

At any points F, f in the lever Aa, let the forces X, Y, &c.; x, y, &c., act perpendicularly on Ff, to turn the lever in opposite directions; and let these forces be such that X would balance P; Y would balance Q, &c.; and also that x would balance p; y balance q, and so on. We have then, from art. 93,



$$P \times CM = X \times CF; \quad Q \times CN = Y \times CF; \text{ \&c.,}$$

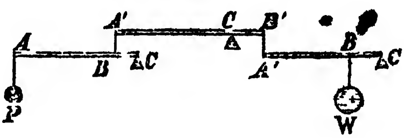
$$\therefore \quad P \times CM + Q \times CN + \text{\&c.} = X \times CF + Y \times CF + \text{\&c.}$$
$$= (X + Y + \text{\&c.}) \times CF.$$

In like manner, $p \times Cm + q \times Cn + \text{\&c.} = (x + y + \text{\&c.}) \times Cf.$
But, by hypothesis,

$$P \times CM + Q \times CN + \text{\&c.} = p \times Cm + q \times Cn + \text{\&c.}$$
$$\therefore \quad (X + Y + \text{\&c.}) \times CF = (x + y + \text{\&c.}) \times Cf;$$

hence the force $X + Y +$ &c. acting at F will balance the force $x + y +$ &c. acting at f (art. 93). But the forces P, Q, &c. balance the former force, and the forces p, q, &c. the latter force; hence it follows that the forces P, Q, &c. will balance p, q, &c.

97. Prop. III.—*In any compound lever the power and weight are in equilibrium, when they are to each other as the continual product of the alternate arms, commencing from the weight, to the continual product of the alternate arms, commencing from the power; the arms being supposed to act perpendicularly upon each other.*



Let the power P act at the extremity of the arm CA; this will produce a pressure at the point B, which we may call Q. Again, the pressure Q acting at B or A' will produce a pressure at B', which call R. Lastly, the pressure R acting at A'' will support the weight W.

Let $CA = a$, $C'A' = a'$, $C''A'' = a''$; also, $CB = b$, $CB' = b'$, $C''B'' = b''$. We have then (art. 94),

$$Pa = Qb; \qquad Qa' = Rb'; \qquad Ra' = Wb'';$$

and multiplying the corresponding terms of these equations together, and omitting the common multipliers, Q and R, we get

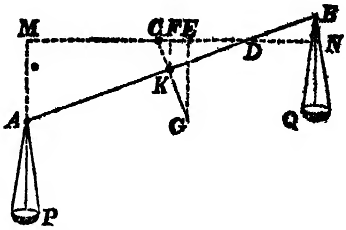
$$Paa'a'' = Wbb'b''. \quad \text{Hence}$$
$$P : W :: bb'b'' : aa'a''.$$

98. *Scholium*.—The machine for determining the weights of carriages and waggons is formed of a composition of levers. The waggon is placed upon a rectangular platform, on a level with the road, which rests at its angles upon a system of four levers, whose fulcra are fixed in solid masonry, a short distance beyond the angular points. These levers converge in the directions of the diagonals towards the centre of the rectangle, and there rest upon another lever, which has its fulcrum near the centre, and passes under the platform. The opposite extremity of this lever is in the weighing-house, where the power is placed for determining the weight of the waggon.

99. The common balance consists of a lever called the beam, having equal arms, from the ends of which the scales are suspended. The requisites of a good balance are the following:—1st. When loaded with equal weights the beam should be perfectly horizontal. 2d. When there is a very slight difference between the weights, the beam should incline per-

ceptibly from the horizontal position, or it should have great *sensibility*. 3d. When the balance is disturbed, it should quickly return to a state of rest; that is, it should have great *stability*.

100. PROP. IV.—*To find how the requisites of a good balance may be obtained.*



Let P and Q be the weights in the scales; A, B the points of suspension; C the fulcrum, or the axis round which the beam revolves; W the weight of the beam and scales; G their common centre of gravity. Also let $AB = 2a$, $CG = h$, $CK = k$, $MDA = \theta$, the angle which the beam makes with the horizon when the whole is in equilibrium. Now, from art. 96, we have

$$P \times CM = Q \times CN + W \times CE,$$

and, since $AK = KB$, therefore $MF = FN = a\cos\theta$. Also, $CE = h\sin\theta$; and $CF = k\sin\theta$; hence

$$P(a\cos\theta - k\sin\theta) = Q(a\cos\theta + k\sin\theta) + Wh\sin\theta,$$

from whence we obtain, by reduction,

$$\frac{\tan\theta}{P - Q} = \frac{a}{(P + Q)k + Wh}.$$

This determines the position of equilibrium.

The first requisite—that the beam shall be horizontal when P and Q are equal—will be satisfied if the arms are equal, and the centre of gravity G lower than the centre of suspension.

To obtain the second requisite, it is evident that for a given difference of P and Q, the sensibility is greater in proportion as $\tan\theta$ is greater; and for a given value of $\tan\theta$, the sensibility is greater in proportion as $P - Q$ is less; hence $\frac{\tan\theta}{P - Q}$ may be considered as the measure of the sensibility; and therefore the second requisite is fulfilled by making $\frac{a}{(P + Q)k + Wh}$ as great as possible. Hence it appears that,

(1). The sensibility is increased by increasing the lengths of the arms.

(2). The sensibility is increased by diminishing the weight of the beam and scales.

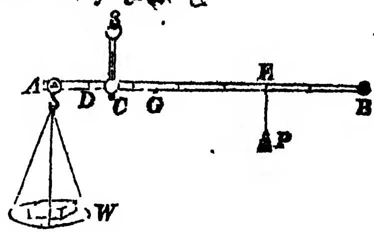
(3). The sensibility is increased by diminishing the distance between the centre of motion and the centre of gravity; and also the distance of the centre of motion from the line joining the points of suspension.

The stability is proportional to the force, which, at a given angle of inclination, tends to restore the equilibrium when it is destroyed. Suppose $P = Q$, then P and Q may be placed at the middle point between A and B, and this force is $[(P + Q)k + Wh]\sin\theta$. Hence, to satisfy the third requisite, this must be made as great as possible. This appears to be incompatible with the second requisite. The

stability of a balance, however, is of less importance than the sensibility, since the eye can judge sufficiently well whether the index of the beam makes equal oscillations on each side of the vertical line; that is, whether the position of rest would be horizontal.

101. Prop. V.—*To show how the common steelyard must be graduated.*

The common or Roman steelyard is a lever with arms of unequal length, by means of which a single weight P is sufficient to determine, from its position, the weight of any other body.



Let AB be the beam of the steelyard, and C its fulcrum. The body W, whose weight is to be found, is suspended at the extremity A of the shorter arm, and the constant weight P is moved along the graduated arm until there is an equilibrium. Let w be the whole weight of the beam, with its hooks, &c., and G its centre of gravity. Suppose that when W is removed, the weight P placed at D would keep the beam horizontal; then, since we may suppose the whole weight of the beam, &c. to be collected at G, we shall have $P \times CD = w \times CG$. Now let P, placed at E, balance the weight W at A; then W balances P at E, together with the beam. Hence

$$W \times CA = P \times CE + w \times CG = P \times CE + P \times CD;$$

and if $CA = a$, $DE = x$, then $Wa = Px$. Hence, if we take $x = a$, $x = 2a$, $x = 3a$, &c., successively, we shall have the corresponding weights of W equal to P, $2P$, $3P$, &c.

Scholium.—In the Roman steelyard, the weight is proportional to the distance of P from the point D, and therefore its graduation is a scale of equal parts. But in the Danish steelyard the fulcrum is moveable and the weight P is fixed. In this case it may easily be shown that the distances of the divisions from A will increase in harmonical progression.

PROBLEMS.

102. Prob. I.—*To determine the weight of a body when the arms of a balance are unequal.*

Let the body be weighed at both ends of the balance; and let the apparent weights, when it is suspended at A and B, be a and b respectively; also, let the true weight of the body be x. Then

$$a : x :: AC : BC; \quad \text{and} \quad x : b :: AC : BC;$$

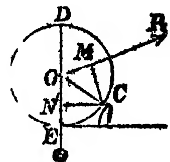
$$\therefore \quad a : x :: x : b;$$

that is, x is a mean proportional between a and b.

Scholium.—The true weight of a body, however, may be readily found with a false balance, independently of calculation. For this purpose, place the body in one scale, and accurately balance it with fine

sand in the other. Then take out the body, and replace it with known weights which will restore the equilibrium. These will evidently be the true weight of the body, independently of any error in the balance.

103. Prob. II.—*To find the force necessary to draw a carriage wheel over an obstacle, supposing the weight of the carriage to be collected at the axis of the wheel.*



Let O be the centre of the wheel, C the obstacle, P the power acting in the direction OP, W the weight of the load acting in the direction OW. Draw CM, CN perpendicular to OP, OW. Then, if the wheel turn over the obstacle, it must turn round the point C; therefore OC may be considered as a lever whose fulcrum is C, and CM, CN are the perpendiculars in the directions of the power and the weight. Hence

$$P : W :: CN : CM :: \sin CON : \sin COM.$$

If, therefore, W and the angle CON be given, P will be the least when $\sin COM$ is greatest, or when COM is a right angle. Also, if W and $\sin COM$ be given, P will vary as $\sin CON$; and if $OE = r$, $NE = h$,

$$\sin CON = \frac{CN}{CO} = \frac{\sqrt{2rh-h^2}}{r} = \sqrt{\left(\frac{2h}{r} - \frac{h^2}{r}\right)} = \sqrt{\frac{2h}{r}}, \text{ nearly,}$$

since h is generally very small compared with r. Hence the power, *ceteris paribus*, varies inversely as the square root of the radius of the wheel.

Problems for Practice.

1. On a lever three feet long a weight of 100lbs. is suspended at one extremity, and $2\frac{2}{3}$ inches from this end is placed a fulcrum; what weight at the other end will preserve the equilibrium? Ans 8lbs.

2. The arms of a bent lever, ACB, of equal length, make an angle at the fulcrum of 135°. To find the position in which the lever will rest when two weights of 6 and 10lbs are suspended at A and B.
Ans. CA makes an angle of 8° 37′ with the horizon.

3. To find the weight of the greater body, when 6lbs. is suspended at A, and CA is horizontal. Ans. 8·485lbs.

4. If the weights 1, 2, 3lbs. be suspended at the distances of 6, 12, and 18 inches from the fulcrum on one arm of a straight lever, and 2, 3, 4lbs. be placed at the distances of 4, 10, and 12 inches from the fulcrum on the other arm; to find where a weight of 1lb. must be placed so as to keep the lever in equilibrium.
Ans. 2 inches from the fulcrum on the first arm.

5. Two men carrying a burthen of 200lbs. weight between them, hung on a pole, the ends of which rest on their shoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet?
Ans. 125lbs. and 75lbs.

6. A piece of timber, 24 feet long, being laid over a prop, is found

to balance itself when the prop is 10 feet from the greater end; but, removing the prop to the middle of the beam, it requires a man's weight of 200lbs standing on the less end, and also a weight of 20lbs. at a distance of 4 feet from this end, to keep it in equilibrium: required the weight of the tree. Ans. 1280lbs.

7. The arms of a bent lever are as 10 to 7, and they are inclined to each other at an angle of 112° 30′; to find the weight which, suspended at the end of the shorter arm, will balance 35lbs at the end of the longer arm, when the inclination of the longer arm to the horizon is twice as great as the inclination of the shorter. Ans. $50\sqrt{2-\sqrt{2}}$ lbs.

8. *ACB* is an arc of 120°, situated in a vertical plane, and resting on a horizontal plane, with its concavity upwards; to find its position when two weights of 8 and 12lbs. suspended at the two extremities of the arc keep it in equilibrium (the arc being supposed to be without weight).

Ans. It rests on a point 79° 7′ from the less weight.

9. To divide the beam of a steelyard of uniform thickness, so that the weights 1, 2, 3, &c. lbs. on the one side shall balance a constant weight of 95lbs. at the distance of two inches, on the other side of the fulcrum; the weight of the beam being 10lbs, and its whole length 36 inches. Ans. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, &c. inches.

10. To graduate a Danish steelyard of the same dimensions as the last, that is, to find the positions of the fulcrum when a given weight of 4lbs. at one end of the beam shall balance 1, 2, 3, &c lbs. at the other end. Ans. $14\frac{2}{5}$, $15\frac{3}{4}$, $16\frac{16}{17}$, 18, $18\frac{18}{19}$, &c. inches.

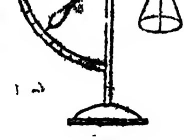
11. A person purchased 2lbs of tea, and suspecting some fraud in the balance, he had 1lb. nominally weighed in one of the scales, and the 2nd lb. in the other scale. How much did he gain or lose by the bargain, supposing the lengths of the arms to be as a to b (or 6 : 5)?

Ans. He gained $\frac{8}{11}$ of an ounce.

12. A ladder is 50 feet long, and weighs 120lbs.; and a person wishing to raise it, places one end against a wall, and then lifts it gradually upwards from the other end. Required the force which he will have to exert at any point of the ladder, supposing that his hands and feet are equally distant from the foot of the ladder; find also the pressure against the wall?

13. In the weighing machine for carriages, suppose the distances of the four points of support from the fulcra of the lever to be each 1 foot, and the length of each of the four levers 10 feet; also, suppose the distance from the fulcrum of the great lever, to the point where it supports the extremities of the other four levers to be $\frac{1}{10}$th of the distance from the fulcrum to the extremity of the lever in the weighing-house: required the weight on the platform which 400lbs. would sustain?

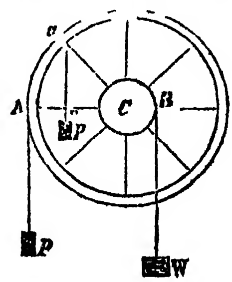
14. In the *bent lever balance*, the arm is terminated by a heavy knob, which moves along a graduated arc of a circle, and from the extremity of the other arm a scale pan is suspended, which contains the substance to be weighed. Show how the arc must be graduated.



II. THE WHEEL AND AXLE, AND TOOTHED WHEELS.

104. The wheel and axle consists of a cylinder moveable about its axis, and a circle so attached to the cylinder that the axis of the cylinder passes through its centre and is perpendicular to its plane. The power is applied at the circumference of the wheel, usually in the direction of a tangent to it, and the weight is raised by a rope which winds round the axle in a plane at right angles to the axis.

105. PROP. VI.—*The wheel and axle are in equilibrium when the power is to the weight as the radius of the axle is to the radius of the wheel.*



The effort of the power to turn the machine round the axis must be the same at whatever point in the axle the wheel is fixed. Suppose it to be placed in such a situation that the power and weight may act in the same plane, and let CA, CB, be the radii of the wheel and axle, at the extremities of which the power and weight act; then ACB may be considered as a lever, whose fulcrum is C; and since AP, BW are perpendicular to CA, CB, we have

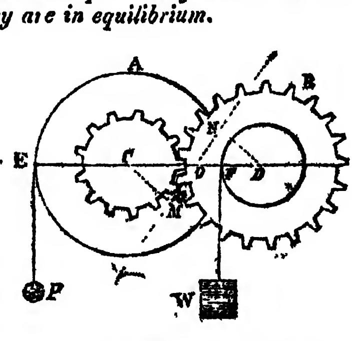
$P : W :: CB : CA ::$ rad. of axle : rad. of wheel.

106. *Cor.* 1.—The power may act by means of a bar, and the wheel may be removed, as in the case of the *capstan* and the *windlass*.

107. *Cor.* 2.—If the power act in the direction ap, draw CE perpendicular to ap, and there will be an equilibrium when $P : W :: CB : CE$.

108. DEF.—If two circles, moveable about their centres, have their circumferences indented or cut into equal teeth all the way round, and be so placed that their edges touch, one tooth of one circumference lying between two of the other; then, if one of them be turned round by any means, the other will be turned round also. Such circles are called *toothed wheels*.

109. PROP. VII.—*In toothed wheels it is required to find the relation between the power and weight when they are in equilibrium.*



Let the wheel A act upon the wheel B at Q, the pressure there exerted will be perpendicular to the surfaces which are in contact at that point; for if this were not the case, the pressure might be resolved into two forces, the one perpendicular to the tangent and the other in the direction of the tangent; but the latter force is not at present taken into consideration. Let the action of A on B be a pressure Q in the direction MQN; then the force Q acting on the wheel B supports the weight W. Also the re-action of B on A will be equal and opposite

to Q; and this is supported by the power P. Hence, if CM, DN be perpendicular to MQN, we shall have (art. 107)

$$P \times CE = Q \times CM; \qquad W \times DF = Q \times DN;$$

$$\therefore \quad P \times CE : W \times DF :: CM : DN; \text{ that is,}$$

$$\text{moment of } P : \text{moment of } W :: CM : DN.$$

110. *Cor.* 1.—If CD meet MN in O, we have, by similar triangles, $CM : DN :: CO : DO$; and if the teeth be small, in comparison to the radii of the wheels, CO, DO will be nearly equal to these radii. Hence

$$\text{moment of } P : \text{moment of } W :: \text{rad. of } A : \text{rad. of } B.$$

111. *Cor.* 2.—Since the intervals of the teeth in the two wheels must be equal, the numbers of teeth will be as the circumferences, and therefore as the radii; hence

$$\text{mom. of } P : \text{mom. of } W :: \text{no. of teeth of } A : \text{no. of teeth of } B.$$

112. *Cor.* 3.—In a combination of toothed wheels, there will be an equilibrium when P is to W as the product of the radii of all the axles to the product of the radii of all the wheels.

113. *Scholium.*—In the construction of the teeth of wheels several things must be attended to, in order to ensure due efficiency in the work. (1). The teeth of one wheel should press in a direction perpendicular to the radius of the other wheel. (2). As many teeth as possible should be in contact at the same time, in order to distribute the pressure amongst them, and thus diminish the pressure upon each tooth. (3). The surfaces of the teeth, in working, should not rub one upon another, and should suffer no jolt, either at the commencement or termination of their mutual contact. All these advantages will be obtained, if the form of the teeth be the curve called the invollute of the circle. The force also, in this case, will continue constant during the motion. Wheels are sometimes turned by simple contact with each other, and sometimes by the intervention of cords, straps, or chains passing over them. In such cases the friction of the surfaces prevents their sliding on each other, and a mutual action takes place similar to that which is supposed in the proposition.

Examples.

1. In the wheel and axle, the radius of the wheel is 15 inches, and the radius of the axle 3 inches; required the power necessary to balance a weight of 180lbs. Required the power also when the thickness of a rope is 1 inch, the power and the weight being supposed to be applied in the axes of the ropes.

2. Let the radii of four axles be 2, 4, 5, and 3 inches, and the radii of the wheel 10, 12, 8, and 15 inches respectively; required the weight which will be balanced by a power of 30lbs.

3. A power of 5lbs. balances a weight of 150lbs. upon a wheel whose diameter is 10 feet; find the diameter of the axle, the thickness of the rope being 2 inches.

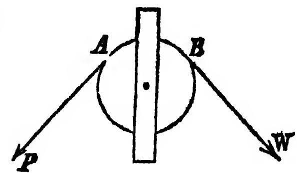
4. The number of teeth in each of three successive wheels is 48, and the number of teeth in each of the pinions or axles is 6; required the weight supported by a power of 20lbs.

III. THE PULLEY.

114. A *pulley* is a small wheel moveable about its centre, in the circumference of which a groove is formed to admit a rope or flexible chain. The pulley is said to be fixed or moveable, according as the centre of motion is fixed or moveable.

115. Prop. VIII.—*In the single fixed pulley there is an equilibrium when the power and weight are equal.*

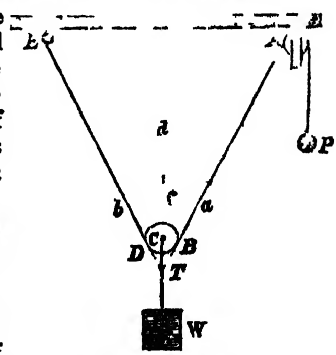
Let a power P and a weight W be equal, and act against each other by means of a perfectly flexible rope, which passes over the fixed pulley AB; then it is obvious that whatever force is exerted by P in the direction PAB, an equal force is exerted by W in the opposite direction WBA, consequently these equal and opposite forces must be in equilibrium.



116. *Cor.*—The great advantage arising from the fixed pulley is, that the force may easily be changed from one direction to another.

117. Prop. IX.—*In the single moveable pulley there is an equilibrium when the power is to the weight as radius to twice the cosine of the angle which either string makes with the direction in which the string acts.*

A cord fixed at E passes under the moveable pulley DB and over the fixed pulley A, and at the extremity of the cord the power P is applied. The weight W is annexed to the centre C of the moveable pulley. Let the cords AB, ED, be produce to meet in T; join CT.



Because the power acts freely along the cord $ABDE$, the tension of this cord must be everywhere the same. And since the forces or tensions at B and D act in the directions BA, DE, they will produce the same effect as if they acted at T (art. 10); and these forces are equal, each being equal to P. Hence, at the point T, there are three forces which are in equilibrium, viz., the two tensions P, P, and the weight W. Let Ta, Tb represent the two tensions, and complete the parallelogram $Tadb$; then Td is equivalent to the forces Ta, Tb, and therefore dT will be proportional to W, and will be vertical, or in the direction TW. Also, it is evident that Td bisects the angle aTb; call the angle 2α. Then

$$P : W :: Ta : Td \text{ or } 2Tc :: \text{rad.} : 2\cos\alpha.$$

118. *Cor.*—When the strings are parallel, $\alpha = 0$, and $\cos\alpha =$ radius. In this case the pulley C is supported by two equal and parallel forces, and the force W, which acts in the opposite direction and keeps them in equilibrium, is equal to their sum. Hence $W = 2P$.

119. Prop. X.—*In a system where the same string passes round all*

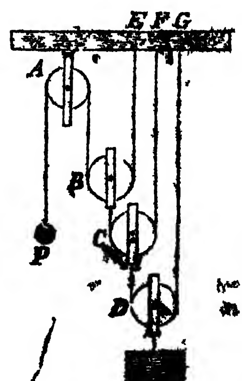
the pulleys, and the parts of the string between the pulleys are parallel, there is an equilibrium when P : W :: 1 : n; n *being the number of strings at the lower block.*

Since the same string passes round all the pulleys, its tension will be every where the same, and equal to the power P. And since each of the strings supports a weight P, they will altogether, supposing them parallel, support a weight nP. Hence $W = nP$.

120. *Cor.*—When the strings are not vertical, the force must be resolved into two, the one vertical and the other horizontal; and the vertical forces alone must be taken.

121. *Scholium.*—In the arrangement of the system in the adjoining diagram, the pulleys are placed below each other, which prevents the weight from being raised to within a considerable distance of the point of support. To remedy this inconvenience they are sometimes placed in separate sheaves by the side of each other; but in this plan it is difficult to keep the strings parallel to each other, and the blocks in their proper places. From the oblique action, also, of the ropes upon the pulleys, they are subject to greater friction and to more wear.

An ingenious contrivance was made by *White* to remove the effects of friction. It will easily be seen, in the figure above, that, if the system be put in motion, whilst 1 inch of cord passes over the pulley A, 2 inches pass over the pulley B, 3 over C, 4 over D, and so on. Hence, if in the solid block A, grooves be cut whose radii are 1, 3, 5, &c., and in the block B, grooves whose radii are 2, 4, 6, &c.; and a string be passed round them, the grooves will answer the purpose of so many distinct pulleys, and every point in each moving with the same velocity as the string in contact with it, the whole friction will thus be moved to the two centres of motion in the blocks A and B. There is great difficulty, however, in making the radii of the wheel exactly of the proper proportion, particularly as the radius of the cord must be added to the wheel; and any deviation from this rule tends to destroy all the advantages which this arrangement appears to offer.



122. Prop. XI.—*In a system where each pulley hangs by a separate string, and the strings are parallel,* P : W :: 1 : 2^n; n *being the number of moveable pulleys.*

In this system a string passes over the fixed pulley A, and under the moveable pulley B, and is fixed at E. Another string is fixed at B, passes under the moveable pulley C, and is fixed at F; and so on.

From art. 122 it appears, when there is an equilibrium, that

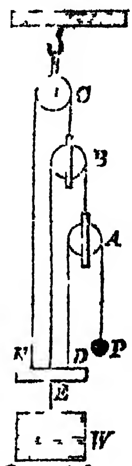
the weight at $B = 2P$.

In like manner,

the weight at $C = 2$ weight at $B = 2^3P$,
the weight at $D = 2$ weight at $C = 2^4P$;
and, if the number of pulleys be n, $W = 2^n P$.

123. *Cor.*—When the strings are not parallel, the power is to the weight, in each case, as the radius to twice the cosine of the angle the string makes with the direction in which the weight acts.

124. Prop. XII.—*In a system of pulleys where the strings are parallel, and each string is attached to the weight,* P : W :: 1 : $2^n - 1$; n *being the number of pulleys.*



2 + 2(n-1)

In this system each string, as PAD, supports the weight partly by its action at D, where it is attached, and partly by its pressure on the next string AB. The weight supported by the string DA is equal to the power P; also the pressure downwards upon A, or the weight which the string AB sustains, is $2P$, therefore the string EB sustains $2P$; and so on. Hence the whole weight sustained is $P + 2P + 4P + \ldots$ to n terms, or

$$W = (1 + 2 + 2^2 \ldots + 2^{n-1})P = (2^n - 1)P.$$

125. *Scholium.*—The two last systems of pulleys, as well as some other arrangements which we have not described, are seldom used. Pulleys are applied not only to overcome resistances, but to produce continued motion. Now, when the weight ascends through a very small space in the last figure, the pulley A, which carries the power, will soon be brought down, and encumbered; and it will then become useless, almost before the resistance has been perceptibly overcome. For this reason, the class of pulleys described in art. 119 is the only arrangement which has been found generally useful in practice; although the friction is greater, and the power considerably less than in the other systems.

Examples.

1. Find the proportion between the power and the weight (in art. 119) when the weight of the pulleys is taken into consideration.

2. If w be the weight of each pulley in art. 122, prove that $W = 2^n P - (2^n - 1)w$.

3. If w be the weight of each pulley in art. 124, prove that $W = (2^n - 1)P + (2^n - n - 1)w$.

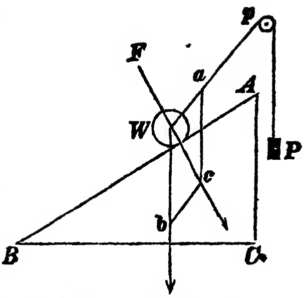
4. Required the tensions of each of the strings in arts 122, 124.

IV. THE INCLINED PLANE.

126. Prop. XIII.—*When a body* W *is sustained upon a plane which is inclined to the horizon,* P : W :: sin α : cos ϵ; α *being the inclination of the plane to the horizon, and* ϵ *the angle which the string makes with the plane.*

Let BC be parallel to the horizon, BA a plane inclined to it; W a body sustained upon the plane by a power acting in the direction WP.

Draw Wb perpendicular to BC, and take Wa, Wb to represent the two forces P and W: complete the parallelogram $Wbca$. Now, since P and W are in equilibrium, their resultant must be perpendicular to the plane AB, and will be supported by the reaction of the plane; for, if the resultant were in any other direction, it might be resolved into two forces, one perpendicular to the plane, and the other in the direction of the plane; and since the plane is supposed to be perfectly smooth, this last force would cause the body to slide along AB, and W would not be at rest. Hence the three forces, the power P, the weight W, and the pressure R, will be represented by the three lines Wa, Wb, Wc; therefore



$$P : W :: Wa : Wb \text{ or } ac :: \sin Wca : \sin aWc.$$

But the angle $Wca = cWb = ABC = \alpha$;

and $\sin aWc = \sin aWF = \cos PWA = \cos \epsilon$; hence

$$P : W :: \sin \alpha : \cos \epsilon.$$

127. *Cor.* 1.—When $\epsilon = 0$ or $\cos \epsilon = 1$,

$$P : W :: \sin \alpha : 1 :: AC : AB.$$

In this case the direction of the power is parallel to the inclined plane, and the weight is the greatest that can be supported by the given power P.

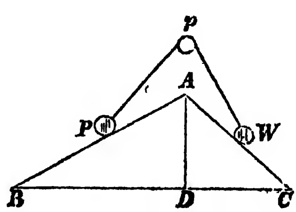
128. *Cor.* 2.—If the power act parallel to the base, $\epsilon = \alpha$,

$$\therefore \quad P : W :: \sin \alpha : \cos \alpha :: AC : BC.$$

129. *Cor.* 3.—If the power act perpendicularly to the horizon, $\epsilon = 90 - \alpha$, therefore $\cos \epsilon = \sin \alpha$ and $P = W$. In this case the weight is entirely sustained by the power, and there is no pressure on the plane.

130. Prop. XIV.—*To find the proportion between two weights,* P and W, *connected by a string passing over the pulley* p *and resting upon the two inclined planes* AB, AC.

Suppose a weight T hanging freely at p, as in the last proposition, to balance P; then will T be the tension of the string Pp. And if the angle $ABC = \alpha$, $APp = \epsilon$; $ACB = \alpha'$, $AWp = \epsilon'$; then (art. 126)



$$P : T :: \cos \epsilon : \sin \alpha.$$

Also, since P [illegible] are in equilibrium, the tension of the string Pp is evidently the same as that of the string Wp;

$$\therefore \quad T : W :: \sin \alpha' : \cos \epsilon'.$$

Hence, multiplying like terms together,

$$P : W :: \cos \epsilon \sin \alpha' : \cos \epsilon' \sin \alpha :: \frac{\cos \epsilon}{\sin \alpha} : \frac{\cos \epsilon'}{\sin \alpha'}.$$

131. *Cor.*—If the string be parallel to the two planes, ι and ι' are each $= 0$; in this case, therefore,

$$P : W :: \sin \alpha' : \sin \alpha :: AB : AC.$$

Examples.

1. A power of 5lbs., acting parallel to a plane, supports a weight of 10lbs.; required the inclination of the plane to the horizon, and the pressure upon the plane. Ans. Incl^n. $= 30°$; pressure $= 8{\cdot}66$lbs.

2. The weight, power, and pressure on an inclined plane are, respectively, as 13, 9, and 6; required the inclination of the plane, and the angle which the direction of the power makes with the plane.
Ans. Incl^n. of the plane $= 37° \; 21'$; dir^n. of the power $= 28° \; 47'$.

3. If P and W sustain each other on two inclined planes AB, AC, (see last figure), by means of a string passing over a pulley at A; and the circumference of a circle be described through the three points A, P, W; prove that P, W will still be in equilibrium, at these points, if the string PpW passes over a pulley p in any part of this circumference.

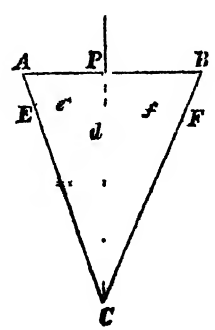
V. THE WEDGE.

132. The *wedge* is a triangular prism, and is used for the purpose of separating obstacles by introducing its edge between them, and then thrusting the wedge forward.

Knives, swords, coulters, nails, &c., are instruments of this kind.

The power in this case is the blow of a mallet, or other such means, upon the back of the wedge, which produces a violent pressure for a short time. The force to be overcome, or W, is the resistance which the parts of the body oppose to their being separated. But as this resistance is never well known, we shall not attempt to determine the ratio between P and W, as in the other cases, but confine ourselves to ascertain the effect which the power exerts in a direction perpendicular to the two sides of an isosceles wedge.

133. Prop. XV.—*An isosceles wedge being introduced between two obstacles; to find its tendency to separate the obstacles when a given force acts perpendicularly upon its back.*



Let ABC be a section of the wedge made by a plane passing through P perpendicular to the edge, and consequently perpendicular to the two sides of the wedge. Let P be the power acting at the back of the wedge; from P draw PE, PF perpendicular to AC, BC. Let Pd represent the force P, and complete the parallelogram $Pedf$. Then the force Pd may be resolved into the two forces Pe, Pf, perpendicular to the sides of the wedge. And since the three sides Pd, Pe, de, of the triangle Pde are, respectively, perpendicular to the sides AB, AC, BC of the triangle ABC, these two triangles are similar. Hence, if we call Q,

Q' the forces which P exerts on the two sides AC, BC, and 2α the angle ACB of the wedge, we have

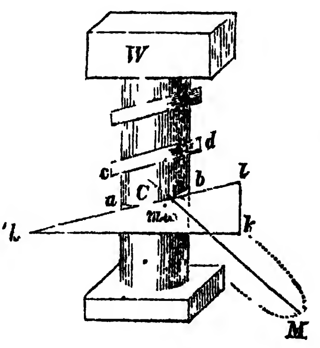
$$P : Q :: Pd : Pe :: AB : AC :: 2\sin\alpha : 1;$$

$$\therefore \quad P = 2Q\sin\alpha.$$

VI. THE SCREW.

134. *The screw* consists of a cylinder AB, with a uniform projecting thread $abcd$ .. traced round the surface, and making a constant angle with lines parallel to the axis of the cylinder. We may conceive it to be formed by taking a right-angled triangle hkl, whose base hk is equal to the circumference of the cylinder, and height kl equal to the distance between the threads measured in a direction parallel to the axis. If this triangle be folded round the cylinder so that hk shall be parallel to the base of the cylinder, hl will form the spiral thread of the screw. This cylinder fits into an equal hollow cylinder, on the inner surface of which there is a groove exactly corresponding to the projecting thread $abcd$. The two screws being thus adapted to each other, either cylinder may be moved round the common axis by a lever perpendicular to that axis, and a motion will be produced in the direction of that axis by means of the thread sliding in the groove.

135. PROP. XVI.—*In a vertical screw when there is an equilibrium,* P : W :: d : 2πr; d *being the distance between the two contiguous threads, measured in a direction parallel to the axis, and* $2\pi r$ *the circumference of the circle which* P *describes.*



Let the screw which sustains the weight W be supposed to be supported by its thread resting on the groove of the external screw. Then we may suppose a portion of the weight to be sustained at each portion of the thread, and the whole weight will be the sum of these portions. Let a weight w be supported at m by means of the arm CM. Then we may conceive m to be a portion of the inclined plane hl, whose height is kl, the distance between two threads, and the base hk equal to the circumference of the cylinder. Call q the power which, acting at m parallel to hk, would by its horizontal pressure sustain the weight w, or prevent the motion of the screw round the axis. Then, since the weight w is sustained upon the inclined plane mn by a power q acting parallel to its base, we have (art. 128),

$$q : w :: \text{height} : \text{base} :: d : 2\pi \times Cm.$$

Now, instead of supposing the force q to act at m, let a power p act at M in a direction parallel to the former force q, and let it produce the same effect at m that q does; then, by the property of the lever,

$$p : q :: Cm : CM :: 2\pi \times Cm : 2\pi \times CM \text{ or } 2\pi r$$

$\therefore$ *ex æquali*, $p : w :: d : 2\pi r$.

In the same manner let the weight w' be supported at any other point by p', acting at the end of an arm equal to CM; w'' by p''; and so on. Then we shall have

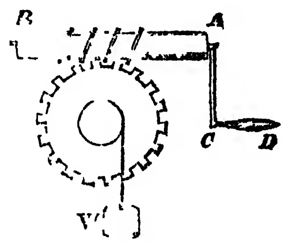
$$p' : w' :: d : 2\pi r; \quad p'' : w'' :: d : 2\pi r, \&c.$$

But the sum of all the partial weights $w + w' + w'' + \&c. = W$; and the sum of all the separate powers acting at M, M', &c., will produce the same effect as a single power $P = p + p' + p'' + \&c.$ acting at an equal distance CM. Hence (Alg. art. 181.)

$$P : W :: d : 2\pi r.$$

Cor.—Instead of supposing the screw to support a weight W acting vertically, we may suppose it employed to produce a pressure W in any direction, and the proportion will be the same as before.

136. Prop. XVII.—*In the endless screw there will be an equilibrium when the power is to the weight, as the distance of the threads multiplied by the radius of the axle to the circumference described by the power multiplied by the radius of the wheel.*



The endless screw is so combined with the wheel and the axle that threads of the screw may work in the teeth fixed in the circumference of the wheel. Let Q represent the power produced by the screw at the circumference of the wheel, then, by art. 125, if d be the distance between the threads, and $2\pi r$ the circumference described by P,

$$P : Q :: d : 2\pi r,$$

and in the wheel and axle (art. 105)

$$Q : W :: \text{radius of axle} : \text{radius of wheel};$$

$$\therefore P : W :: d \times \text{radius of axle} : 2\pi r \times \text{radius of wheel}.$$

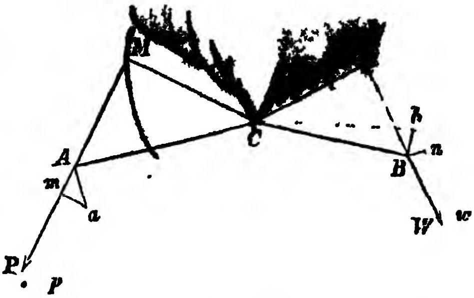
137. Prop. XVIII.—*If a power and weight be in equilibrium in any machine, and the whole be put in motion; the power : weight :: weight's velocity in the direction of its action : power's velocity in the direction of its action.*

The proof of this important proposition, which is only a particular case of the principle of *virtual velocities*, will be most simply derived from an enumeration of the cases of the different mechanical powers.

In the application of the rule two things must be attended to.

1. The velocity of the power or weight must be estimated in the direction in which it acts. Thus, if Ww (see fig. page 193) represent the velocity of W on an inclined plane, and wn be drawn in the direction of gravity, and Wn perpendicular to wn, then wn is the velocity of W in a vertical direction, or in the direction in which it acts. Also, Wn is its velocity in a horizontal direction.

2. That part only of the power or weight must be estimated which is effective.



(1) *The lever.* Let ACB be a lever kept at rest by a power and weight acting in the directions AP, BW; and let CM, CN be perpendiculars from the fulcrum upon these directions. Let the lever move uniformly through a very small angle into the position aCb. A and B will describe circular arcs Aa, Bb, which will be as the velocities of the points A and B, and ultimately these arcs may be taken for straight lines. Draw am, Bn perpendicular to AP, bw, then will Am, bn be proportional to the velocities estimated in the directions of the forces.

Now, considering Aa as a straight line, CAa will ultimately be a right angle; hence

$$CAM + aAm = \text{a right angle} = CAM + ACM;$$

$$\text{therefore the angle } aAM = ACM.$$

Hence the triangles CAM, Aam are similar. In like manner the triangles CBN, Bbn may be proved to be similar. Hence we have these proportions,

$$Am : Aa :: CM : CA; \quad Aa : Bb :: CA : CB,$$

$$Bb : bn :: CB : CN;$$

therefore, *ex equali*,

$$Am : bn :: CM : CN :: W : P;$$

$$\therefore \quad P\text{'s vel.} : W\text{'s vel.} :: W : P.$$

(2) *In the wheel and axle.* If the power be made to descend through a space equal to the circumference of the wheel with a uniform motion, the weight will be uniformly raised through a space equal to the circumference of the axle, and these circumferences are as the radii. Hence

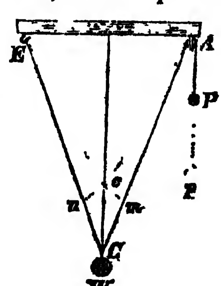
$$P\text{'s vel.} : W\text{'s vel.} :: \text{rad. axle} : \text{rad. wheel} :: W : P.$$

(3) *In the single fixed pulley.* If the weight be uniformly raised 1 inch, the power will uniformly describe 1 inch in the direction of its action. Hence

$$P\text{'s vel.} : W\text{'s vel.} :: W : P.$$

(4) *In the single moveable pulley with parallel strings.* If the weight be raised 1 inch, each of the strings is shortened 1 inch, and the power describes 2 inches, therefore

$$P\text{'s vel.} : W\text{'s vel.} :: 2 : 1 :: W : P.$$



(5). *In the single moveable pulley with strings not parallel.* If the pulley at W be considered as a point, and W be raised through a very small space Cc, P will descend through a space

$$= (AC+EC)-(Ac+Ec) = Cm+Cn,$$

taking $Am = Ac$ and $En = Ec$. But when the angle CAc is extremely small, cm and cn

may be considered ultimately perpendicular to AC and EC; hence, putting the angle $ACe = \alpha$,

$$Pp = Cm + Cn = 2Cc \cos\alpha;$$

consequently $Pp : Cc :: 2\cos\alpha : 1$;

$\therefore$ P's vel. : W's vel. :: W : P.

If the pulley be of finite magnitude, the proportion will still be the same, for when the weight moves through a very small space Cc, the angles at C and c will ultimately be equal, and, therefore, the part of the string which adheres to the pulley remains unaltered; and consequently the space described by P is the same.

(6). *In the system of pulleys* (art. 119), if the weight be raised 1 inch, each of the strings at the lower block is shortened 1 inch, and the power describes n inches: therefore

P's velocity : W's velocity :: n : 1 :: W : P.

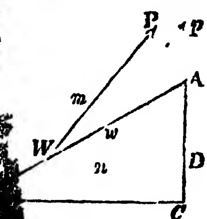
(7). *In the second system of pulleys* (art. 122), if W be raised through 1 inch, C is raised through 2 inches, B is [illegible] $\times$ 2 inches, A through $2 \times 2 \times 2$ inches, and so on, [illegible] will be lowered 2^n inches; hence

P[illegible] : W[illegible] :: 2^n : 1 [illegible] P.

(8) *In the third system* (art. 124), [illegible] ed 1 inch, the pulley B will descend 1 inch; the pulley [illegible] 1 inch because W is raised 1 inch, and 2 inches because [illegible] y B descends 1 inch; therefore it will descend altogether 1 [illegible]; the next pulley will descend 1 inch because W is raised [illegible] 2 $\times$ (1 + 2) inches in consequence of the descent of A; [illegible] l descend 1 + 2 + 4 inches; and similarly for any number of [illegible] herefore

P's velocity : W's velocity :: 1 + 2 + [illegible] c. : 1 :: W : P.

(9) *In the inclined plane.* Let W be [illegible] through a small space Ww, and let WP, [illegible] be drawn in the direction of the power. Draw WD parallel to BC, and wm, wn perpendicular to WP, WD; then Wm, wn, are ultimately as the velocities of the power [illegible] weight in the directions in which they act. But $wWn = ABC = \alpha$, and $PWA = \epsilon$, therefore $Wm = Ww \cos\epsilon$, $wn = Ww \sin\alpha$; hence



P's vel. : W's vel. :: $\cos\epsilon$: [illegible] : P.

(10) *In the screw.* If the power uniformly [illegible] the circumference of the circle (art. 141), the weight is uniformly raised through the distance between two contiguous threads; therefore

P's vel. : W's vel. :: $2\pi \times CM$: ac :: W : P.

(11) *In any combination of machines.* If in any compound machine, P, Q; Q, R; .... V, W; be the power and weight in each case where there is an equilibrium, and p, q, ... v, w be the respective velocities in the directions in which the forces act, we have, from the preceding articles,

$$Pp = Qq; \quad Qq = Rr; \quad Rr = Ss \ldots . Vv = Ww;$$

and therefore $Pp = Ww$. Hence

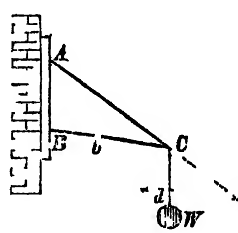
$$P\text{'s vel.} : W\text{'s vel.} :: p : w :: W : P.$$

CHAP. V.—ROOFS, ARCHES, AND BRIDGES.

It is our intention, in the present chapter, to apply the doctrine of equilibrium to explain the manner in which the thrusts and strains usually act in roofs and arches. Although the subject is of the utmost importance to the military and civil engineer, we can only consider the general principles, and refer the student, for many practical details, to the 1st volume of Robison's Mechanical Philosophy, and Tredgold's Elementary Principles of Carpentry.

THE EQUILIBRIUM OF ROOFS.

138. Prop. I.—*If* ABC *be a frame fixed against a wall, it is required to calculate the strains on the beams* CA, CB *by a weight W suspended from* C.



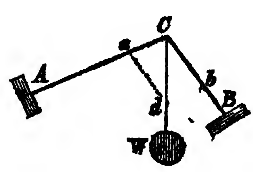
Take CD to represent the weight W and draw da, db parallel to CB, CA; and let da meet AC, produced in a. Then $Cadb$ is a parallelogram, and the force Cd is equivalent to the two forces Ca, Cb. The force Ca acts in the direction AC, and, therefore, the beam is in a state of tension: but Cb acts in the direction CB, or tends to compress it. If, then, we put the angle $ACW = \alpha$, $BCW = \beta$, and $ACB = \gamma$, we have

$$Ca : Cd :: \sin Cda : \sin Cad :: \sin\beta : \sin\gamma,$$

$$Cb : Cd :: \sin Cdb : \sin Cbd :: \sin\alpha : \sin\gamma,$$

$$\therefore \quad \text{tension of } CA = W\frac{\sin\beta}{\sin\gamma}; \quad \text{thrust on } CB = W\frac{\sin\alpha}{\sin\gamma}.$$

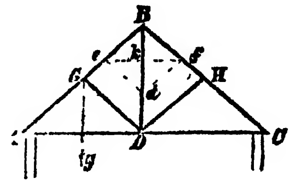
Hence it appears that the strain on any piece is proportional to the sine of the angle, which the straining force makes with the other piece directly, and to the sine of the angle which the pieces make with each other inversely.



139. *Scholium.*—When any force tends to stretch a beam or draw it asunder, the place of the beam might be supplied by a rope, and it is called a *tie*; but when the beam is compressed or crushed by a force, it is called a *strut* or a *brace*. It is easy, in any particular case, to distinguish a tie from a strut; for if

the direction of the force Wd fall within the angle formed by the pieces strained, as in the adjoining figure, then both pieces are compressed. But if it fall within the angle formed by producing the direction of either of the sustaining pieces, as in the first figure, this beam is in a state of tension.

140. Prop. II.—*In a simple isosceles truss roof it is required to calculate the tension of the tie-beam.*



Let AB, BC, be two beams of the roof, connected by the tie-beam AC. Let W be the weight of each sloping beam, and the portion of the roof supported by it; and let G, H be their centres of gravity, or the points at which these weights may be supposed to act. Also, let the weight on the vertex, arising from a longitudinal beam or other appendages $= w$, the angle $BAC = \alpha$, $AB = a$, $AG = b$.

The weight W of the roof AB may be supposed to be collected at G, its centre of gravity; and this weight is supported at the points A and B. Hence, by the property of the lever,

weight W : part of this weight sustained at B :: AB : AG,

$$\therefore \quad \text{pressure at } B \text{ arising from the roof } AB = W\frac{b}{a};$$

similarly, $\quad$ pressure at B arising from the roof $BC = W\frac{b}{a}$;

Hence, if we put B for the whole pressure at B, we have

$$B = w + \frac{2Wb}{a}.$$

Draw BD perpendicular to AC, and take Bd to represent this pressure at B; then, if we complete the parallelogram $Bedf$, Be, Bf will represent the pressures or thrusts in the directions BA, BC (art. 41). Resolve the force Be into the two Bk, ke; ke will represent the horizontal thrust at A, which we may call H. We have then

$$H : B :: ke : Bd \text{ or } 2Bk :: \text{rad} : 2\tan\alpha;$$

$$\therefore \quad H = \tfrac{1}{2}B\cot\alpha = \left(\tfrac{1}{2}w + \frac{Wb}{a}\right)\cot\alpha.$$

This measures the horizontal thrust of the roofing against the walls, supposing the tie-beam to give way.

141. *Cor.* 1.—If the rafter AB support a covering of uniform thickness, G will bisect AB. Let A be the weight of a portion of this roof equal in length to AD, then $A = W\cos\alpha$; therefore

$$H = (\tfrac{1}{2}w + \tfrac{1}{2}W)\cot\alpha = \tfrac{1}{2}w\cot\alpha + \tfrac{1}{2}A\operatorname{cosec}\alpha.$$

Hence it appears that the greater α is, or the steeper the roof is, the less will be the horizontal thrust.

142. *Cor.* 2.—When AC is great, and loaden with a floor and ceiling, it is apt to bend in the middle. To counteract this tendency a small piece of timber BD, called a *king-post*, is suspended at B, and

connected by an iron strap or other means to the beam AC. Half the weight of the beam AC may now be supposed to be sustained at D by BD, and the king-post must be made sufficiently strong to bear this pull or strain. The weight of the king-post, and half the weight of the tie-beam, are therefore supported at B, and must be included in the value of B.

143. Prop. III.—*To determine the strains on the braces* DG, DH, *drawn from* D *to the middle of* BA, BC. (See the last figure.)

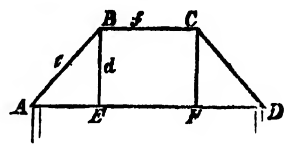
The pressure at G may be considered equal to half the weight of the roof AB, or $\frac{1}{2}W$: for the weight of AG is equally supported at A and G, and the weight of BG equally supported at B and G. Also, the direction of this pressure is vertical or parallel to BD. Let this be resolved into two forces, in the directions GA, GD. The pressure in the direction GA will be sustained at A, and will have no effect in straining the brace GD; but the force in the direction GD will be entirely effective. Call this thrust T, then (art. 16)

$$\tfrac{1}{2}W : T :: \sin AGD : \sin AGg :: \sin 2\beta : \sin \beta;$$

$$\therefore \quad T = \frac{W}{2} \cdot \frac{\sin \beta}{\sin 2\beta} = \frac{W}{4 \cos \beta} = \frac{W}{4 \sin \alpha}.$$

144. *Cor.*—If GD represent the thrust at G, HD will also represent the thrust at H; and these two forces are equivalent to the force BD, which, therefore, is the strain on the king-post arising from the weights of the roofs AB, BC.

145. Prop. IV.—*To determine the horizontal thrust on the tie-beam* AD *of the roof* ABCD; *of which the rafters* AB, CD *are equal, and* BC *horizontal.*



Let the weight of each of the rafters AB, CD, with their covering, be W, and the weight of BC be W'; also, let the angle at $A = \alpha$. Then (art. 140)

vertical pressure at B arising from the roof $AB = \frac{1}{2}W$,

do. from the roof $BC = \frac{1}{2}W'$.

Hence the whole pressure at $B = \frac{1}{2}(W + W')$.

Take Bd to represent this pressure, and complete the parallelogram $Bedf$. The vertical pressure Bd is equivalent to the two thrusts Be, Bf; and since Bde is a right angle, Be is the thrust at A in the direction BA, and de is the horizontal thrust at A. Hence

$$H : \tfrac{1}{2}(W + W') :: de : Bd :: \text{rad} : \tan \alpha;$$

$$\therefore \quad H = \tfrac{1}{2}(W + W') \cot \alpha.$$

Cor. 1.—The force which tends to crush the rafter BA is

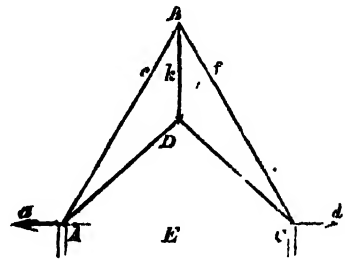
$$Be = \tfrac{1}{2}(W + W') \operatorname{cosec} \alpha;$$

and the force which tends to crush the strut-beam BC is Bf or ed, which is equal to the horizontal thrust at A.

Cor. 2.—Additional stiffness is given to this roof by adding the two

ties or side-posts BE, CF; which are connected with the tie-beam, either by mortises or straps.

146. PROP. V.—*In a roof where two stretchers* DA, DC *are substituted for the tie-beam, it is required to determine the strain upon the king-post* BD.



Let BA, BC be the two rafters of the roof; and let W be the weight of each rafter, and the portion of roof supported by it; also, let w be the additional weight at B. If the centres of gravity of AB, BC bisect these lines, W will be the pressure downwards at B, arising from the roof ABC (art. 140); and $W+w$ will be the whole pressure downwards at B: put $W + w = B$. Let BD represent the pressure B, and complete the parallelogram $BeDf$; then Be, Bf represent the thrusts on BA, BC, and ke represents the horizontal thrust at the point A. Now we may consider AB as a lever, moveable round the point B as a fulcrum, and pulled at A in the direction Aa by the force ke, and resisted by the reaction of the stretcher DA, pulling in the direction AD. Let H = horizontal thrust at A, S = strain in the direction AD, α = angle BAE, β = angle BAD: then

$$H : B :: ke : BD \text{ or } 2Bk :: \text{rad} : 2\tan\alpha;$$

$$\therefore \quad H = \tfrac{1}{2}B\cot\alpha.$$

Also, from the properties of the lever, when there is an equilibrium (art. 94),

$$H \times BA\sin\alpha = S \times BA\sin\beta;$$

$$\therefore \quad S = H\frac{\sin\alpha}{\sin\beta} = \tfrac{1}{2}B\frac{\cos\alpha}{\sin\beta}.$$

Similarly, the pull or strain on $DC = S$, and the two strains S, S, in the directions DA, DC, produce a strain on the king-post at D, in the direction DE, equal to

$$2S\cos ADE = 2S\sin(\alpha-\beta) = B\frac{\cos\alpha\sin(\alpha-\beta)}{\sin\beta}.$$

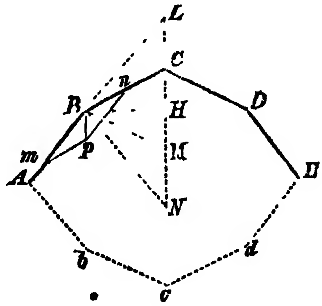
147. *Cor.*—If, instead of DA, DC, two ties, DE, DF, were introduced, making an angle θ with BD, we might, in this case, also consider AB as a lever, moveable round the fulcrum B, and kept at rest by the force H acting at A in the direction Aa, and the reaction S of the tie DE. Putting, therefore, $AB = a$, $BD = k$; we have

$$Ha\sin\alpha = Sk\sin\theta, \quad \text{hence}$$

$$S = H\frac{a\sin\alpha}{k\sin\theta} = \frac{B}{2}\,\frac{a\cos\alpha}{k\sin\theta};$$

$$\therefore \quad \text{strain on } BD = 2S\cos\theta = B\frac{a\cos\alpha\cot\theta}{k}.$$

148. Prop. VI.—*To calculate the conditions of equ librium of any number of beams,* AB, BC, &c. *forming a frame-work in a vertical plane.*



Let W, W', &c. be the weights of AB, BC, &c. and their coverings; and w, w', &c. the additional weights supported at the angles B, C, &c. Also, let the angle $ABp = \beta$, $CBp = \beta'$. Now, if the centres of gravity of AB, BC, &c. be in the middle points of these lines, the pressure at B, arising from the roof $AB = \frac{1}{2}W$, and from the roof $BC = \frac{1}{2}W'$. Hence, if B be the whole vertical pressure at B, we have

$$B = w + \tfrac{1}{2}(W + W').$$

Let Bp represent the pressure B. Complete the parallelogram $Bmpn$, then will Bm, Bn represent the pressures or thrusts on the beams BA, BC. Call these thrusts T, T'; also, put H = horizontal thrust at A. We have then

$$T : B :: Bm : Bp :: \sin Bpm : \sin Bmp$$
$$:: \sin\beta' : \sin(\beta + \beta').$$

Hence $$T = B\frac{\sin\beta'}{\sin(\beta+\beta')}, \text{ and, therefore,}$$

$$H = B\frac{\sin\beta\sin\beta'}{\sin(\beta+\beta')} = \frac{B}{\cot\beta + \cot\beta'}.$$

149. *Cor.* 1—The horizontal pressures at B, arising from the thrusts on BA and BC, are evidently equal and opposite. Also, the horizontal pressure at B, arising from the pressure on BC, must be equal to the horizontal pressure at C, arising from the pressure on CB. Hence the horizontal pressure at each angle is the same.

150. *Cor.* 2.—If, from any point B, BH be drawn horizontally to represent the constant horizontal pressure, and BL, BC, BM, &c. be drawn parallel to AB, BC, CD, &c., meeting a vertical line passing through H, these lines will represent respectively the thrusts on the beams; for the triangle BLC is similar to the triangle Bmp, and so on with the others. Hence, also, it follows that LC, CM, MN represent the whole vertical pressures at the angles B, C, D.

151. *Cor.* 3.—If the whole figure $ABCDE$ be supposed to be inverted, and to come into the position $AbcdE$ in the same vertical plane as before; and if cords be substituted for the beams, and weights be suspended at the angles b, c, &c., equal to the corresponding vertical pressures at B, C, &c., the equilibrium will still subsist. For since the angles at the points B, b are respectively equal, and the vertical forces are equal, the tensions in the directions bA, bc will be equal to the pressures in the directions BA, BC. In like manner it may be shown that all the other tensions at c, d, &c. are equal to the pressures at C, D, &c.; and since the pressures at B, C, &c. balance each other, the tensions at b, c, &c. will also keep each other at rest. In the latter

case, however, the equilibrium is *stable*, but in the former case it is *unstable*. (See art 69.)

THE EQUILIBRIUM OF ARCHES.

152. By an *arch* is meant a number of bodies, of the form of wedges, supported by their mutual pressures, and the pressures of the two extreme bodies against fixed obstacles.

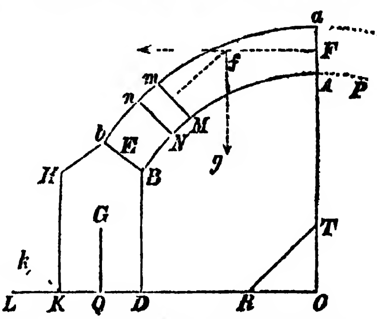
These wedges are called *voussoirs*, and the voussoir which is at the top or *crown* of the arch is called the *key-stone*.

The surfaces which separate the voussoirs are called *joints*.

The external curve of the arch is called the *extrados*, and the internal curve the *intrados*.

The solid mass against which the lowest voussoir on each side rests is called the *pier* or *abutment*.

153. Prop. VII.—*To find the conditions of equilibrium of an arch.*



Let $AamM$ be any portion of the arch; then the forces which keep it at rest are, 1st, Its own weight W, which acts in a vertical line fg passing through its centre of gravity; 2d, The horizontal pressure of the opposite arch aP, which acts at every part of the surface Aa, but is equivalent to a single force acting at some point F, which we may suppose to be the middle of Aa; and 3d, The reaction of the arch mB. When the resultant of the first two forces falls within the joint Mm, and is perpendicular to it, there will be an equilibrium. But if the resultant be not perpendicular to Mm, the voussoirs will slide along each other, since the joints are supposed to be perfectly smooth: and if the resultant falls without Mm, there will be a rotatory motion about that extremity of Mm towards which the resultant falls.

In the horizontal line OD, take OR to represent the horizontal pressure at F, and draw RT perpendicular to the direction of Mm; then the three sides of the triangle ORT are parallel to the three forces which keep the arch aM at rest, and therefore they are proportional to these forces (art. 18). Hence, putting $H =$ the horizontal pressure at F; $P =$ the pressure at Mm; $W =$ the weight of the arch aM; and also, $\alpha =$ the angle which the joint Mm makes with the vertical; we have

$$H : W : P :: OR : OT : RT;$$

$$\therefore \quad W = H \tan\alpha; \qquad P = H\sec\alpha; \qquad \text{and } P = \frac{W}{\sin\alpha}.$$

154. *Cor.* 1.—The pressure TR may be resolved into the two forces TO, OR, of which $TO = W$, the weight of the arch, and $OR = H$, the pressure at F. Hence the horizontal pressure at each joint is the same, and is equal to the horizontal force at F.

155. *Cor.* 2,—If W' = the weight of the arch aN, and α' the angle which the joint Nn makes with the vertical, then the weight of the voussoir $MmnN = W' - W = H(\tan\alpha' - \tan\alpha)$.

156. *Cor.* 3.—If β be the angle which the joint Bb makes with the vertical, and A the weight of the half-arch aB,

$$\text{the pressure upon the pier } BK = \frac{A}{\sin\beta}.$$

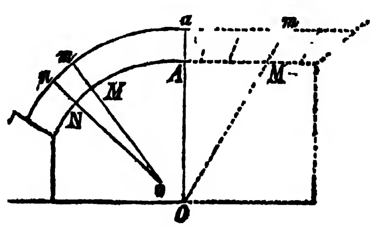
157. Prop. VIII.—*To find the pressure exerted to overturn the pier of an arch.* (See the last figure.)

If the pier be overturned by the pressure of the arch, it will turn about the point K. We may, therefore, consider the pier as a lever, whose fulcrum is K; and the pressure of the arch acting at some point E in the direction EL, perpendicular to Bb, tends to overturn the pier, and the weight of the pier acting at its centre of gravity G in the direction GQ, tends to keep it steady. Draw Kk perpendicular to EL or parallel to Bb; and let β be the angle which the joint Bb makes with the vertical, then $\beta = HKk = ELK$; therefore $Kk = KL\sin\beta$. Now the pressure on the joint $Bb = \dfrac{A}{\sin\beta}$ (art. 156). And in order that the pier may stand,

$$\text{weight of pier} \times KQ \text{ must be} > \text{pressure in direction } EL \times Kk.$$

$$> \frac{A}{\sin\beta} KL\sin\beta > A \times KL.$$

158. Prop. IX.—*Given the intrados, to find the lengths of the voussoirs when they are pressed only by their own weights.*



Let AM, am be the intrados and extrados of an arch; Mm, Nn two joints extremely near each other; and let them be produced until they meet each other in O. Let α, α' be the angles which Mm, Nn make with the vertical, then the angle $MON = \alpha' - \alpha$ also, let $OM = r$, $Mm = v$.

Now the weight of the voussoir $MmnN$ is proportional to $\tan\alpha' - \tan\alpha$ (art. 166). But the weight of this voussoir is also as the area $MmnN$; and the area OMN being considered as a triangle

$$= \tfrac{1}{2}OM \times ON \times \sin(\alpha' - \alpha),$$

and the area $Omn = \tfrac{1}{2}Om \times On \times \sin(\alpha' - \alpha)$;

hence, therefore, $\tfrac{1}{2}[Om \times On - OM \times ON] \times \sin(\alpha' - \alpha)$ is proportional to $\tan\alpha' - \tan\alpha$. Let c^2 be such a constant quantity that

$$(Om \times On - OM \times ON) \times \sin(\alpha' - \alpha) = c^2(\tan\alpha - \tan\alpha)$$

$$\left(\frac{\sin\alpha'}{\cos\alpha'} - \frac{\sin\alpha}{\cos\alpha}\right) = c^2\frac{\sin(\alpha' - \alpha)}{\cos\alpha\cos\alpha'};$$

$$\text{then}\quad Om \times On - OM \times ON = \frac{c^2}{\quad}$$

But since the joints are supposed to be extremely near each other, we have, ultimately, $Om = On$, $OM = ON$, and $\alpha' = \alpha$; therefore

$$\frac{c^2}{\cos^2 \alpha} = Om^2 - OM^2 = (r + v)^2 - r^2 = 2rv + v^2;$$

from whence we obtain

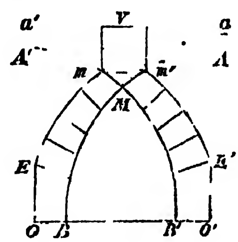
$$v = \sqrt{(c^2 \sec^2 \alpha + r^2)} - r.$$

And since the intrados is given, and the position of the joints, r is known at every point, and therefore v is the length of the voussoir. Hence the form of the extrados is known.

We have here supposed the joints to be extremely near each other, or the voussoirs to be indefinitely thin. If we suppose any number of these small voussoirs to be united and become one larger voussoir, it is manifest that the equilibrium will still subsist.

159. *Cor.*—Let the intrados be a straight line (see the dotted part of the figure), then $c^2 = (Om^2 - OM^2) \cos^2 \alpha = Om^2 \cos^2 \alpha - OA^2$; therefore $Om^2 \cos^2 \alpha = c^2 + OA^2$. Hence $Om \times \cos \alpha$ is a constant quantity, and therefore the extrados is also a straight line.

160. Prop. X.—*The intrados of an arch being composed of two equal circular arcs* BM, B'M ; *and the joints being perpendicular to the curve; to find the extrados that there may be an equilibrium.*



Let BM, $B'M$ be two equal arcs, whose centres are O', O, and let them be continued to the highest points A and A'. Also let Ea, $E'a'$ be the corresponding extrados to these arcs. Let Mm, Mm' be the joints passing through M; if now we suppose the parts Ma, Ma' to be removed, and a voussoir V to be placed at M, which will exert the same pressures on Mm, Mm', it is evident that the arches Bm, $B'm'$ will still be in equilibrium.

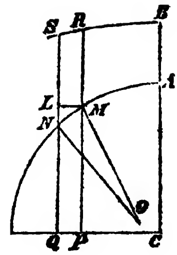
Let H be the horizontal pressure at Aa, and α the angle which Mm makes with the vertical, then $H \sec \alpha$ is the pressure on Mm (art. 153). Now, if V be the weight of the wedge VM, it is manifest that $\frac{\frac{1}{2}V}{\sin \alpha}$ is the pressure on each of the surfaces Mm, Mm'. Hence

$$\frac{\frac{1}{2}V}{\sin \alpha} = H \sec \alpha; \quad \therefore \quad V = 2H \tan \alpha.$$

161. In the preceding propositions we have supposed the voussoirs to be acted on by their own weights alone. We will now suppose a large superincumbent weight to be placed upon the voussoirs, and this weight not to be affected by lateral pressure, but to press solely in a vertical direction. In this case, since the weight of the voussoir and the pressure act in the same direction, the pressure may be considered as simply an addition of weight to the voussoir.

162. Prop. XI.—*If every part of an arch be pressed vertically by the weight immediately above it, it is required to find the height of the extrados above the intrados.*

If the length of the voussoir be small compared with MR, t l



weight of the column above the voussoir will be nearly equal to that of $MRSN$; and when the voussoirs are very small, this weight is nearly proportional to $MR \times LM$. Suppose the joints to be perpendicular to the intrados, and let α, α' be the angles which the joints at M and N make with the vertical; then, from art. 163, the pressure on MN is as $\tan \alpha' - \tan \alpha$, therefore $MR \times ML$ is proportional to $\tan \alpha' - \tan \alpha$. Let c^2 be such a constant quantity that

$$MR \times ML = c^2 (\tan \alpha' - \tan \alpha);$$

then, since

$$LM = MN \cos LMN = MN \cos \alpha = OM(\alpha' - \alpha) \cos \alpha;$$

$$\therefore \quad LM \times MR = OM \times MR (\alpha' - \alpha) \cos \alpha.$$

Also $$\tan \alpha' - \tan \alpha = \frac{\sin \alpha'}{\cos \alpha'} - \frac{\sin \alpha}{\cos \alpha} = \frac{\sin(\alpha' - \alpha)}{\cos \alpha \cos \alpha'}:$$

$$\therefore \quad OM \times MR (\alpha' - \alpha) \cos \alpha = c^2 \frac{\sin(\alpha' - \alpha)}{\cos \alpha \cos \alpha'}.$$

And, because LM is extremely small, $\sin(\alpha' - \alpha) = \alpha' - \alpha$; and $\cos \alpha' = \cos \alpha$, very nearly; also, OM is ultimately the radius of curvature at the point M; hence

$$MR = \frac{c^2}{OM \times \cos^3 \alpha} = \frac{c^2 \sec^3 \alpha}{\text{rad. curv.}}.$$

163. *Cor.* If the intrados be a semicircle whose radius is r,

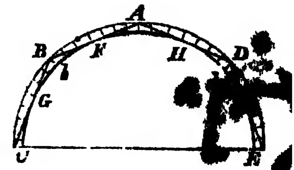
$$MR \text{ varies as } \sec^3 \alpha, \text{ or inversely as } PM^3,$$

and therefore $PM^3 : CA^3 :: AB : MR$. When $\alpha = 90°$, or when $PM = 0$, MR is infinite.

Scholium.

164. In the preceding propositions we have investigated the conditions of the equilibrium of an arch when it is prevented from falling, in consequence of the voussoirs not sliding along each other. But there are other circumstances which must be taken into consideration in the construction of arches. We have first supposed the joints to be perfectly smooth or polished, and to sustain each other merely by the equilibrium of their vertical pressure; if this be not the case, the voussoirs may be prevented from sliding in each other and the arch may stand, although the above conditions be not satisfied. To make the arch still stronger, the voussoirs are sometimes *joggled* or united together so that an arch really never gives way for the want of equilibrium in the vertical pressure of the voussoirs. According to the views of Professor Robison, " An arch, when exposed to a great overload on the crown (or indeed on any part) divides of itself into a number of parts, each of which contains as many arch-stones as can be pierced (so to speak) by a straight line, and it may then be considered as nearly in the same situation with a polygonal arch of long stones butting on each other, like so many beams in a Norman roof." After mentioning some circumstances con-

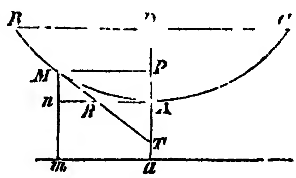
nected with the failure of a considerable arch, he says, "straight lines can be drawn within the arch-stones from A to B and D, and from those points to C and E. Each of the portions ED, DA, AB, BC, resist as if they were of one stone, composing a polygonal vault $EDABC$. When this is overloaded at A, A can descend in no other way than by pushing the angles B and D outwards, causing the portions BC, DE to turn round C and E. This motion must raise the points B and D, and cause the arch-stones to press on each other at their inner points b and d. This produced the copious splintering at those joints immediately preceding the total downfal. The splintering, which happened a fortnight before, arose from this circumstance—that the lines AB and AD, along which the pressure of the overload was propagated, were tangents to the soffit of the arch in the points F, H, and G, and therefore the strain lay all on these corners of the arch-stones, and splintered a little from off them till the whole took a firmer bed." Upon making the experiment with models of arches of chalk, he found, when he overloaded the models at A, the arch always broke at some place B considerably beyond another point F, where the first chipping had been observed.



THE CATENARY AND SUSPENSION BRIDGES.

165. Prop. XII.—*If a perfectly flexible chain, of uniform density and thickness, be suspended from two fixed points,* B *and* C; *to find the equation of the curve when it is in equilibrium.*

Let BAC be the given chain suspended at the points B and C; A the lowest point, or the point where the direction of the curve is horizontal. Through A draw AP vertically; and from any point P draw the ordinate PM perpendicular to AP. Let $AP = x$, $PM = y$, the arc $AM = s$; also, let a be the length of a portion of chain which is equal to the tension at A. Now, if we suppose the part AM to become rigid, after it has assumed the form of equilibrium, it will evidently be supported in the same manner, and the tensions at A and M will be the same as before. Considering AM, therefore, as a rigid body, it is kept at rest by three forces, the tensions at M and A acting in the directions of the curve, or in the directions of the tangents MR, AR, and the weight of the chain acting in a vertical direction; and since the directions of these forces are respectively parallel to the three sides of the triangle MPT, the forces will be proportional to these sides. Hence



$$\text{weight of } AM : \text{tension at } A :: TP : PM,$$

and therefore, by the differential calculus, $s : a :: dx : dy$;

$$\therefore \quad \frac{dy}{dx} = \frac{a}{s}; \text{ and } \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} = \frac{\sqrt{a^2 + s^2}}{s}.$$

Hence $dx = \dfrac{sds}{\sqrt{a^2 + s^2}}$. Take the integral of this equation, supposing that $s = 0$ when $x = 0$; we then have

$$x + a = \sqrt{a^2 + s^2}.$$

From whence also we find

$$s^2 = x^2 + 2ax \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We have also $\dfrac{dy}{dx} = \dfrac{a}{s} = \dfrac{a}{\sqrt{x^2 + 2ax}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$

Hence integrating again, and supposing x and y to vanish together,

$$\frac{y}{a} = \log \frac{x + a + \sqrt{x^2 + 2ax}}{a} \ldots\ldots\ldots\ldots (2),$$

and, if e be the base of the Naperian system of logarithms, we have, Alg. art. 391.

$$e^{\frac{y}{a}} = \frac{x + a}{a} + \frac{\sqrt{x^2 + 2ax}}{a}.$$

Transposing $\dfrac{x + a}{a}$, and squaring both sides of the equation,

$$e^{\frac{2y}{a}} - 2e^{\frac{y}{a}} . \frac{x + a}{a} + \frac{(x + a)^2}{a^2} = \frac{x^2 + 2ax}{a^2};$$

from whence we obtain,

$$x + a = \tfrac{1}{2}a \left(e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right) \ldots\ldots\ldots\ldots (3).$$

Also, from equation (1), $s^2 = (x + a)^2 - a^2$, therefore

$$s = \tfrac{1}{2}a \left(e^{\frac{y}{a}} - e^{-\frac{y}{a}}\right) \ldots\ldots\ldots\ldots\ldots (4).$$

This curve is called the *catenary*.

166. *Cor.* 1.—The tension at M : weight of AM :: TM : TP; and if t be the length of a portion of chain which is equal to the tension at M,

$$t : s :: TM : TP :: ds . dx; \text{ therefore } sds = tdx,$$

but $s^2 = x^2 + 2ax$, consequently $sds = xdx + adx$. Hence

$$t = x + a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

167. *Cor.* 2.—If we suppose the tension at M to be balanced by means of a portion Mm of the chain passing over a pulley at M and hanging freely, $Mm = x + a = Mn + a$; therefore $nm = a$. Hence it follows, that if, at different points of the curve B, M, &c., the tension be balanced by means of portions of the chain hanging freely, all their lower extremities will be in the same horizontal line.

168. *Scholium.*—From these equations Mr. Davies Gilbert has calculated various tables, which are given in the Philosophical Transactions for 1826, for their more ready application to suspension bridges. These formulæ, however, may be simplified in the case of suspension bridges, where the span is very great compared with the deflexion of the chain.

169. Prop. XIII.—*To find an approximate equation to the catenary, when the abscissa is small compared with the ordinate.*

We have, from equation (a),

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 + 2ax}} = \frac{a}{\sqrt{2ax}} \left(1 + \frac{x}{2a}\right)^{-\frac{1}{2}}$$

$$= \sqrt{\frac{a}{2x}} \left(1 - \frac{x}{4a} + \&c.\right)$$

And taking the integral from $x = 0$, $y = 0$; and neglecting all the terms after the second,

$$y = \sqrt{\frac{a}{2}} \left(2x^{\frac{1}{2}} - \frac{2}{3} \frac{x^{\frac{3}{2}}}{4a}\right) = \sqrt{2ax} \left(1 - \frac{x}{12a}\right).$$

Squaring this equation and again neglecting the third term, we get

$$y^2 = 2ax - \tfrac{1}{3}x^2 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Hence also $a = \dfrac{3y^2 + x^2}{6x}$ .... (7).

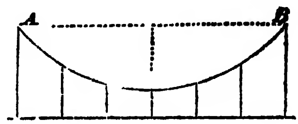
We have likewise, from equation (1),

$$s = \sqrt{2ax + x^2} = \sqrt{(y^2 + \tfrac{4}{3}x^2)}$$

And again expanding and neglecting all the terms after the second,

$$s = y + \frac{2x^2}{3y} \text{ .... (8).}$$

170. Ex. 1.—*Let the span proposed for a suspension bridge be* 800 *feet, and let the adjunct weight of suspension rods, roadway, &c., be taken at one-half of the weight of the chains; and let it be determined to load the chains at the point of their greatest strain, that is, at the point of suspension, with one-sixth part of the weight they are theoretically capable of sustaining.*



The weight which an iron bar, whose transverse section is 1 square inch, can support when pulled in the direction of its length, is 66,250lbs. (See Table in the next page.) Now this is equal to the weight of a bar of iron about 19,000 feet long and one inch square. Davies Gilbert assumes 14,800 feet to be the length of this bar. But the strain arises not only from the weight of the chain, but also from the weight of the roadway, &c., which is supposed to be half the weight of the chain, thus making the whole strain = $\frac{3}{2}$ times the strain of the iron. Hence, therefore, $\frac{2}{3}$ of this length, or 9867 feet, will be the utmost length of the chain of the suspension bridge, with its weights &c., which it can bear. The tension of the chain, therefore, at the points of support, is by hypothesis $\frac{1}{6}$ of 9867 feet = 1644·5 feet. We have, therefore, y = half the span = 400 feet; and t = 1644·5 feet.

From equations (5) and (6) we have

$$y^2 = 2ax - \tfrac{1}{3}x^2 = 2x(t - x) - \tfrac{1}{3}x^2.$$

Hence, substituting for y and t their values above, and solving the quadratic equation, we get

$$x = 50\cdot4 \text{ feet}; \quad a = t - x = 1594\cdot1 \text{ feet};$$

$$s = y + \frac{2x^2}{3y} = 404\cdot23 \text{ feet.}$$

Having obtained the value of a, we can find the corresponding values of y to as many values of x as we please; and these values of x will be the lengths by which the suspending rods exceed the length of the suspending rod at the lowest point of the curve.

171. Ex. 2.*—*In the Hungerford Suspension Bridge, the central span between the piers is* 676·5 *feet, the droop or deflection of the chain* 50 *feet, the number of links* 1280, *their weight* 352 *tons; to find the strains at the highest and lowest points.*

Here x = 50 feet, $y = \frac{1}{2}$ of 676·5 = 338·25 feet

$$s = y + \frac{2x^2}{3y} = 343 \text{ feet}; \qquad a = \frac{3y^2 + x^2}{6x} = 1152\cdot5 \text{ feet};$$

$$t = a + x = 1202\cdot5 \text{ feet.}$$

Sir Howard Douglas supposes the entire weight of the chain, platform, and a full load upon upon it, estimated at 100lbs. per square foot, to be 1000 tons. In this case we shall have

$$\text{Weight of 1 foot of the chain} = \frac{1000}{2s} = 1\cdot457 \text{ tons};$$

tension at A, or weight of a = 1680 tons;

tension at B, or weight of t = 1753 tons.

* See a pamphlet on "Metropolitan Bridges and Westminster Improvements; by Sir Howard Douglas, Bart., M.P."

CHAP. VI.—STRENGTH OF MATERIALS AND FRICTION.

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172. There are several strains to which a piece of solid matter may be exposed, and in which the mechanical effort to produce fracture, and the resistance opposed to it by the particles of the body, are differently exerted. We shall, however, only notice the three following, which come more immediately under the attention of the engineer:

1st. A body may be torn asunder, as in the case of ropes, king-posts, tie-beams, &c.

2nd. It may be broken across, as in the case of joists or levers of any kind.

3rd. It may be crushed, as in the case of pillars, posts, &c.

I. DIRECT COHESION OF DIFFERENT BODIES.

173. When a body is stretched in the direction of its length, its resistance will depend upon its natural power of cohesion, and the area of the transverse section. This will manifestly be true, since the cohesion of each particle is supposed to be alike, and therefore the whole force of cohesion must be proportional to their number, or to the area of the section. Many experiments have been made by different persons to ascertain the power which is necessary to overcome this direct cohesion; but the results are not very consistent with each other. The following, however, may be considered as the mean practical results which may be depended upon.

We shall take for the measure of cohesion the number of pounds avoirdupois which are just sufficient to tear asunder a prism or cylinder of one inch square. From this it will be easy to compute the strength corresponding to the area of any other section.

	lbs. av		lbs. av.		lbs av.
Oak..........	13000	Teak......	15000	Steel........	115500
Beech	14900	Pear	9000	Iron wire.....	103500
Alder	14200	Walnut....	81000	Malleable iron	66250
Chestnut, Span.	13300	Poplar.....	5600	Cast iron.....	18500
Ash..........	15200	Fir	10970	Copper.......	33800
Elm..........	13500	Scotch pine	7800	Gun metal....	36300
Acacia........	20600	Cedar......	5000	Yellow brass..	18000
Mahogany.....	8000	Larch......	10200	Copper wire...	61200
Box..........	20000	Norway pine	7300	Chain cable...	88152

Hemp rope, one inch in circumference, 6166lbs.

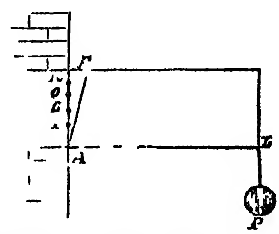
174. There are many curious facts, which are difficult to be explained, respecting the direct cohesion of metals. Thus, if a metal be forged, or frequently drawn through a small hole in a steel plate, its cohesion is very much increased, and at the same time its density is augmented. But in lead the density is diminished, whilst its cohesion is more than tripled by this operation. Gold, silver, and brass, have their cohesion nearly tripled; copper and iron have it more than doubled. It is also remarkable that almost all the mixtures of metals are more

tenacious than the metals themselves; and the change of tenacity depends in a great measure on the proportion of the ingredients. See the first volume of Robison's Mechanical Philosophy.

The ultimate strengths of different materials have been given above, but two-thirds of these weights will in general impair their strength after some time; and the engineer cannot with safety suspend more than half these weights in his constructions.

II. TRANSVERSE STRENGTH OF BEAMS.

175. PROP. I.—*To determine the transverse strain of a beam fixed horizontally in a wall, when a weight is suspended at its extremity; the weight of the beam being neglected.*



Let BAL be a beam projecting horizontally from a wall, in which it is firmly fixed; and let the weight P be suspended at the extremity L. Let the body be supposed to break in the vertical section BA, perpendicular to AL, by means of the strain exerted upon it by P. Now, the beam BL may be considered as a bent lever, in which the force P tends to turn it round a horizontal axis passing through A perpendicular to AL, and the force of cohesion of all the particles in AB tends to resist this separation. Let m represent any indefinitely small area of the transverse section at m, whose distance from the axis passing through A is equal to x, and let φ represent the force of attraction of a single particle at m, then $\varphi \times m$ will be the force of attraction of the area m; and, from the property of the lever (art. 108), $\varphi m \times x$ will be the force of m to resist fracture. Hence, if we suppose the section AB to be made up of the elementary portions m, m', m'', &c., whose distances from the axis of fracture passing through A are x, x', x'' &c., and that the force of each particle in m, m', m'', &c. is φ, φ', φ'', &c., respectively, the whole cohesive force will be represented by $\varphi mx + \varphi' m'x' + \varphi'' m''x'' +$ &c. Now this must be equal to the energy of the power employed to break it. Let the length AL be called l, then $P \times l$ is the corresponding energy of the power. This, therefore, gives us

$$Pl = \varphi mx + \varphi' m'x' + \varphi'' m''x'' + \text{\&c.},$$

for the equation of equilibrium corresponding to the section AB.

176. PROP. II.—*To determine the transverse strain according to the hypothesis of Galileo.*

According to this hypothesis, every particle at the instant of fracture is exerting its full force of cohesion. Let this force be f, we have then, in the last proposition, $f = \varphi = \varphi' =$ &c.; therefore

$$Pl = f(mx + m'x' + m''x'' + \text{\&c,}) = fMh,$$

M being the area of the transverse section, and h the distance of the centre of gravity from the fulcrum A. But fM evidently expresses the absolute cohesion of the section of fracture, therefore

$$Pl = \text{absolute cohesion} \times h.$$

177. *Cor.*—If the solid is a rectangular beam, whose depth $AB=a$, and breadth $= b$, then $h = \frac{1}{2}a$. Therefore, in this case, $Pl = \frac{1}{2}fa^2b$.

178. Prop. III.—*To determine the transverse strain according to the hypothesis of Dr. Hooke.*

From the observations made by this philosopher, the attractive forces of different particles are proportional to the distances to which they are removed from each other; and Dr. Robison states that this is universally true *in all moderate extensions*. If, then, we suppose that the extensions of the fibres are proportional to their distances from A, which seems not improbable, it will follow that the cohesive force of any particle at m will be as Am. Hence, if f be the attractive force of a particle at B, $\varphi : f :: x : a$, therefore

$$\varphi = \frac{fx}{a}; \text{ in like manner } \varphi' = \frac{fx'}{a} \quad \varphi'' = \frac{fx''}{a}, \text{ \&c.}$$

$$\therefore \quad Pl = \frac{f}{a}(mx^2 + m'x'^2 + m''x''^2 + \text{\&c.})$$

Now, it will be proved, in a future part of this work, that if G be the centre of gravity, O the centre of oscillation of the transverse section AB vibrating about the axis of fracture, $h = AG$, $o = AO$, and $M = m + m' + m'' + \text{\&c.}$, then

$$mx^2 + m'x'^2 + m''x''^2 + \text{\&c.} = Mho; \quad \text{and} \quad Pl = fM\frac{ho}{a}.$$

Let AI be taken a fourth proportional to AB, AO, and AG, we shall have $AI = \frac{ho}{a}$; and if we put $AI = i$,

$$Pl = fMi.$$

Let f represent the full force of cohesion of the particle at B, when it is on the point of breaking, it is evident that a total fracture will immediately ensue; for the strain which overcame the full cohesion of the particle at B, and the proportional part of the cohesion of all the other particles, must be more than sufficient to overcome the full cohesion of the particle next within B, and a corresponding portion of the cohesion of a smaller number of particles which remain. Hence it follows, that the strength of any beam, according to these principles, is to its strength, according to the hypothesis of Galileo, as AG to AI, or as AB to AO. This hypothesis is frequently attributed to Leibnitz.

179. *Cor.* 1.—Since the adhesive forces are all acting in a direction parallel to each other, their resultant R is equal to their sum; therefore

$$R = \frac{fmx}{a} + \frac{fm'x'}{a} + \frac{fm''x''}{a} + \text{\&c.} = \frac{f}{a}Mh.$$

Also, the sum of the momenta to resist the fracture, by the last article, is equal to $\frac{f}{a}Mho = Ro$. Thus it appears that the

momentum of actual force of cohesion = absolute cohesion $\times AI$,
or, = actual cohesion $\times AO$.

The actual force of cohesion is considerably less than the absolute

cohesion, for it is only the extreme particles at B which are exerting their full cohesive force at the instant of fracture.

180. *Cor.* 2.—If the solid is a rectangular beam, whose depth $AB = a$, and breadth $= b$, then $h = \frac{1}{2}a$, and $o = \frac{2}{3}a$; consequently $Pl = \frac{1}{3}fa^2b$. The strength, therefore, in this case, is to the strength, according to Galileo, as 2 : 3.

181. *Cor.* 3.—Hence, a joist laid on its narrow edge b is stronger than when laid on its flat side a, on either supposition, in the proportion of a to b.

182. PROP. IV.—*If a weight* P *be placed on any part of a horizontal beam supported at the ends, the strain on that part will be proportional to the rectangle* AP × PB

Since the weight P is supported by the two props, the pressures on A and B will be $P\frac{PB}{AB}$ and $P\frac{AP}{AB}$. Now, if we conceive two weights equal to these two pressures to be placed on the beam at A and B, and a fulcrum to be placed at P, it is manifest that the beam will be subject to the same strains as before, for the three forces acting at A, P, and B, in the last case, are equal to the three former forces acting in opposite directions. But in the last case the weight $P\frac{PB}{AB}$ acting at A will manifestly produce a strain on the fulcrum at P proportional to the weight multiplied by the arm of the lever AP, or to $P\frac{AP \times PB}{AB}$. The pressure $P\frac{AP}{AB}$ acting at B merely serves to balance the pressure at A, and the strain is the same as if PB were firmly fixed in a wall. Hence, if P and AB be given, the strain is proportional to $AP \times PB$ and is the greatest when P is in the middle of AB.

183. *Cor.*—The strain at any other point C arising from the pressure $P\frac{PB}{AB}$ acting at A is $P\frac{PB}{AB} \times AC$.

184. PROP. V.—*To find the strain at any point* C, *when the weight is uniformly distributed along the beam.*

Let the weight of any indefinitely small portion Pp of the weight $P = m$. Put $AC = a$, $CB = b$, $AB = l$, and $PB = x$. We have then the strain at C arising from the elementary portion m

$$= m\frac{AC \times PB}{AB} = \frac{a}{l}mx;$$

and, if we suppose the weight on CB to be made up of all the elementary portions m, m', m'', &c., whose distances from B are x, x', x'', &c., we shall have the whole strain at C arising from the weight on CB

$$= \frac{a}{l}(mx + m'x' + m''x'' + \&c.) = \frac{a}{l}Mh,$$

M being $= m + m' + \&c.$, and $h =$ the distance of the centre of gravity of M from the point B. In like manner we shall find the strain

at C arising from the weight M' on $AC = \frac{b}{l} M'h'$, h' being the distance of the centre of gravity of M' from the point A. Hence

$$\text{whole strain at } C = \frac{Mah}{l} + \frac{M'bh'}{l};$$

and if the weight be uniformly distributed along AB, $h = \frac{1}{2}b$, and $h' = \frac{1}{2}a$, therefore the

$$\text{whole strain at } C = \frac{(M + M')\frac{1}{2}ab}{l} = P\frac{ab}{2l}.$$

185. PROP. VI.—*To determine the strain of a beam arising from its own weight.*

When a beam projects from a wall, every section is strained by the weight of all that projects beyond it. This is the same as if this part were collected at its centre of gravity. Therefore the strain on any section is proportional to the weight which projects beyond it multiplied into the distance of its centre of gravity from this section.

If, therefore, we suppose two similar cylinders, or any two similar beams, projecting from a wall, the strengths of the beams are as the cubes of the diameters, while the strains are as the fourth powers of the diameters; because the weights are as the cubes, and the levers by which these weights act in producing the strain are as the lengths, or as the diameters.

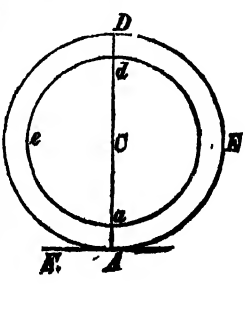
186. *Cor.*—Hence it appears that, in similar bodies of the same texture, the force which tends to break them in the larger bodies increases in a higher proportion than the force which tends to preserve them entire; and, therefore, although a small beam may be abundantly strong, a similar larger beam may even break by its own weight. From these principles it is inferred that there are necessarily limits in all the works of nature and art which they cannot surpass in magnitude. The cohesion of an herb could not support it if it were increased to the size of a tree; nor could an oak support itself if it were considerably greater than its present size.

Hence we may easily solve the following problems.

187. PROBLEM I.—*To find the strongest joists that can be cut out of a cylindrical piece of timber.*

Since the strength varies as a^2b, it will easily be found that the strongest joist is when $b = \frac{2}{3}r\sqrt{3}$; and that the strength of this joist is to that of a square joist cut out of the same timber as 10 to 9, nearly.

188. PROBLEM II.—*To compare the strength of a hollow cylinder with one that is solid, having equal length and equal quantity of materials.*



Let DEA be a section of the hollow cylinder, and C the centre of the section. Put $CA = R$, $Ca = r$, then it will be afterwards shown that the sum of all the mx^2 in the circle dea, measured from the line $AF = \pi r^2(R^2 + \frac{1}{4}r^2)$, and the sum of all the mx^2 in the circle $DEA = \pi R^2(R^2 + \frac{1}{4}R^2)$. Hence the sum of all the mx^2 in the ring $DdAa$

$$= \pi R^2(R^2 - r^2) + \tfrac{1}{4}\pi(R^4 - r^4) = \pi(R^2 - r^2)\,[R^2 + \tfrac{1}{4}(R^2 + r^2)]$$
$$= \tfrac{1}{4}M(5R^2 + r^2);$$
$$\therefore\quad Pl = \frac{f}{a}(mx^2 + m'x'^2 + \&c.) = \frac{fM}{8R}(5R^2 + r^2)$$
$$= fM\left(\frac{5R}{8} + \frac{r^2}{8R}\right).$$

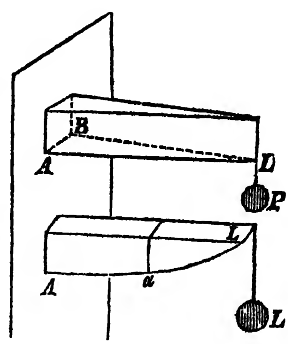
Also, if ϱ be the radius of the solid cylinder having an equal length and an equal quantity of materials, and P' the weight which would just break it,

$$P'l = fM \times \frac{5\varrho}{8},$$

$$\therefore \text{ strength of hollow cyl. : do. solid cyl. } :: P : P' :: \frac{5R}{8} + \frac{r^2}{8R} : \frac{5\varrho}{8},$$
$$:: R + \frac{r^2}{5R} : \sqrt{R^2 - r^2}.$$

189. *Scholium.*—This property of hollow tubes is also accompanied with greater stiffness. It is evident that the fracture will take place as soon as the particles at D are separated beyond the utmost limit of cohesion. This is a constant quantity, and the piece bends until this degree of extension is produced in the outermost fibre. It follows that the less is the distance between A and D, the more will the beam bend before it breaks. Greater depth, therefore, makes a beam not only stronger but also stiffer. Hence we see the wisdom of Providence in having formed the stalks of corn and the feathers and bones of animals hollow. Our best engineers also now imitate Nature, in making many parts of their machines hollow, such as their axles of cast-iron, &c.; and modern philosophical instrument makers now form the axles and framings of their great astronomical instruments in the same manner.

190. PROBLEM III.—*To determine the figure of a beam in order that it may be equally strong in every part, or that the strength may be everywhere proportional to the strain.*



1. Suppose, first, that the strain arises from a weight suspended at one extremity, while the other end is firmly fixed in a wall; and suppose that all the sections are rectangles equally deep. Since Pl is proportional to a^2b (art. 179), and P and a^2 are the same throughout, l will be proportional to b, and therefore the horizontal section ALB will evidently be a triangle.

2. If the beam be of uniform breadth, or b be constant, l will be proportional to a^2. Hence the curve LaA will be the common parabola, having the vertex L and the axis LD.

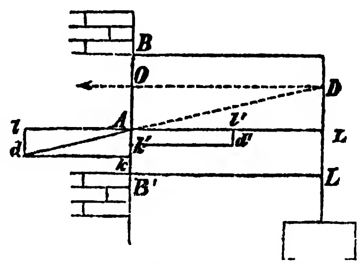
3. If the sections are all squares, or other similar figures, l will be

proportional to a^3; and therefore the depths will be the ordinates of a curve called a cubical parabola.

191. We have here supposed that the weight is placed at the extremity of the beam. If the weight be uniformly distributed over the beam, or if the strain arises from the weight of the beam itself, the figures will be different from the former; but our limits will not permit us to pursue the subject farther.

Scholium.

192. In the preceding propositions we have supposed that, at the instant of fracture, all the particles were in a state of extension. This is manifestly impossible, unless the particles be absolutely incompressible, for, in order to stretch the fibres at B, there must be a reaction, and some fulcrum to support the pressure of the lever. Thus, if BAL be the beam, A the fulcrum about which it turns, and all the fibres in AB be in a state of extension; there is some point O at which the resultant R, or the accumulated cohesion of all the forces, may be supposed to act. Let p be the weight suspended at L which balances this resultant, and let OD be drawn parallel to AL. Then, since the force p balances the resultant R, by the property of the lever



$$p : R :: AO : AL :: DL : DO.$$

Also, because the forces p and R act in the directions DL, DO, they may be supposed to act at the point D; and since they are proportional to DL, DO, their resultant will be proportional to DA, which is manifestly the pressure at the point A. Take Ad to represent this pressure at A; we may again resolve it into two forces Al, Ak, of which Ak is equivalent to the weight p, and is supported by the lateral cohesion of all the particles between A and B, for the weight p tends not only to turn the beam round the fulcrum A, but also to draw it downwards. The other component force Al is equivalent to the sum of all the cohesive forces in AB; and this must be supported by the resistance of those particles which are in a state of actual compression.

Suppose, then, all the particles in AB' to be compressed; and let R' be the resultant of all the resisting forces. Let p' be the weight at L which will balance R'. Then p' and R', as in the former case, will produce a pressure at A. Let Ad' represent this pressure; this may again be resolved into Ak', Al'; of which Ak', as before, is equivalent to p', and is opposed by the lateral resistance of the particles between A and B', and Al' is equivalent to R'. Now it is evident that these two forces R and R' must balance each other, or that $Al=Al'$, for the forces Ak, Ak' can have no tendency either to augment Al or Al'; hence $R = R'$.

193. These remarks are fully verified by experiments. Duhamel took 16 bars of willow, and supporting them by props under the ends, he broke them by weights hung on the middle; and he found the average weight which broke 4 bars to be 45lbs. He then cut 4 of them a third through on the upper side, and filled up the cut with a thin piece of

harder wood stuck in pretty tight, and he found the average weight which broke these bars to be 51lbs. He cut other 4 half through, and found the average weight was 48lbs. The remaining 4 were cut two-thirds through, and their mean strength was 42lbs. Other experiments agreed with these results. From this it appears that more than half the thickness contributed nothing to the strength. Mr. Barlow also made a number of experiments with reference to this point, and he found that "the beams in most cases showed very distinctly, after the fracture, what part of the section had been compressed and what part had experienced tension; the compressed fibres being always broken very short, having been first crippled by the pressure to which they had been exposed, while the lower part was drawn out in long fibres frequently 5 or 6 inches in length." See Robison's Mechanical Philosophy, and Barlow's Essay on Strength and Stress of Timber.

194. Prop. VII.—*To determine the strength of beams when the particles are both extensible and compressible.*

Let BAL be the vertical section of the given beam. Suppose the particles in the line AB to be in a state of extension, and those in the line AB' to be in a state of compression. Draw AL in a horizontal direction, then the particles which are at the point A, being situated between those which are extended and those which are compressed, are manifestly in a state of neutrality; and, therefore, the horizontal line drawn through A perpendicular to the plane BL has been denominated the *neutral axis*. Now, if we suppose, as in art. 177, that the attractive forces are as the extensions, we shall have the whole resistance to fracture in $AB = fM\dfrac{ho}{a}$; f being the absolute cohesion of a fibre at B, M the area of the section AB, $a = AB$, h the distance of its centre of gravity, and o the distance of its centre of oscillation from the point A. Let f' represent the resisting force of compression of a fibre at B', at the instant of fracture; and let M', a', h', o' represent the same quantities with respect to AB' as the unaccented letters in the section AB. If, then, the resisting forces be as the compressions, we shall have the whole resistance to fracture in $AB' = f'M'\dfrac{h'o'}{a'}$, and consequently

$$\therefore\quad Pl = fM\frac{ho}{a} + f'M'\frac{h'o'}{a'}.$$

But because $R = R'$ (art. 198), and also $R = fM\dfrac{h}{a}$, $R' = f'M'\dfrac{h'}{a'}$, it follows that

$$Pl = fM\frac{h}{a}(o + o').$$

195. *Cor.*—If the section be a rectangle, and the extensions be equal to the compressions at the instant of fracture, $a = a'$, $h = h' = \frac{1}{2}a$, $o = o' = \frac{2}{3}a$, therefore $Pl = \frac{2}{3}fa^2b$ where a is half the depth. If we put a for the whole depth, or substitute $\frac{1}{2}a$ for a, we have $Pl = \frac{1}{6}fa^2b$. Thus it appears that the strength on this supposition is only half its value when the body is incompressible. The reason of this is sufficiently

apparent, for only a portion of the section is exerting cohesive forces at the instant of fracture, while the remaining part merely serves as a fulcrum to the lever.

196. The following numbers have been taken from Barlow; and it must be remembered, that they represent the utmost weights which the body can bear without straining, and that only half these numbers can be used with safety by the engineer. If l = length of beam in feet; a = depth in inches; b = breadth in inches; s = the strength derived from experiment given in the table, and estimated in pounds avoirdupois; then the weight P required to break the beam will be $\frac{a^2b}{l}s$.

The beam is supposed to be fixed at one end and loaded at the other; if it be supported at both ends and loaded in the middle, the strength will be four times as great.

	Strength.		Strength.		Strength.
Teak	205	Beech	130	Birch	160
English oak	126	Elm	85	Christiana deal	130
Canadian oak	147	Fir	92	Memel deal	144
Dantzic oak	122	Larch	71	Cast iron	674
Ash	169	Acacia	156	Malleable iron	750

III. RESISTANCE OF COLUMNS, STRUTS, &c., TO COMPRESSION.

197. This strain is the most difficult of any to be explained on mechanical principles, and little more has hitherto been done than to obtain certain formulæ for the use of the engineer, which have been derived solely from experiment. Thus, in a rectangular column, if a be the depth or greater dimension of the section in inches, b the breadth or less dimension in inches, l the length in feet, P the weight in pounds, and e the number given in the table, then

$$ePl^2 = ab^3.$$

And in the case of a round column or cylinder, if d be the diameter,

$$Pl^2 \times 1{\cdot}7e \doteq d^4.$$

The numbers given in this table are those which may be used in practice.

English oak	·0015	Acacia	·00152	Riga fir	·00152
Beech	·00195	Mahogany Spain	·00205	Memel fir	·00133
Alder	·0023			Norway spruce	·00142
Chestnut, green	·00267	do. Honduras	·00161	Weymouth pine	·00157
Ash	·00168	Teak	·00118		
Elm	·00181	Cedar, Lebanon	·0053	Larch	·0019

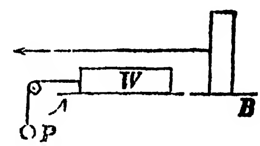
FRICTION.

198. In our investigations of the problems of equilibrium, we have supposed the surfaces of bodies to be perfectly smooth. But, in practice, all bodies are more or less rough; and, therefore, the results which we

have obtained will be more or less modified by the effects of the roughness.

The friction of a body on a surface is measured by the least force which will put the body in motion along the surface. Thus, if the weight W rest on the horizontal table AB, and it be drawn horizontally by the force P; then, if P will put W in motion, and any smaller force will not put it in motion, P measures the friction of W upon the table.

199. Many experiments have been made by Coulomb, Ximenes, &c., and in later times by the French Académie des Sciences (1834), to determine the measure and varieties of friction; and the following are results which they have deduced:—



(1). *The friction is independent of the extent of the surfaces in contact, so long as the pressure continues the same.*—Let W be a rectangular parallelopiped, and let it be placed on its broader side on the horizontal table AB, and let the force be ascertained which, acting horizontally, will first put the body in motion. Again, let it be placed on its narrower side, and put in motion as before. It will be found that the force requisite for this purpose is also the same as in the first case; and therefore the friction is independent of the extent of surface in contact.

When the surfaces in contact are very small, this law gives the friction much too great.

(2). *The friction is proportional to the pressure, when other circumstances are the same.*—This is not exactly true, for the friction corresponding to large pressures is a little less than this law would give.

These two laws are true when the body is on the point of moving, and also when it is actually in motion; but in the former case the friction is much greater than in the latter. As soon as the force has overcome the friction and put the body in motion, the friction is instantly diminished. Thus, in new wood planed, the friction amounts to $\frac{1}{2}$ the pressure when it is at rest, and is immediately reduced to $\frac{1}{8}$th the pressure as soon as motion begins.

(3). *The friction is independent of the velocity when the body is in motion.*

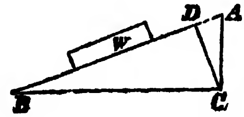
It follows, from these laws, that if R be the pressure of the body perpendicular to the surface, the friction may be represented by fR; f being an invariable coefficient for the same substance, which is called the coefficient of friction.

200. When the body is on the point of moving, the values of f in the following substances is found from experiments to be as below:—

Iron on oak	·62	Brass on brass	·20
Cast-iron on oak	·49	Wrought iron on cast-iron	·19
Oak on oak (fibres parallel)	·48	Cast-iron on elm	·19
Leather belts on wooden pulleys	·47	Brass on iron	·16
Leather belts on cast iron do.	·28		

Oil and grease considerably diminish friction; fresh tallow reduces it to half its value.

201. PROP. VIII.—*To determine by experiment the value of the coefficient of friction.*



Let AB be an inclined plane, whose inclination may be altered at pleasure. Let the weight W be placed upon it, and let the plane be gradually raised from a horizontal position until W begins to slide down the plane. Let AB be the position of the plane immediately before the body begins to slide.

In this position the body is kept in equilibrium by three forces: its weight, which acts in a direction parallel to AC; the resistance of the plane, which is parallel to CD; and the friction, which necessarily acts along the inclined plane, and is therefore parallel to AD. These forces, therefore, are as these three lines; and, consequently,

$$\text{friction} : \text{pressure} :: AD : DC :: AC : BC;$$

therefore $$fR : R :: AC : BC.$$

Hence $$f = \frac{AC}{BC} = \text{tangent of the angle } B.$$

202. *Cor.*—In all cases we may resolve the pressure on any surface into two forces, the one perpendicular to the surface, which will be supported by the re-action of the surface, and the other parallel to the surface; and if this last be less than the friction, the equilibrium will not be disturbed.

CHAP. VII.—MISCELLANEOUS PROBLEMS.

203. The following problems will serve to illlustrate the previous parts of this subject, and to bring more immediately before the notice of the student the general principles upon which questions of equilibrium can be solved.

204. *Equilibrium of three forces.*—When three forces keep a rigid body in equilibrium, each of them passes through the intersection of the other two, and is equal and opposite to the resultant of the other two. For if P, Q, R be the three forces which keep the body at rest, the forces P, Q produce the same effect as if they acted at A, their point of intersection (art 10); and, therefore, they cannot be kept in equilibrium by a force which does not pass through A.

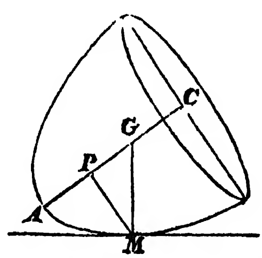
205. *Equilibrium on a surface.*—When a body rests on a given surface, it will touch it either in one point or in several points. In every case the body must be supposed to be acted on by forces perpendicular to the surface at the points where it is in contact, or by the re-action of the surface at those points. For if the force should be inclined to the surface, at any of these points, it might be resolved into two forces, one perpendicular to the surface, and the other parallel to it; and since the

surface is supposed to be perfectly smooth, this last force would cause the body to slide along it, and it would not be at rest.

206. *Equilibrium on two points.*—When a body is supported on its surface resting on two points, the re-action at each point will be in the direction of a perpendicular to the surface; and these perpendiculars must meet in the vertical line passing through the centre of gravity.

The same remarks may be made when a body rests on two surfaces, or on a point and a surface.

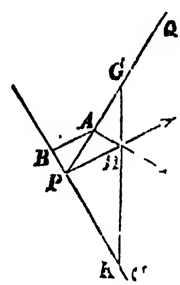
207. Prob. I.—*A paraboloid rests upon a horizontal plane; to find its position.*



Let G be the centre of gravity, and M the point of contact; then the line GM must be vertical, for the body may be supposed to be collected in its centre of gravity, and it will then be supported by the re-action which acts in the direction MG. It is evident, also, that G is a point in the axis, since the centre of gravity of every indefinitely thin circular section parallel to the base must be in the centre of the section, or in the axis of the paraboloid. Draw MP perpendicular to AC; and since MP is an ordinate, and MG evidently a normal, therefore $GP = \frac{1}{2}$ parameter $= p$. Hence, if GP be taken equal to p, and the ordinate PM be drawn, M will be the point on which the figure will rest.

Cor.—If the paraboloid is homogeneous, it will be proved, in the integral calculus, that $AG = \frac{2}{3}AC$; if, therefore, $\frac{2}{3}AC > p$, or $AC > \frac{3}{2}p$, there is always a point M on which the solid will be in equilibrium.

208. Prob. II.—*A given beam* PQ, *considered as a straight line, rests upon a given point* A, *with its end against the inclined plane* BC; *to find the position in which it will rest.*



Because the re-action of the plane BC is in the direction PH, perpendicular to BC; and the re-action of the point A is in HA, perpendicular to PQ, the point of intersection H of these forces must be in the vertical line, passing through G, the centre of gravity of PQ; hence GHK is vertical.

Let $AB = a$, $PG = b$,

angle $BAP = APH = \theta$; and also,

$\alpha =$ inclination of $BC = 90° - BKH = PHK$.

Then $AP = a \sec\theta$; $PH = AP \sec\theta = a\sec^2\theta$.

Also, $b : a\sec^2\theta :: \sin PHG : \sin PGH :: \sin\alpha : \sin(\alpha - \theta)$.

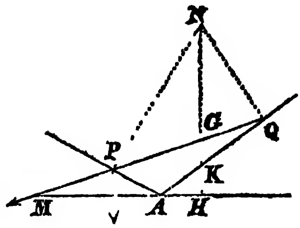
Hence $a \sin\alpha \sec^2\theta = b\sin(\alpha - \theta)$;

from which equation θ is to be determined.

Cor.—If the plane BC be vertical, $\alpha = 90°$, and $\cos\alpha = 0$;

$$\therefore \quad a\sec^2\theta = b\cos\theta; \quad \text{or} \quad b\cos^3\theta = a.$$

209. Prob. III.—*A given beam* PQ, *considered as a line, is supported on two given inclined planes* AP, AQ; *to find the position in which it will rest.*



Let G be the centre of gravity of the beam, and let PN, QN be drawn perpendicular to the planes, meeting one another in N. Because the re-actions of the planes AP, AQ are in the directions PN, QN, a vertical line, drawn through G, will pass through their intersection N. Let PQ meet the horizontal line AH in M. Now the angles QAH, QNH are evidently equal, for each of them is the complement of AKH; in like manner the angle $PAM = PNK$.

Put $PG=a$, $GQ=b$; $\angle\ PAM=PNH=\alpha$; $\angle\ QAH=QNH=\beta$; angle $M=\theta$: then $\angle APQ=\alpha+\theta$; and $\angle AQP=\beta-\theta$,

$$\therefore PG : GN :: \sin PNG : \sin NPG :: \sin\alpha : \cos(\alpha+\theta),$$

$$GN : GQ :: \sin NQG : \sin QNG :: \cos(\beta-\theta) : \sin\beta,$$

hence, multiplying corresponding terms,

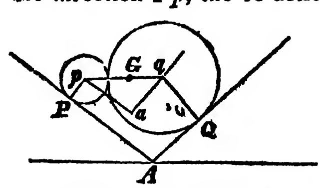
$$a : b :: \sin\alpha\cos(\beta-\theta) : \sin\beta\cos(\alpha+\theta),$$

from which proportion the student will find

$$\tan\theta = \frac{a\cot\alpha - b\cot\beta}{a+b}.$$

210. Prob. IV.—*To find the position of two spheres* p, q, *which touch each other and rest on two inclined planes* AP, AQ.

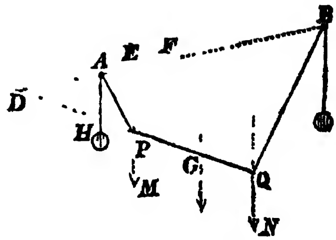
Join p, q their centres; draw pP, qQ to the points of contact; and pa, qa parallel to PA, QA. The sphere p is kept at rest by three forces, the re-action of the plane AP in the direction Pp, the re-action of q in the direction qp, and its weight p acting at p. Similarly, q is kept at rest by the re-actions of AQ and p, and its own weight q. Now, when there is an equilibrium, the re-action of q in the direction qp will be equal to the re-action of p in the direction of pq; hence it is evident that pq will be supported in the same way a a beam pq resting in planes ap, aq, parallel to AP, AQ: and, therefore, we may find its position by the last article.



Let G be the centre of gravity of the two spheres, and $pG=a$, $Gq=b$; then $\dfrac{a}{a+b}=\dfrac{q}{p+q}$, $\dfrac{b}{a+b}=\dfrac{p}{p+q}$; hence

$$\tan\theta = \frac{q\cot\alpha - p\cot\beta}{p+q}.$$

211. Prob. V.—*A given beam PQ hangs by two strings of given lengths AP, BQ, from two given fixed points A, B; to find its position when it rests.*



Let G be the centre of gravity of PQ; put $PG = a$, $GQ = b$, $PQ = c$, $AB = e$; also, $AP = p$, $BQ = q$.

Likewise put $\angle\, BAP = \alpha$, $ABQ = \beta$, $ADP = \delta$, $AEP = \epsilon$. Then will

$$APM = \alpha + \epsilon, \qquad APQ = \pi - (\alpha - \delta), \; MPQ = \pi - (\delta + \epsilon)$$

$$BQN = \pi - (\epsilon - \beta), \; BQP = \pi - (\beta + \delta), \; NQP = \delta + \epsilon.$$

And if PF be drawn parallel to BQ, $\angle\, APF = \pi - (\alpha + \beta)$.

Now, if perpendiculars be let fall upon BQ from A and P, we shall see immediately that

$$p \sin(\alpha + \beta) + c \sin(\beta + \delta) = e \sin\beta \ldots\ldots\ldots\ldots (1).$$

and in like manner, if perpendiculars be let fall upon AP from B and Q, we shall have

$$q \sin(\alpha + \beta) + c \sin(\alpha - \delta) = e \sin\alpha \ldots\ldots\ldots\ldots (2).$$

We have here three unknown quantities, α, β, δ. To obtain a third equation, we must take into consideration the condition of equilibrium. This may be done in several ways; the following, perhaps, is as simple as any.

Let W, the weight of PQ, be resolved into two parallel forces, $W\dfrac{b}{c}$, $W\dfrac{a}{c}$, acting at P and Q; call these forces M and N. Then the point P may be supposed to be kept at rest by three forces: the weight M, the tension of the string AP, and the strain upon the beam PQ; call this last strain T. Then

$$M : T :: \sin APQ : \sin APM :: \sin(\alpha - \delta) : \sin(\alpha + \epsilon).$$

In like manner the point Q is kept at rest by the three forces, N, the tension of BQ, and the strain upon QP, which is manifestly the same as the strain at P, acting in an opposite direction; therefore

$$T : N :: \sin BQN : \sin BQP :: \sin(\epsilon - \beta) : \sin(\beta + \delta).$$

Hence, multiplying the corresponding terms of these two proportions, and remembering that $M : N :: b : a$, we have

$$b : a :: \sin(\alpha - \delta)\sin(\epsilon - \beta) : \sin(\beta + \delta)\sin(\alpha + \epsilon),$$

$$\therefore\; a \sin(\alpha - \delta)\sin(\epsilon - \beta) = b \sin(\beta + \delta)\sin(\alpha + \epsilon) \ldots\ldots (3).$$

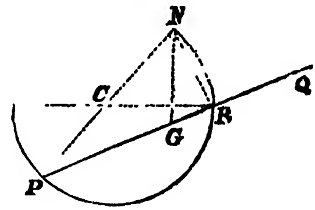
And these three equations (1), (2), (3), will give the three unknown quantities, α, β, δ.

212. *Cor.*—If P and Q be the tensions of the strings AP, BQ, we have

$$P : M :: \sin MPQ : \sin APQ :: \sin(\delta + \epsilon) : \sin(\alpha - \delta),$$
$$Q : N :: \sin NQP : \sin BQP :: \sin(\delta + \epsilon) : \sin(\beta + \delta).$$

Hence, if the beam be supported by strings which go over fixed pulleys at A and B, and have given weights P, Q attached to them, the values of α, β, δ may be determined from the two last proportions, and equation (3).

213. PROB. VI.—*A uniform beam* PQ, *considered as a line, is placed with one end* P *inside a hemispherical bowl, and a point* R *in it resting on the edge of the bowl; find* PR *when* PQ = 3 *times the radius.*



Draw the diameter PN, and join NR; then will NR be perpendicular to PR, and the re-actions of the surface and the point R will be in the directions PN, RN. A vertical line, therefore, drawn from N will pass through G, the centre of gravity of PQ.

Put the angle $CPR = CRP = \theta$; then $CRN = CNR = 90^\circ - \theta$; $RNG = \theta$; and $CNG = 90^\circ - 2\theta$. Also, put $CP = r$, $PG = \frac{3}{2}r$, $PR = x$, and $GR = x - \frac{3}{2}r$. Hence

$$PG : GN :: \sin PNG : \sin NPG :: \cos 2\theta : \sin\theta,$$
$$GN : GR :: \sin NRG : \sin RNG :: 1 : \sin\theta.$$

And multiplying corresponding terms together we get

$$PG : GR :: \tfrac{3}{2}r : x - \tfrac{3}{2}r :: \cos 2\theta : \sin^2\theta;$$

and since $x = 2r\cos\theta$, $\cos 2\theta = 2\cos^2\theta - 1$, $\sin^2\theta = 1 - \cos^2\theta$;

$$\therefore \quad 3r : 4r\cos\theta - 3r :: 2\cos^2\theta - 1 : 1 - \cos^2\theta.$$

Hence, multiplying and reducing,

$$8\cos^2 - 3\cos\theta = 4.$$

From which equation we find $\cos\theta = \cdot 919$ and $PR = 1\cdot 838r$.

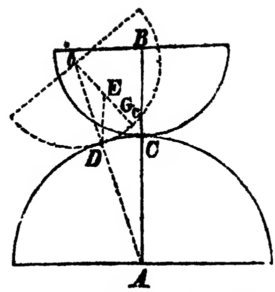
STABLE AND UNSTABLE EQUILIBRIUM.

214. We have already seen, in art. 64, that if a body be made to deviate from the position of equilibrium, it has, in some cases, a tendency to return to it; but, in other cases, if the position of a body be changed in the smallest degree, it has a tendency to move farther from the position of equilibrium, until it assume some new position. The following problem will serve to illustrate this distinction.

215. PROB. VII.—*A spherical segment rests upon a sphere; to find in what case the equilibrium will be stable.*

When the body is at rest, it must evidently touch the sphere at the highest point, and its centre of gravity will be in the vertical line, passing through the point of contact. Let C be this point, A the centre of the sphere, and B the centre of the sphere of which the segment is a part.

Let the segment come into any other position touching the sphere



in D; so that the radius BC comes to the position bc. Draw DE vertical; then DE is parallel to AC, and in the plane CAD; and will, therefore, meet bc in some point E. Let G be the centre of gravity of the segment in the second position, then it is evident that, if G be situated between E and C, the body will have a tendency to return to its first position; but if G fall beyond E, it will have a tendency to recede farther from it.

Let $CA = a$, $CB = b$, and the arc $CD = s$; then the arc Dc is manifestly also $= s$, since each point of Dc has been applied to the arc DC. Hence the angle $EDb = CAD = \frac{s}{a}$, and the angle $cbD = \frac{s}{b}$.

Now $\quad bE : DE :: \sin bDE : \sin EbD :: \sin \frac{s}{a} : \sin \frac{s}{b}$.

And when the angles are diminished indefinitely, the sines of the angles are proportional to the angles themselves; therefore

$$bE : DE :: \frac{s}{a} : \frac{s}{b} :: b : a;$$

$$\therefore\ bE + DE : DE :: a + b : a.$$

But when CD is extremely small, ED becomes Ec, and $bE + ED$ becomes bc.

Hence $\qquad Ec = \dfrac{ab}{a + b}$;

and the equilibrium will be stable if cG be less than this.

216. *Cor.* 1.—If the body rest on the concave surface of a sphere, we shall find, in the same manner, that the equilibrium will be stable, if cG be less than $\dfrac{ab}{a - b}$.

217. *Cor.* 2.—If the surfaces have any curvilineal form, the same expressions will apply; a and b representing, in this case, the radius of curvature of the body and of the surface at the point of contact.

If a be infinite, we have the case of a curve surface resting on a horizontal plane, and the equilibrium will be stable if cG be less than $\dfrac{ab}{a + b}$ or less than $\dfrac{b}{1 + \dfrac{b}{a}}$ or $< b$.

PROBLEMS FOR EXERCISE.

1. From a given rectangle $ABCD$, of uniform thickness, to cut off a triangle CDP, so that the remainder, when suspended at P, shall have the sides AP, BC horizontal.

2. A solid, composed of a cone and a hemisphere on the same base, with their axes in opposite directions, rests on a horizontal plane; to find its dimensions that it may rest on the hemispherical end in all positions.

3. When a body is supported on three vertical props, to find the pressure on each.

4. C is the centre, CA the horizontal radius, and A the highest point of a quadrant in a vertical plane. A string passes over a pulley at A, supporting two weights, P and Q $(= 2P)$, the former hanging freely, and the latter resting on the concave arc of the quadrant. Find the position of equilibrium. Ans. Angle $PAQ = 32° \; 32'$.

5. A string passes over an indefinitely small pulley at the focus of a parabola whose axis is vertical. The weight P, hanging freely, supports Q on the convex arc of the parabola. Show that $P = Q$, and that there is an equilibrium in all positions.

6. A uniform beam PQ rests with one end P on a smooth vertical wall; the other end Q is supported by a string fastened to a point A in the wall. If the beam be 2 feet and the string 3 feet in length, find AP.

7. A sphere, whose weight is W, rests between two inclined planes; required the perpendicular pressure upon the planes.

8. Two weights P, Q are connected by a string passing over the convex circumference of a vertical circle, whose centre is C, and highest point Z; required the position of the weights when they are in equilibrium, and the length of the string is one-fourth of the circumference.

9. Two iron rods, of equal thickness, whose lengths are a and b, rest against each other on a smooth horizontal plane; the upper ends resting against two smooth vertical walls that are parallel. Prove that, if c be the distance between the walls, and α, β the inclinations of the rods to the horizon,

$$a \cot \alpha = b \cot \beta; \quad \text{and} \quad a \cos \alpha + b \cos \beta = c.$$

10. A uniform beam AB, moveable about A, leans upon a prop CD, situated in the same vertical plane; show that if $AB = 2a$, $CD = b$, $\angle BAC = \alpha$, $\angle ACD = \beta$, then the strain upon the prop, or the pressure perpendicular to $CD = \dfrac{Wa \sin 2\alpha \cos(\alpha + \beta)}{2b \sin \beta}$.

11. A given weight P is suspended from the rim of a uniform hemispherical bowl, placed on a horizontal plane; show that if W = weight of bowl, c = distance between its centre and centre of gravity, and θ the inclination of the axis to the vertical; then $\tan \theta = \dfrac{Pr}{Wc}$, when the bowl is at rest.

12. An isosceles right-angled triangle, whose hypothenuse $= 2a$, rests in a vertical plane with the right-angle downwards, between two points at a distance b from each other in the same horizontal line; to determine its positions of equilibrium.

Ans. Inclination of the base to the horizon $= 0$, or $\cos^{-1}\left(\dfrac{a}{3b}\right)$.

13. A uniform beam [illegible]hose weight is W, rests with one end P on a horizontal plane AP, and the other end on an inclined plane AQ, whose angle of inclination to the horizon is 60°. If a string AP (which is $= AQ$) prevent the beam from sliding, what is the tension of this string? $T = \sqrt{3}\, W$

PART II.—DYNAMICS.

CHAP. I.—DEFINITIONS AND PRINCIPLES.

218. Dynamics is that branch of Mechanics which relates to the action of force producing motion. The most simple principles to which this science can be reduced appear to be those laws of motion laid down by Sir I. Newton; and they have, therefore, generally been adopted by writers in this country as the foundation of all their demonstrations.

219. By *motion* is understood the act of a body's changing place; and it is divided into two kinds, *absolute* and *relative*.

A body is said to be in absolute motion when it is actually transferred from one point in fixed space to another; and to be relatively in motion when its situation is changed with respect to surrounding bodies.

220. When a body always passes over equal parts of space in equal successive portions of time, its motion is said to be *uniform*. When the successive portions of space described in equal times continually increase, the motion is said to be *accelerated;* and to be *retarded* when they continually decrease. Also, the motion is said to be *uniformly* accelerated or retarded when the increments or decrements of the spaces described in equal successive portions of time are always equal.

221. The degree of swiftness or slowness of a body's motion is called its *velocity*, and it is *measured* by the space *uniformly described* in a unit of time, as, for instance, in one second.

In *variable* motions the velocity is measured by the space which *would have been described* in a unit of time, if the motion had continued uniform from that point; or had, at that point, ceased to increase or decrease.

222. The *momentum* of a body is the product of its velocity and quantity of matter.

223. *Accelerating force* is measured by the velocity uniformly generated in a given time, no regard being had to the quantity of matter moved.

224. *Moving force* is measured by the momentum uniformly generated in a given time; and is equal to the product of the accelerating force and the quantity of matter.

225. Prop. I.—*In uniform motion the space described in any time is equal to the product of the velocity and the time.*

Let v be the velocity, then, by the last article, v is the space described in one second. And, since the motion is uniform, v is the space described in the next second, or $2v$ is the space described in two seconds. In like manner, $3v$ is the space described in three seconds; and generally tv in t seconds. If, therefore, s be the whole space described,

$$s = tv.$$

226. *Cor.*—Since the motion is uniform, this equation is evidently true also, when t is a fractional number.

227. First Law of Motion.—*A body in motion, not acted on by any external force, will move in a straight line, and with a uniform velocity.*

It is difficult to *demonstrate* this law satisfactorily by direct experiment, because there is no place where a body is entirely uninfluenced by external causes. But if it appear from experience that the more we remove these causes, the more nearly uniform is the motion of the body, we may justly conclude that any deviation from the first direction and the first velocity must be attributed to the agency of external causes; and that there is no tendency in matter itself to increase or diminish any motion impressed upon it.

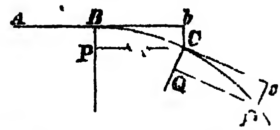
Now, if a ball be thrown along a rough pavement, its motion will soon cease. Its motion will continue longer if it be bowled upon a smooth bowling-green; and if be thrown along a smooth sheet of ice, it will preserve both its direction and its motion for a considerable time. In these cases the causes which retard the body's motion are friction and the resistance of the air, and in proportion as the former is diminished, the motion becomes more nearly uniform and rectilineal.

When a wheel is accurately constructed, and a rotatory motion about its axis communicated to it, the motion will continue for a considerable time; if the axis be placed upon friction wheels, the motion will continue longer; and if the apparatus be placed under the receiver of an air-pump, the motion will continue without visible diminution for a very long time.

From these and similar experiments we infer that, if we could entirely remove the external causes which retard the motion of bodies, the velocity would continue undiminished for ever; and, therefore, that the first law of motion is true.

228. This property of all bodies to resist any change, either from a state of rest or from a state of uniform rectilineal motion, is called their *inertia* or *vis inertiæ*.

229. Second Law of Motion.—*When a force acts upon a body in motion, the change of motion in magnitude and direction is the same as if the force acted upon the body at rest.*



Thus, if a body, considered as a point, move uniformly from A to B in $1''$, by the first law of motion, it would in the next second describe Bb in the same straight line equal to AB. At B let a force in the direction BP act uniformly upon it for $1''$; the force being such that it would in $1''$ cause the body to describe BP from rest. Then, at the end of this second, the body will be found at C, the opposite angle of parallelogram $CQDc$.

The truth of this law is shown from such experiments as the following:—

A ball rolled along the horizontal deck of a ship in motion will move precisely in the same manner as if the vessel were at rest.

A ball dropped from the main-topmast of a vessel in motion will fall at the same place on the deck as if the vessel were at rest.

If a ball be projected horizontally at a certain height, and at the same instant another ball be suffered to fall freely from rest, from the same place, the two balls will strike the ground at the same instant of time.

Since the earth revolves on its axis from west to east, if this law of motion were not true, bodies struck in a direction north or south would not move in that direction. Likewise, the oscillations of a pendulum would be performed in different times, according as it vibrated in a north and south plane, or in an east and west plane; which is not found to be true.

230. Third Law of Motion.—*When pressure communicates motion to a body, the momentum generated in a given short time is proportional to the pressure.*

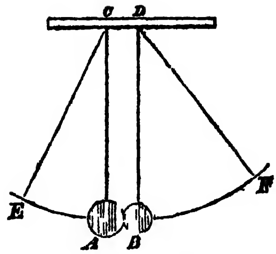
The experiments which most satisfactorily prove the truth of this law of motion are made with great accuracy by means of a machine invented by Atwood, for the purpose of examining the motions of bodies when acted upon by constant forces. This machine may be simply described as a single fixed pulley, with its axis placed on friction wheels, to diminish the friction. Let two bodies, P, P, equal in weight, be placed in two similar and equal boxes, which are connected by a string passing over the pulley, then these will exactly balance each other. Now let a weight p be added to either of them, then it is found that the velocity generated in a given time is always proportional to $\frac{p}{2P+p}$; that is, if the whole weight moved $2P+p$ be the same, the velocity is as p, the weight which puts the whole in motion; and if p be constant, the velocity is inversely as $2P+p$, the weight moved. There is a correction necessary to be introduced on account of the inertia of the wheels, which Atwood has done. Since the velocity is proportional to $\frac{p}{2P+p}$, it follows that p varies as $(2P+p) \times$ vel., and, therefore, varies as the momentum generated in a given time, or varies as the moving force (art. 225).

231. Prop. II.—*In the direct impact of two bodies the momentum gained by one is equal to the momentum lost by the other.*

Impact is only a pressure of short duration, increasing from nothing to a finite magnitude, and then decreasing to nothing again. Now, when two bodies impinge upon each other, the pressures of each arising from the contact of their surfaces must be equal and in opposite directions. Hence, by this third law, the momenta added to the two bodies in opposite directions will be equal.

232. Momentum gained and lost are sometimes called *action* and *re-action;* and in this sense *action and re-action are equal and opposite.*

This is the form in which the law has been given by Newton, and he has proved it from the following experiment:—Let A and B be two balls suspended by equal and parallel threads from two points C and D in the same horizontal line. Let a small steel point be fixed in A, so that when A and B come in contact, they will be prevented from separating. Then it is found that if A and B are drawn through the arcs



AE, *BF*, so that the velocities acquired by the bodies shall be inversely as the masses, they will impinge and exactly destroy each other's motion, a small allowance being made for the resistance of the air. If one of the balls be moved through a greater arc, the two balls will move together after impact in the direction of the greater momentum; and Newton found that the velocity with which they move is that which will be shown in the next chapter to result from the third law of motion.

The method of ascertaining the velocity with which the balls strike each other will be shown in a future part of this work.

233. *Cor.*—It follows, from this law of motion, that if a person in a boat pulls a vessel towards him, the boat itself will also be drawn towards the ship, with such a velocity that its momentum shall be equal to the momentum of the vessel. Also, if two bodies, a magnet and a piece of iron, attract one another, they will approach each other with velocities inversely proportional to their masses; and they will meet in their centre of gravity, which remains at rest all the time.

234. *Scholium.*—These laws of motion are the simplest principles to which dynamics can be reduced, and upon them the whole theory depends. They do not admit of accurate proof by experiment, in consequence of the many causes of error which are necessarily introduced; but they are established upon the firmest basis, from the following considerations. In assuming these laws to be true, and applying them to the investigation of the motions of the heavenly bodies, innumerable results have been obtained, after operations more or less intricate and complex; and these have, in every case, been found to agree with observation. It follows, therefore, that the principles from which they were derived must be true.

235. Prop. III.—*With uniform accelerating forces, the velocity generated in any time is equal to the product of the force and the time.*

Let f be the accelerating force; then f is the velocity generated in one second of time. And since the force is uniform, f will also be the velocity added in the next second; and $2f$ will be the whole velocity generated in 2 seconds. In like manner $3f$ will be the velocity at the end of 3 seconds; and generally tf will be the velocity at the end of t seconds. Hence, if v be the velocity,

$$v = ft.$$

Since the velocity generated in equal fractions of a second is equal, this equation is also true when t is a fraction.

CHAP. II.—COLLISION OF BODIES.

236. The impact of two bodies is said to be *direct* when their centres of gravity move in the same straight line, and this line passes through the point of impact. When this is not the case, the impact is said to be *oblique*.

237. A *hard* body is one which is not susceptible of compression by any finite force. An *elastic* body is one which is susceptible of compression, and recovers its former figure again after having been compressed. When a body recovers its figure with a force equal to that with which it was compressed, it is said to be *perfectly elastic*. When the force with which it recovers itself is less than the force of compression, it is said to be *imperfectly elastic*. A *soft* body is one which does not alter its figure after compression.

238. Prop. I.—*If the impact of two inelastic bodies be direct, it is required to determine their motion after impact.*

Let A and B be the quantities of matter in the two bodies, a and b their velocities. When they move in the same direction, let A overtake B; then will A continue to accelerate B's motion, and B will continue to retard A's, till their velocities are equal, when they will cease to act upon each other; and since there is no force to separate them, they will move on together, and their common velocity, by the first law of motion, will be uniform. Also, the momentum before impact (art. 223) is $Aa + Bb$, and since B gains by the impact (art. 231) as much momentum as A loses, $Aa + Bb$ is equal to the momentum after impact.

When the bodies move in opposite directions, if A's force be greater than B's, the whole force of B will be destroyed; and A's not being destroyed, A will communicate velocity to B, and B by its reaction will retard A, until the velocities are equal, as in the former case. In this case Bb, the momentum of B, will be destroyed, and therefore A's momentum will be diminished by the quantity Bb. Thus, when the bodies begin to move in the same direction, $Aa - Bb$ is their whole momentum, and as much momentum as is afterwards communicated to B so much is lost by A, therefore $Aa - Bb$ is equal to the whole momentum after impact.

Let v be the common velocity after impact, then $(A + B)v$ is the whole momentum, consequently $(A+B)v=Aa\pm Bb$, and $v=\dfrac{Aa\pm Bb}{A+B}$. In which expression the positive sign is to be used when the bodies move in the same direction before impact, and the negative sign when they move in opposite directions.

239. *Cor.* 1.—The *positive* sign may be made applicable to all cases, by supposing a and b to be the velocities in the *same* direction, and considering them *negative* in the opposite direction.

240. *Cor.* 2.—The velocity lost by A is equal to

$$a - v = a - \frac{Aa + Bb}{A + B} = \frac{B(a - b)}{A + B},$$

The velocity gained by B in the direction of A's motion is equal to

$$v - b = \frac{Aa + Bb}{A + B} - b = \frac{A(a - b)}{A + B}.$$

241. *Cor.* 3.—If A and B move in opposite directions, the velocity gained by B is $v + b$ or $\dfrac{A(a+b)}{A+B}$; in this case B first loses its own velocity b in the direction of its motion, and then acquires the velocity v in the direction of A's motion.

242. *Cor.* 4.—The whole momentum lost by A and gained by B, when they move in the same direction, is $\dfrac{AB(a+b)}{A+B}$.

243. PROP. II.—*In the direct impact of two imperfectly elastic bodies the force of restitution is to the force of compression in a ratio which is nearly constant for bodies of the same nature.*

When two bodies A and B move in the same direction, and A impinges on B, A will continue to accelerate B's motion, and B to retard A's motion, until their velocities are equal; and if the bodies were inelastic, they would then cease to act upon each other, and move on together. But during the impact the bodies have been compressed, and, when their velocities become equal, they endeavour to recover their former figure, and rebound from each other in consequence of their spring or elasticity. That such a change of figure actually takes place may be easily shown. If an ivory ball, stained with ink, be made to touch an unstained ball, the spot received by the latter will be very small; but, if one of the balls be made to impinge upon the other, the spot will be enlarged, and the greater the force of impact the greater will be the surface stained; hence it is evident that one or both of the balls has been compressed, and afterwards recovered its spherical figure.

Many experiments have been made to determine the laws with which bodies recover their figure in inelastic bodies. The following are the conclusions deduced from a series of experiments lately made by Mr. Hodgkinson, of Manchester:—

(1). All rigid bodies are possessed of some degree of elasticity; and among bodies of the same nature, the hardest are generally the most elastic. Thus, among metals, the force of restitution in lead is to the force of compression as 1 to 5, or as ·20 to 1; therefore the decimal fraction ·20 expresses the force of elasticity in lead. In brass the elasticity is ·36; in bell-metal ·59; in cast-iron ·66; in steel ·79; and their hardness follows the same order. Again, the elasticity of malleable clay is ·17; of stone 39; of hard-baked clay ·39; of glass 94; which numbers are nearly in the same order as their hardness.

(2). There are no perfectly hard inelastic bodies.

(3). The force of restitution is to the force of compression in a ratio which is nearly constant; but it decreases a little as the velocity increases.

(4). (5) The force of restitution is the same whatever be the ratio and magnitudes of the bodies.

(6). In impacts between bodies differing very much in hardness, the common elasticity is nearly that of the softer body. Thus, the elasti-

of cork is ·65, and the elasticity of lead struck against the softer cork is ·57, but the elasticity of lead struck against the harder body steel is ·19.

(7). When the hardness differs in a less degree, the resulting elasticity is made up of the elasticity of both. Thus, the elasticity of brass struck against steel is ·52.

244. PROP. III.—*When two bodies impinge upon each other directly with given velocities, and the force of compression is to the force of restitution in a given ratio; it is required to determine their velocity after impact.*

Let A and B be the bodies, a and b their velocities before impact, and 1 to e the ratio of the force of compression to the force of restitution; then it appears, from the last article, that the velocity gained by B, and the velocity lost by A, during the compression, are the same as if the bodies were inelastic. But during this period the bodies are compressed by the stroke, and, by hypothesis, the force with which each recovers its former figure is equal to e times the force with which it was compressed. Let α be the velocity lost by A, and β the velocity gained by B, during the compression; then, during the restitution, A will lose an additional velocity equal to $e\alpha$; therefore, altogether, A will lose the velocity $(1 + e)\alpha$. In like manner B will gain the velocity β during compression and $e\beta$ during the restitution, therefore B will gain altogether the velocity $(1 + e)\beta$; hence, if x be the velocity of A, and y the velocity of B, after impact, we have, art. 210,

$$x = a - (1 + e)\alpha = a - (1 + e)\frac{B(a - b)}{A + B},$$

$$y = b + (1 + e)\beta = b + (1 + e)\frac{A(a - b)}{A + B}.$$

245. *Cor.* 1.—The relative velocity after impact is equal to

$$y - x = b - a + (1 + e)\frac{(A + B)(a - b)}{A + B} = e(a - b);$$

∴ relative vel. before impact : relative vel. after impact :: 1 : e.

246. *Cor.* 2.—When the bodies are perfectly elastic $e = 1$. In this case,

$$x = a - \frac{2B(a - b)}{A + B}; \qquad y = b + \frac{2A(a - b)}{A + B}.$$

Also, $y - x = a - b$, or the relative velocity before impact is equal to the relative velocity after impact.

247. *Cor.* 3.—Since the sum of the momenta before impact is equal to the sum of the momenta after impact,

$$Aa + Bb = Ax + By, \quad \therefore \quad A(a - x) = B(y - b).$$

Also, when the bodies are perfectly elastic,

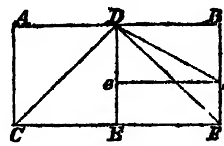
$$y - x = a - b; \quad \therefore \quad a + x = y + b.$$

Hence, by multiplication, $A(a^2 - x^2) = B(y^2 - b^2)$; or

$$Aa^2 + Bb^2 = Ax^2 + By^2.$$

248. *Cor.* 4.—In perfectly elastic bodies, if $A = B$, and A impinges upon B at rest, A will remain at rest after the impact, and B will move on with A's velocity. If A be greater than B, they will both move in the direction of A's motion; if A be less than B, B will move in the direction of A's motion, and A will be reflected back.

249. Prop. IV.—*If an imperfectly elastic body impinges upon a perfectly hard plane* AB, *it is required to determine its motion after impact.*



Let CD represent the velocity of the impinging body; draw CE parallel, and DE perpendicular to AB. Take DE to De as the force of compression to the force of restitution. Draw ef parallel to AB, and take EF, ef, each equal to CE; join DF, Df. Now, the force of impact is proportional to CD, and this may be resolved into two forces AD, ED, of which AD remains unaffected by the impact; but the force ED will be destroyed during the compression, and by the elasticity of the body a force proportional to De will be communicated to it. Hence it appears that the body, after impact, has two motions, which are represented by De and AD or DB; therefore the body will describe Df, after reflection, in the same time that it described CD before incidence. Hence

$$\text{vel. before incidence} : \text{vel. after reflection} :: CD : Df$$

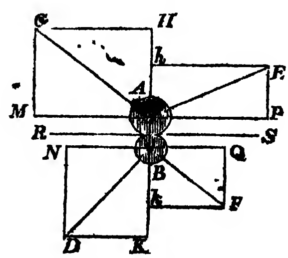
$$:: \frac{CE}{\sin CDE} : \frac{ef}{\sin eDf} :: \sin EDf : \sin EDC$$

$$:: \text{sine of reflection} : \text{sine of incidence.}$$

250. *Cor.* 1.—If the body is inelastic, all the force ED will be destroyed, and the body will move after impact with the velocity DB.

251. *Cor.* 2.—If the body is perfectly elastic, DF will be the velocity after impact, and the angle of reflection EDF is equal to the angle of incidence EDC.

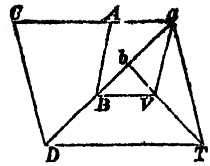
252. Prop. V.—*To determine the motion of two spherical bodies after impact, when they impinge upon each other obliquely.*



Let RS be the plane which touches the bodies at the point of impact. Through the centres A, B draw the line HK; also draw MP, NQ parallel to RS. Let CA, DB represent the velocities of the bodies before impact; resolve CA into the two MA, HA, and DB into the two NB, KB. Now, the velocities MA, NB, which are parallel to the plane RS, will not be affected by the impact, but the velocities HA, KB which are perpendicular to it, are the velocities with which the bodies impinge directly upon each other, and their effects must be calculated by art. 244. Let Ah, Bk be the velocities after impact, thus determined: take $AP = AM$ and $BQ = BN$, complete the parallelograms Ph,

and draw the diagonals AE, BF; these lines will represent the velocities of A and B after impact.

Note. The situation of the plane RS may be found by the following construction, which we will leave to the student to demonstrate. Let CA, DB be produced to meet in a; and let Db be described by B in the same time that Ca is described by A. Complete the parallelogram $aCDT$, join bT, and take $aV =$ sum of the radii of the two bodies. Draw VB parallel to DT, and complete the parallelogram $aVBA$. Then A, B will be the centres of the two spheres when they meet.



Problems for Practice.

1. The weights of A and B are 6lbs. and 10lbs., and they move in the same direction with velocities of 8 feet and 6 feet per second; required their velocities after impact, 1st, When the bodies are inelastic; 2nd, When they are perfectly elastic; 3rd, When the force of elasticity is $\frac{1}{3}$ force of compression.

Ans. (1) 6·75 ft.; (2) 5·5 and 7·5 ft.; (3) 6·33 and 7 ft.

2. A and B are perfectly elastic, and A, with a velocity of 20 feet per second, strikes B at rest; find their velocities after impact, 1st, When $A = B$; 2nd, When $A = 4B$; 3rd, When $A = \frac{1}{4}B$.

3. The weights of A and B are 3 and 5lbs., and their velocities are 7 and 9 feet per second, in opposite directions; required their velocities after impact, in the same cases as in the first example.

Ans. (1) $-$ 3 ft.; (2) $-$ 13 and $+$ 3 ft.; (3) $-$ 6·33 and $-$ 1 ft.

4. If there be a row of perfectly elastic bodies, A, B, C, D, &c., at rest, and a motion be communicated to A, and thence to B, C, D, &c., then, if the bodies are all equal, they will all remain at rest after impact, except the last, which will move off with a velocity equal to that with which the first moved.

5. If these bodies decrease in magnitude, they will all move in the direction of the first motion, and the velocity communicated to each succeeding body will be greater than that which was communicated to the preceding. If they increase in magnitude, they will be all reflected back except the last.

6. The velocity communicated from A through B to C, when the bodies are perfectly elastic, is greater than the velocity communicated immediately from A to C, if B be greater than one of the bodies A and C, and less than the other.

7. The velocity communicated through B is the greatest when B is a mean proportional between A and C.

8. Five balls, whose elasticity is $\frac{5}{6}$, are in geometrical progression, having the common ratio $\frac{1}{2}$. These balls are placed contiguous to each other, and the first ball impinges upon the second with a velocity of 3 feet per second; to find the velocity of the last body after impact.

Ans. 6·69 ft. per second.

9. To find in what direction a perfectly elastic ball must be projected from a given point, that, after reflexion from a given plane, it may hit a given mark.

10. To find the same when the ball is imperfectly elastic.

CHAP. III.—UNIFORMLY ACCELERATED MOTION AND GRAVITY.

253. It appears, from article 235, that when a body is impelled in a straight line by a constant accelerating force, the velocity communicated to it is proportional to the time of its motion; and that if v represent the velocity, t the time, and f the accelerating force measured by the velocity generated in a unit of time, then $v = ft$.

By some writers all accelerating forces are compared with the force of gravity, which is called 1: thus, a force which is equal to twice the force of gravity is called 2; and so on.

254. Prop. I.—*If a body be moved from a state of rest by a constant accelerating force, the space described from the beginning of the motion is as the square of the time.*

Let the time t be divided into n portions, each equal to τ, so that $t = n\tau$. Then, by the last article, the velocities at the end of the times

$$\tau, \quad 2\tau, \quad 3\tau, \ldots\ldots\ldots \; n\tau, \text{ will be}$$
$$f\tau, \quad 2f\tau, \quad 3f\tau, \ldots\ldots\ldots \; nf\tau.$$

Now, if the body had moved with the uniform velocity $f\tau$ during the first portion of time τ, the space described would have been $f\tau \times \tau = f\tau^2$ (art. 226). In like manner, if the body had moved with the uniform velocities $2f\tau$, $3f\tau$, &c., during each of the preceding intervals of time τ, the spaces described would have been $2f\tau^2$, $3f\tau^2$, &c.; therefore the whole space described on this supposition would have been

$$f\tau^2 + 2f\tau^2 + 3f\tau^2 \ldots\ldots \; + nf\tau^2$$
$$= f\tau^2 (1 + 2 + 3 \ldots\ldots + n) = f\tau^2 \frac{n(n+1)}{2}$$
$$= \frac{f\tau^2 n^2}{2}\left(1 + \frac{1}{n}\right) = \frac{ft^2}{2}\left(1 + \frac{1}{n}\right).$$

Let the portions of time be diminished indefinitely, or the number of intervals n be increased without limit, then will the motion approach to a continually accelerated motion, and the space described in this case will be the value of $\frac{ft^2}{2}\left(1 + \frac{1}{n}\right)$, when n becomes indefinitely great. Hence, therefore, $s = \frac{1}{2}ft^2$, because the fraction $\frac{1}{n}$ ultimately vanishes; and, therefore, the space described from the beginning of the motion is as the square of the time.

255. *Cor.* 1.—Since $s = \frac{1}{2}ft^2$ and $t = \frac{v}{f}$ (art. 235), we have $s = \frac{v^2}{2f}$, and also $s = \frac{1}{2}tv$. Hence, when the accelerating force is constant, we have the four following equations of motion.

$$\left.\begin{array}{ll} v = ft; & s = \frac{1}{2}tv \\ s = \frac{1}{2}ft^2; & s = \frac{v^2}{2f} \end{array}\right\} \dots\dots\dots (1).$$

256. *Cor.* 2.—The space described by a body uniformly accelerated from rest is half the space described in the same time with the last acquired velocity; for tv is the space described in the time t with the velocity v.

257. *Cor.* 3.—The spaces described in

1 second,	2 seconds,	3 seconds,	4 seconds,	&c., are
$1 \times \frac{1}{2}f$,	$4 \times \frac{1}{2}f$,	$9 \times \frac{1}{2}f$	$16 \times \frac{1}{2}f$,	&c.;

and, therefore, the spaces described in the

1st second,	2nd second,	3rd second,	4th second,	&c., are
$1 \times \frac{1}{2}f$,	$3 \times \frac{1}{2}f$,	$5 \times \frac{1}{2}f$,	$7 \times \frac{1}{2}f$,	&c.

258. Prop. II.—*When a body is projected with a given velocity* V, *and acted on in the same direction by a constant force* f; *to determine the equations of motion.*

The space described in the time t, with the velocity V, will be equal to Vt; and the space described in the same time by the constant force f is equal to $\frac{1}{2}ft^2$. But, by the second law of motion, when any force is exerted on a body in motion, the effect is the same as if it acted upon the body at rest; and, therefore, since the force acts in the direction in which the body is moving, the whole space described will be equal to the sum of the spaces described by each motion separately; that is,

$$s = Vt + \tfrac{1}{2}ft^2 \dots\dots\dots\dots\dots (2).$$

259. *Cor.* 1.—If the body be projected in a direction opposite to that in which the force acts, we have, for the same reason,

$$s = Vt - \tfrac{1}{2}ft^2 \dots\dots\dots\dots\dots (3).$$

260. *Cor.* 2.—In the same manner it may be shown that

$$v = V + ft; \quad \text{or} \quad v = V - ft \dots\dots\dots (4),$$

according as the body is projected in the direction of the force or in the opposite direction.

261. *Cor.* 3.—If a body fall through any space, from a state of rest, by the action of a uniform force, and then be projected in the opposite direction with the velocity acquired, and move until that velocity is destroyed, the whole spaces described in the two cases are equal. For the velocity destroyed in any time is equal to the velocity generated in the same time, and therefore the whole times of motion in the two cases are equal. Also, if equal times be taken from the beginning of the motion in the former case, and from the end of the motion in the latter, the velocities at those instants are equal. Since, then, the whole times of motion are equal, and also the velocities at all corresponding points of time, the whole spaces described are equal.

262. *Cor.* 4.—By squaring the first equation in cor. 2,

$$v^2 = V^2 + 2Vft + f^2t^2, \text{ or } v^2 - V^2 = 2f(Vt + \tfrac{1}{2}ft^2).$$

$$\therefore \quad v^2 - V^2 = 2fs \ldots\ldots\ldots\ldots\ldots (5).$$

If we had taken the second equation in cor. 2, we should have found $V^2 - v^2 = 2fs$.

263. Prop. III.—*The force of gravity, at any given place, is a uniform force, and accelerates all bodies equally.*

The same body, by its gravity, always produces the same effects under the same circumstances; thus, it will at the same place bend the same spring in the same degree, it will also fall through the same space in the same time, if the resistance of the air be removed; therefore, the force of gravity is uniform. Also, all bodies in an exhausted receiver fall through the same space in the same time; consequently, gravity accelerates all bodies equally.

From the latest experiments that have been made of the vibrations of pendulums in small circular arcs, it appears that, in the latitude of London, at the level of the sea, a body would in a vacuum fall through 193·14 inches, or through $16\frac{1}{10}$ feet, in a second nearly. Hence the velocity generated in that time is 32·2 feet nearly; this quantity is usually represented by the letter g.

Hence we can easily solve all problems relating to bodies accelerated or retarded by the force of gravity, by substituting g for f in the preceding formulæ.

Problems for Practice.

1. Find the space through which a body falls in 7 seconds, and the velocity acquired, supposing that $g = 32{\cdot}2$ feet.
Ans. Space = 788·9 feet; velocity = 225·4 feet.

2. Required the space described by this body in the last second.
Ans. 209·3 feet.

3. How far must a body fall to acquire a velocity of 1000 feet per second, and what will be the time of its falling? Ans. 1552·8 feet.

4. An arrow shot perpendicularly upwards from a bow returned again in 10 seconds; required the velocity of projection, and the height to which it rose. Ans. Velocity = 161; height = 402·5 feet.

5. A heavy ball was observed to fall through 100 feet in the last second but one: required the height from which it fell.
Ans. 341·4 feet.

6. A body is projected perpendicularly upwards with a velocity of 100 feet per second; required its situation at the end of 10 seconds.
Ans. 610 feet below the point of projection.

7. A body is projected perpendicularly downwards with a velocity of 50 feet per second; where will the body be at the end of 8 seconds?
Ans. 1430·4 feet below the point of projection.

8. With what velocity must a body be projected downwards, from a height of 150 feet, that it may describe it in 2 seconds?
Ans. Velocity = 42·8 feet.

9. The space described by a heavy body in the 4th second of its fall, is to the space described in the last second but 4, as 1 to 3; what was the whole space described by the body? Ans. 3622·5 feet.

10. A body begins to fall from rest from the top of a tower 200 feet high at the same time that a body is projected upwards from the bottom of the tower, with a velocity that would carry it 400 feet high; to find the point where the two bodies will meet. Ans. 25 feet from the top.

11. Suppose a body to have fallen through a feet when another body begins to fall from a point b feet below it; how far will the latter body fall before it is overtaken by the former? Ans. $\frac{b^2}{4a}$.

12. A person drops a stone into a well, and after three seconds hears it strike the water; to find the depth of the surface of the water, supposing that the velocity of sound $= 35g = 1127$ feet. Ans. 133·66 feet.

264. PROP. IV.—*When two bodies hang over a fixed pulley, to determine their motion, the inertia of the pulley and the string being neglected.*

Let P and Q be two unequal bodies hanging over a fixed pulley, and connected together by a string; then it is evident, from art. 230, that the moving force is in this case proportional to the excess of P above Q, that is to $P - Q$. But the accelerating force is as the moving force divided by the quantity of matter, and, therefore, is as $\frac{P-Q}{P+Q}$. When $Q = 0$, the body falls freely, and the accelerating force is the force of gravity. Hence the accelerating force in this case (f) : accelerating force of gravity (g) :: $\frac{P-Q}{P+Q}$: $\frac{P}{P}$; therefore

$$f = \frac{P-Q}{P+Q}g \quad \ldots\ldots\ldots\ldots (6).$$

265. PROP. V.—*To find the force which accelerates a body's motion down an inclined plane.*

Let AB be the inclined plane, BC its base, parallel to the horizon, P any body moving down the inclined plane. From P, the centre of gravity of the body, draw Pd vertically to represent the pressure P arising from the force of gravity; also, draw Pe parallel and Pf perpendicular to AB, and complete the parallelogram ef. Now the force Pd is equivalent to the two Pe, Pf; of which Pf is supported by the re-action of the plane. The other pressure Pe is wholly employed in accelerating the motion of P; and

this pressure : P :: Pe : Pd :: AC : AB.

Hence, if we put $AB = l$, $AC = h$, and the angle $ABC = \alpha$, the pressure which produces motion in the inclined plane $= P\frac{h}{l} = P \sin \alpha$. Also, the quantity of matter moved is P. Therefore, as in the last

article, the accelerating force down the inclined plane (f) : accelerating force of gravity $(g) :: \frac{P \sin \alpha}{P} : \frac{P}{P}$; consequently

$$f = g \sin \alpha = \frac{gh}{l} \dots\dots\dots\dots (7).$$

Hence the accelerating force down the inclined plane is constant, and the equations of motion will be derived from substituting this value for f in equations (1).

266. *Cor.* 1.—Since $v = \sqrt{2fs}$, if $s = l$, the length of the inclined plane,

$$v = \sqrt{\left(\frac{2gh}{l} \cdot l\right)} = \sqrt{2gh}.$$

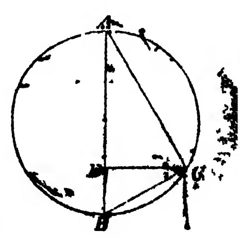
Also, the velocity acquired in falling through the perpendicular height $AC = \sqrt{2gh}$; hence *the velocity which a body acquires in falling down the length of an inclined plane is equal to the velocity acquired in falling down its perpendicular height.*

267. *Cor.* 2.—Since $t = \sqrt{\frac{2s}{f}}$, if $s = l$,

$$t = \sqrt{\left(2l \times \frac{l}{gh}\right)} = \frac{l}{\sqrt{\frac{1}{2}gh}}.$$

If, therefore, h be given, t varies as l.

268. Prop. VI.—*If a circle be situated in a vertical plane, the times of descent down all chords drawn through the highest or lowest points are equal; and the velocities acquired in falling down them are proportional to their length.*



1. Let ABC be the circle, AB the vertical diameter, AC any chord drawn through A, and CD parallel to the horizon. We have then, by the last article,

$$\text{time down } AC = \sqrt{\frac{2AC^2}{g \cdot AD}} = \sqrt{\frac{2AB}{g}}.$$

(Geom. prop. 72). And since this is independent of the position of C, the times of descent down all chords are equal to one another, and equal to the time of falling freely down the diameter AB.

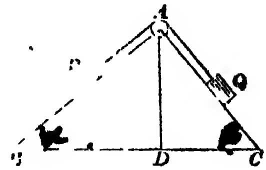
2. We have also (art. 266) the velocity acquired down AC equal to

$$\sqrt{2g \times AD} = \sqrt{2g \frac{AC^2}{AB}} = AC\sqrt{\frac{2g}{AB}}.$$

And since g and AB are constant, the velocity acquired down any plane AC is as the length AC.

In the same manner may this proposition be proved with respect to the plane CB.

269. PROP. VII.—*To find the accelerating force when a heavy body draws another along an inclined plane.*



Let the body P descend down the inclined plane AB, and draw the body Q up the inclined plane AC. Then the pressure of P in the direction PB is equal to $P \sin \alpha$, and the pressure of Q in the direction QC is $Q \sin \beta$; and if these pressures be equal, P and Q will be in equilibrium. But if $P \sin \alpha$ be $> Q \sin \beta$, P will descend and draw Q up the inclined plane. Now, it is only the pressure $P \sin \alpha - Q \sin \beta$ which produces motion. And since the whole mass moved is $P + Q$, it may be shown, as in art 264, that the accelerating force

$$f = \frac{P \sin \alpha - Q \sin \beta}{P + Q} g.$$

270. *Cor.* 1.—If P hangs freely, $\alpha = 90°$, therefore

$$f = \frac{P - Q \sin \beta}{P + Q} g.$$

271. *Cor.* 2.—If P hangs freely, and Q is placed on a horizontal plane, $\alpha = 90°$ and $\beta = 0$, therefore

$$f = \frac{P}{P + Q} g.$$

Problems for Practice.

1. If P and Q hang over a fixed pulley, and $P = 81$ ounces, $Q = 80$ ounces; to find the space descended by P in 10 seconds, and the velocity acquired. Ans. Space = 10 feet; velocity = 2 feet.

2. A weight P of 1 ounce drags a weight Q of 6 lbs. 3oz. along a horizontal table; to find the space described in 10 seconds.
Ans. 16·1 feet.

3. How far will a body descend from rest in 4 seconds upon an inclined plane, whose length is 400 feet and height 300 feet?
Ans. 193·2 feet.

4. How long would a body be in falling down 100 feet of a plane whose length is 150 feet and height 60 feet? Ans. 3·9 seconds.

5. The length of an inclined plane is 100 feet and its elevation 60°; to find the time of falling down it and the velocity acquired.
Ans. Time = 2·6 seconds; velocity = 74·4 feet.

6. A body is projected up an inclined plane, whose height is $\frac{1}{6}$th of its length, with a velocity of 50 feet per second; find its place and velocity after 6 seconds are elapsed.
Ans. Its place is 203·4 feet from the bottom, and its velocity 17.8 feet upwards.

7. To mark out upon an inclined plane a part equal to the height, so

that a body in descending down the whole length of the plane shall describe this part in the same time that it would fall through the height.

Ans. The upper extremity is $\frac{(l-h)^2}{4l}$ from the top of the plane.

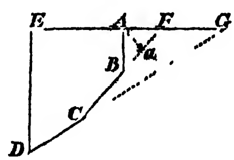
8. To find the straight line of quickest descent from a given point to a given plane.

9. To find the slope of a roof when the breadth is given, so that rain may descend in the shortest time possible.

10. To find the straight line of quickest descent from a given point to a given circle.

CHAP. IV.—MOTION UPON A CURVE, AND THE VIBRATIONS OF SIMPLE PENDULUMS.

272. PROP. I.—*If a body descend down a system of inclined planes, the velocity acquired is equal to that which would be acquired in falling through the perpendicular height of the system, supposing that no motion is lost in passing from one plane to another.*



Let $ABCD$ be the system of planes; draw AE parallel and DE perpendicular to the horizon; produce CB, DC till they meet AE in F and G. The velocity acquired in falling from A to B is equal to that which a body would acquire in falling from F to B (art. 266); and since, by the supposition, no velocity is lost in passing from one plane to another, the body will begin to descend down BC with the same velocity, whether it fall down AB or FB; consequently, the velocity acquired at C will be the same in either case. In the same manner it may be shown that the velocity acquired at D is equal to that which would be acquired in falling through GD; but this velocity is equal to the velocity acquired in falling down the perpendicular height ED; therefore, the velocity acquired in falling down the whole system $ABCD$ is equal to the velocity acquired in falling down the perpendicular height ED.

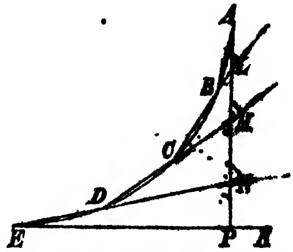
273. *Cor.*—Draw Aa perpendicular to BF; then, when a body passes from AB to BC, the velocity before impact (v) : velocity after impact :: AB : Ba :: rad : cos ABa. Hence, if we put the angle $ABa = \alpha$; the velocity after impact $= v \cos \alpha$, and the velocity lost

$$= v - v \cos \alpha = v(1 - \cos \alpha) = 2v \sin^2 \frac{\alpha}{2}.$$

274. PROP. II.—*If a body fall from rest down a curve surface which is perfectly smooth, the velocity acquired is equal to that which would be acquired in falling through the same perpendicular height.*

Let AE be any curve surface. Suppose the arc to be divided into n parts, AB, BC, &c. Draw the tangents AP, BR; and the chords

AB, BC, &c.; and let these chords be produced to meet AP in the points L, M, &c. Let the angles BAL, ABL, BCM, &c. be represented by α, β, γ, &c., and the angle APR by ϕ; then it is manifest that



$$\alpha + \beta + \gamma + \delta + \ldots\ldots = \phi.$$

Let v, v', v'', &c., be the actual velocities acquired at the points B, C, &c., and V the actual velocity acquired at E. Then, by the last article,

$$\text{whole velocity lost} = 2v\,(\sin\tfrac{1}{2}\beta)^2 + 2v'(\sin\tfrac{1}{2}\gamma)^2 + 2v''(\sin\tfrac{1}{2}\delta)^2 + \text{\&c.}$$

Now this is evidently less than

$$2V(\tfrac{1}{2}\beta)^2 + 2V(\tfrac{1}{2}\gamma)^2 + 2V(\tfrac{1}{2}\delta)^2 + \text{\&c.}$$

And if β be the greatest of these angles, this is less than

$$\tfrac{1}{2}V\beta^2 + \tfrac{1}{2}V\beta\gamma + \tfrac{1}{2}V\beta\delta + \text{\&c.},$$

or less than $\frac{1}{2}V\beta(\beta + \gamma + \delta + \text{\&c.})$ or less than $\frac{1}{2}V\beta\phi$.

Let the arcs AB, BC, &c. be diminished indefinitely, or the number of parts be increased without limit; then it is evident that the angles β, γ, δ, &c. will become indefinitely small, and, therefore, the whole velocity lost is diminished without limit. But when the number of planes becomes indefinitely great, the system approximates to a curve as its limit, in which, therefore, no velocity is lost. Hence the whole velocity acquired is equal to that which a body would acquire in falling through the same perpendicular altitude.

275. *Cor.* 1.—If a body be projected up a curve, the perpendicular height to which it will rise is equal to that through which it must fall to acquire the velocity of projection.

For the body in its ascent will be retarded by the same degrees that it was accelerated in its descent.

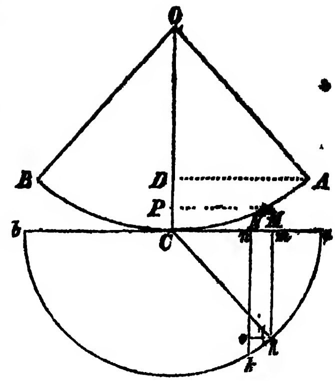
276. *Cor.* 2.—The same proposition is true if the body be retained in a curve by a string which is in every point perpendicular to it. For the string will now sustain that part of the weight which was before sustained by the curve.

277. Def.—A *pendulum* consists of a heavy body suspended by a thread, and made to vibrate in a vertical plane. When the body is supposed to be reduced to a material point, and the thread to be devoid of weight, it is called a *simple pendulum*.

278. Def.—The time elapsed from the commencement of the motion until all the velocity is lost in the ascent of the body, is called the *time of an oscillation*; and the angle through which the body has moved is called the *amplitude*.

279. Prop. III.—*To find the time of an oscillation in a small circular arc.*

Let a body descend from A, and be retained in the circular arc ACB by means of the thread OA, supposed to be without weight. Let the body come to M, and let the arc MN be indefinitely small; then we may conceive this ultimately to be described with the velocity at M con-



tinued uniform. Call this velocity v, then the time through $MN = \frac{MN}{v}$. But the velocity at M is equal to the velocity acquired in falling from D to P (art 274), therefore

$$v^2 = 2g \times DP = 2g\,(CD - CP)$$
$$= 2g \frac{(\text{chord } CA)^2 - (\text{chord } CM)^2}{2OC}$$
$$= \frac{g}{l}\left\{(\text{arc } CA)^2 - (\text{arc } CM)^2\right\}$$

very nearly; since the arc CA is supposed never to exceed 2 degrees.

Take Ca = arc CA, and from the centre C, with the radius Ca, describe the semicircle ahb; take Cm, Cn, equal to the arcs CM, CN, respectively; draw mh, nk perpendicular and he parallel to Ca; join Ch. We have then

$$v^2 = \frac{g}{l}(Ca^2 - Cm^2) = \frac{g}{l} \times mh^2, \text{ and } v = mh\sqrt{\frac{g}{l}}.$$

$$\therefore \text{ time through } MN = \frac{MN}{v} = \frac{mn}{mh}\sqrt{\frac{l}{g}}.$$

Now, when MN or mn is diminished indefinitely, hk will ultimately be a straight line, and the triangles Chm, ehk, will be ultimately similar, therefore

$$Ch : mh :: hk : he \text{ or } mn; \text{ and } \frac{mn}{mh} = \frac{hk}{Ch}.$$

$$\therefore \text{ time along } MN = \frac{hk}{Ca}\sqrt{\frac{l}{g}}.$$

Now, as this is true for every indefinitely small portion of the arc CA, and the sum of all the hk's is manifestly equal to the semicircumference ahb; we have the time of describing $ACB = \frac{ahb}{Ca}\sqrt{\frac{l}{g}}$; and since $ahb : Ca :: \pi : 1$, therefore

$$t = \pi\sqrt{\frac{l}{g}}.$$

where t is the time of the oscillation.

280. *Cor.*—In the latitude of London, the length of the seconds pendulum, in a vacuum, is found by experiment to be 39·1393 inches; calling this value L, we have

$$g = \pi^2 L = 386{\cdot}29 \text{ inches} = 32{\cdot}19 \text{ feet}$$

281. Prop. IV.—*To find the number of vibrations which a given pendulum will gain in a day: (1) By shortening the length of the pendulum; (2) By increasing the force of gravity.*

(1). Let t be the time of one vibration, n the number of vibrations, and N the number of seconds in a day. Then

$$nt = N; \quad \therefore \; n = \frac{N}{\pi}\sqrt{\frac{g}{l}}.$$

Also, let λ be the quantity by which the length of the pendulum is diminished, and ν the corresponding number of seconds which it gains a day; then $n + \nu = \frac{N}{\pi}\sqrt{\left(\frac{g}{l-\lambda}\right)}$;

$$\therefore \; \frac{n+\nu}{n} = \sqrt{\left(\frac{l}{l-\lambda}\right)} = \left(1 - \frac{\lambda}{l}\right)^{-\frac{1}{2}} = 1 + \frac{\lambda}{2l}, \text{ nearly;}$$

since λ is supposed to be very small compared with l. Hence

$$\nu = \frac{n\lambda}{2l}.$$

(2). Let l be constant, and g be increased by a small quantity γ; and let ν again be the corresponding increment of n. Then

$$n + \nu = \frac{N}{\pi}\sqrt{\left(\frac{g+\gamma}{l}\right)},$$

$$\therefore \; \frac{n+\nu}{n} = \sqrt{\left(\frac{g+\gamma}{g}\right)} = 1 + \frac{\gamma}{2g}, \text{ nearly;}$$

neglecting the second and higher powers of $\frac{\gamma}{g}$. Hence

$$\nu = \frac{n\gamma}{2g}.$$

282. *Cor.* 1.—The force of gravity without the earth's surface varies inversely as the square of the distance from the centre, when the latitude is the same. If, therefore, r be the radius of the earth, h the height of any place above the surface, and γ the diminution of gravity; then

$$\frac{g-\gamma}{g} = \frac{r^2}{(r+h)^2}; \quad \therefore \; \frac{n-\nu}{n} = \frac{r}{r+h} = 1 - \frac{h}{r}, \text{ nearly.}$$

$$\text{Hence} \quad \nu = \frac{nh}{r}.$$

283. *Cor.* 2.—The force of gravity also varies in different latitudes, the increment above the force at the equator being nearly as the square of the sine of the latitude. Hence, if 39·0265 inches be the length of the seconds pendulum at the equator, 0·1608 inches the difference between that and the length at the poles; the length of the pendulum (l) in any latitude L, is

$$l = 39{\cdot}0265 + 0{\cdot}1608 \sin^2 L.$$

Examples.

1. What is the time of vibration of a pendulum whose length is 50 inches? Ans. 1·239 seconds.

2. How many vibrations will a pendulum 36 inches long make in an hour? Ans. 3753.

3. If a clock loses 30 seconds in 12 hours, how much must the pendulum be shortened to make it keep true time? Ans. ·055 inch.

4. Required the length of a pendulum that vibrates sidereal seconds; the length of the sidereal day being $23^{hr}\ 56^{m}\ 4^{s}$. Ans. 38·925.

5. A pendulum which vibrates seconds at the equator, when carried to the pole gains 5 minutes a day; to find the proportion of the equatorial and polar gravity. Ans. $144 : 145\frac{1}{576}$.

6. A pendulum which oscillates seconds is carried to the top of a mountain one mile in height; to find the number of seconds which it would lose in a day, supposing the radius of the earth to be 4000 miles. Ans. 21·6 seconds.

284. Prop. VII.—*To find the time of an oscillation in a circular arc of any magnitude.*

Suppose a body to fall from A to M, as in article 279. Let $CP = x$, arc $CM = s$, $CD = h$. It will be proved, in the Differential Calculus, that $dt = \frac{-ds}{v}$; the minus sign being prefixed to ds, because s decreases when the time t increases. Also, $ds = \frac{ldx}{\sqrt{2lx - x^2}}$; and, from the last article, velocity at $M\,(v) = \sqrt{2g \times DP}$ $= \sqrt{2g(h - x)}$. Hence, therefore,

$$dt = \frac{-ldx}{\sqrt{[(2lx - x^2)\,2g\,(h - x)]}}.$$

This equation can only be integrated by means of a series. To obtain a converging series, we have

$$dt = \sqrt{\frac{l}{g}}\,\frac{-dx}{2\sqrt{(hx - x^2)}}\left(1 - \frac{x}{2l}\right)^{-\frac{1}{2}}$$

$$= \sqrt{\frac{l}{g}}\,\frac{-dx}{2\sqrt{(hx - x^2)}}\left\{1 + \frac{1}{2}\,\frac{x}{2l} + \frac{1\,.\,3}{2\,.\,4}\,\frac{x^2}{4l^2} + \&c.\right\}$$

Taking the integral of each of these terms separately from $x = h$ to $x = 0$, we obtain

$$\int \frac{-dx}{\sqrt{(hx - x^2)}} = \pi; \qquad \int \frac{-x\,dx}{\sqrt{(hx - x^2)}} = \tfrac{1}{2}h\pi,$$

$$\int \frac{-x^2dx}{\sqrt{(hx - x^2)}} = \frac{1\,.\,3}{2\,.\,4}\,h^2\pi; \quad \&c.$$

Substituting these values above, and multiplying by 2 to obtain the time of an entire oscillation, we have

$$t = \pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2\frac{h}{2l} + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\frac{h^2}{4l^2} + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\frac{h^3}{8l^3} + \&c.\right]$$

285. *Cor.* 1.—When the arcs of vibration are small, h is small compared with l; we have, then, in taking the first term only, $t = \sqrt{\frac{l}{g}}$; the same expression as that which has been deduced in article 279.

286. *Cor.* 2. If great accuracy be required, we may take two terms of the series; we shall then have $t = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{h}{8l}\right)$. But $h \times 2l =$ (chord $AC)^2$,

$$\therefore\ \frac{h}{8l} = \frac{(\text{chord } AC)^2}{16l^2} = \frac{(\text{chord } \alpha)^2}{16} = \frac{\alpha^2}{16} \text{ nearly;}$$

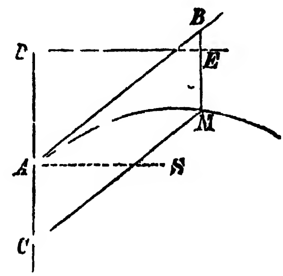
α being put for the arc corresponding to the angle COA, whose radius $= 1$. Hence

$$t = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{\alpha^2}{16}\right).$$

287. *Scholium.* It has been proved by Poisson that the resistance of the air has no sensible effect on the time of an oscillation in a small circular arc. It increases the time of descent by a small quantity $\frac{akl}{3}\sqrt{\frac{l}{g}}$ (the resistance being supposed $= kv^2$); but it diminishes by an equal quantity the time of ascent; so that the time of vibration is the same as in a vacuum. The amplitudes of the successive vibrations are, however, continually diminished; on this account, therefore, the times of vibration will be slightly affected. See Poisson (art. 275).

CHAP. V.—MOTION OF PROJECTILES.

288. Prop. I.—*A body projected in any direction not perpendicular to the horizon will describe a parabola; supposing that the motion is not affected by the resistance of the air.*



Let a body be projected from A in the direction AB. From A draw AC perpendicular to the horizon, and let AB be the space the body would describe with the velocity of projection continued uniform during the time t, and AC the space through which the force of gravity would cause it to descend in the same time. Complete the parallelogram AM; then, because the motion in the direction AB neither accelerates nor retards the approach of the body to the line CM (art. 229), therefore, at the end of the time t, the body will be in the line CM. By the same mode of reasoning, it appears that the body will be in the line BM at the end of this time; consequently it will be at M, the point of their intersection, at the end of the time t. Let V be the velocity of projection; then, since AB is the space which would be described in the time t with the velocity V continued uniform, $AB = Vt$. Also, since AC is the space through which the body would fall by the force of gravity in the time t, $AC = \frac{1}{2}gt^2$. Hence

$$\therefore\ t^2 = \frac{2AC}{g} = \frac{AB^2}{V^2} = \frac{CM^2}{V^2};$$

$$\therefore\quad CM^2 = \frac{2V^2}{g}AC \dots\dots\dots (1)$$

Hence the curve AM is a parabola, of which AC is a diameter, CM an ordinate, and $\frac{2V^2}{g}$ the parameter. (Parabola, prop. 8.)

289. *Cor.* 1.—If, in AC produced, AD be taken $= \frac{1}{4}$ of the parameter at $A = \frac{1}{4}$ of $\frac{2V^2}{g} = \frac{V^2}{2g}$, and DE be drawn at right angles to AD, DE will be the directrix to the parabola. For the distance of any point in the parabola from the directrix is equal to the distance of this

point from the focus; and this distance is equal to $\frac{1}{4}$ of the parameter at A (Parabola, art. 209).

290. *Cor.* 2.—Since $AD = \dfrac{V^2}{2g}$, AD is the space through which the body would fall to acquire the velocity of projection V (art. 255). The line AD is often called the *impetus*, or the height due to the velocity, and in future we shall put it $= h$.

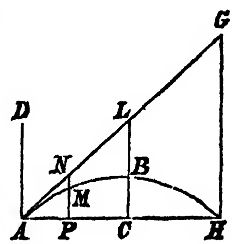
291. *Cor.* 3.—The velocity at any other point M in the curve is equal to the velocity acquired in falling from the directrix DE. For M may be considered as the point of projection; and, in this case, the velocity is equal to that acquired in falling from E (art. 290).

292. *Cor.* 4.—The horizontal velocity of the body is constant, for it is evidently not affected by gravity.

293. *Cor.* 5.—Since AB is parallel to the ordinate CM, it is a tangent to the parabola at A (Par. def. 5). Hence, if the angle BAS be made $= BAD$, and $AS = AD$, S will be the focus of the parabola.

The focus and the directrix being given in position, the parabola can easily be constructed (Con. Sec. art. 238).

294. Prop. II.—*To find the equation to the curve referred to horizontal and vertical co-ordinates.*



Let $AP = x$, $PM = y$, the angle $NAP = \alpha$, $V =$ the velocity of projection, and $t =$ the time of describing AM. Then

$$AP = AN \cos\alpha; \quad \text{or} \quad x = Vt\cos\alpha.$$

Also, $PN = x\tan\alpha$; and $MN = \frac{1}{2}gt^2$;

$$\therefore \; y = PN - MN = x\tan\alpha - \tfrac{1}{2}gt^2.$$

Substituting the value of t derived from the equation $x = Vt\cos\alpha$, we obtain

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha},$$

which is the equation to the curve. If we substitute h for $\dfrac{V^2}{2g}$, as in art. 289, we shall have

$$y = x\tan\alpha - \frac{x^2}{4h\cos^2\alpha} \;\ldots\ldots\ldots\ldots\; (2),$$

from which equation all the properties respecting projectiles may easily be derived.

295. Def.—If the curve meet the horizontal plane which passes through the point of projection in H, the line AH is called the *horizontal range*, and sometimes simply the range. Also, the time of describing ABH is called the *time of flight*. (See last figure.)

296. Prop. III.—*Having given the velocity and direction of projection, to find the time of flight, the horizontal range, and the greatest height to which the body will rise above the horizontal plane.*

(1) Let the body be projected from A in the direction AG. Let

ABH be the path of the projectile, and AH the horizontal range. And, as before let the velocity of projection $= V$, the angle $GAH = \alpha$, and the time of flight $= T$.

Now $\quad GH = AG \sin\alpha = TV \sin\alpha$. But $GH = \frac{1}{2}gT^2$.
Hence $\quad \frac{1}{2}gT^2 = TV \sin\alpha$; therefore

$$T = \frac{2V}{g}\sin\alpha \quad \ldots\ldots\ldots\ldots\ldots (3).$$

(2) Also, $AH = AG\cos\alpha = TV\cos\alpha = \frac{2V}{g}\sin\alpha \times V\cos\alpha$; and if we put $AH = R$, the impetus, $\frac{V^2}{2g} = h$, and $2\sin\alpha\cos\alpha = \sin 2\alpha$, we obtain

$$R = 2h\sin 2\alpha \quad \ldots\ldots\ldots\ldots\ldots (4).$$

(3) If the point C bisect AH, the greatest height H is evidently

$$CB = \tfrac{1}{2}CL = \tfrac{1}{4}GH \text{ (Par. art. 200). But,}$$

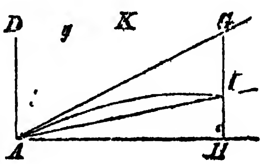
$$GH = \tfrac{1}{2}gT^2 = \frac{2V^2}{g}\sin^2\alpha = 4h\sin^2\alpha, \text{ from equation (3);}$$

$$\therefore \quad H = h\sin^2\alpha. \quad \ldots\ldots\ldots\ldots\ldots (5).$$

297. *Cor.* 1.—When the velocity of projection is given, the range varies as $\sin 2\alpha$, and is therefore the greatest when $2\alpha = 90°$, or the elevation $= 45°$. In this case $R = 2h$.

298. *Cor.* 2.—If the velocity of projection be given, the elevation necessary to hit a given mark in the horizontal plane, passing through the point of projection, will be found from equation (4). Since $\sin 2\alpha = \sin(180° - 2\alpha)$, there will always be two values of α which will satisfy this equation, let the other value of α be α', then $2\alpha' = 180 - 2\alpha$, and $\alpha' = 90° - \alpha$; therefore α and α' are complements to each other.

299 Prop IV.—*Having given the velocity and direction of projection; to find the time of flight, and the range on an oblique plane, which passes through the point of projection.*



(1) Let the body be projected from A in the direction AG. Let AI be the inclined plane passing through A, the arc AI the path of the projectile, and AH horizontal. Put the angle $IAH = \iota$, $GAH = \alpha$, the range $AI = R$, and the time of describing the arc $AI = T$. We have then

$$GI : AG :: \sin GAI : \sin AIG.$$

But $GI = \frac{1}{2}gT^2$; $AG = TV$; $\sin GAI = \sin(\alpha - \iota)$;
and $\quad \sin GIA = \sin AIH = \cos\iota$; hence

$$\tfrac{1}{2}gT^2 : TV :: \sin(\alpha - \iota) : \cos\iota,$$

$$\therefore \quad T = \frac{2V}{g}\,\frac{\sin(\alpha - \iota)}{\cos\iota} \ldots\ldots\ldots\ldots (6).$$

(2) Again, $AI : AG :: \sin AGI : \sin AIG$.

But $AG = TV. = \dfrac{2V^2}{g} \cdot \dfrac{\sin(\alpha - \iota)}{\cos \iota} = 4h\dfrac{\sin(\alpha - \iota)}{\cos \iota}$;

$\sin AGI = \cos GAH = \cos \alpha$; and $\sin AIG = \cos \iota$. Hence

$$R : 4h\frac{\sin(\alpha - \iota)}{\cos \iota} :: \cos \alpha : \cos \iota;$$

$$\therefore \quad R = 4h\frac{\sin(\alpha - \iota)\cos \alpha}{\cos^2 \iota} \ldots\ldots\ldots(7).$$

300. *Cor.*—If the plane be a descending plane, or AI fall below AH, the angle ι must be considered negative.

301. Prop. V.—*To find the direction in which a body must be projected from a given point, with a given velocity, to hit a given mark.*

Because $2\sin(\alpha - \iota)\cos \alpha = \sin(\alpha + \overline{\alpha - \iota}) - \sin(\alpha - \overline{\alpha - \iota})$
$= \sin(2\alpha - \iota) - \sin \iota$,

we have, from equation (7),

$$R = 2h\frac{\sin(2\alpha - \iota) - \sin \iota}{\cos^2 \iota},$$

$$\therefore \quad \sin(2\alpha - \iota) = \frac{R\cos^2 \iota}{2h} + \sin \iota \ldots\ldots (8).$$

From which equation $2\alpha - \iota$, and, therefore, α, or the angle HAG, may be found. If this value of $2\alpha - \iota$ be θ, and θ be less than 90°, there will be another value of $2\alpha - \iota$, namely, $180° - \theta$, which will also satisfy equation (8). Let α' be the other value of α, then

$$2\alpha - \iota = \theta; \qquad 2\alpha' - \iota = 180° - \theta.$$

Adding these two equations together,

$$2\alpha + 2\alpha' - 2\iota = 180°; \qquad \therefore \quad \alpha + \alpha' = 90° + \iota.$$

Hence, if AK be drawn bisecting the angle DAI,

$$2 \angle HAK = HAD + HAI = 90° + \iota = \alpha + \alpha',$$

and, therefore, the two directions of projection AG, Ag, will evidently make equal angles with AK.

302. *Cor.*—The range is evidently the greatest when $\sin(2\alpha - \iota)$ is greatest; that is, when $2\alpha - \iota = 90°$, or $\alpha = 45° + \frac{1}{2}\iota$. Hence AK will be the direction of projection when the range is the greatest.

Scholium.

303. The theory of projectiles, given in this chapter, depends upon three suppositions, which are all inaccurate: 1st, That the force of gravity in every point of the curve described is the same. 2d, That it acts in parallel lines. 3d, That the motion is performed in a non-resisting medium. The two former of these, indeed, differ insensibly from the truth; but the resistance of the air affects the motion of heavy bodies so materially as to render the parabolic theory nearly useless in practice. From experiments, which have been made with great care, it appears, that when the velocity is about 2000 feet per second, the air's resistance

is nearly 100 times as great as the weight of the ball; and that the greatest horizontal range is less than a mile, whereas, according to the theory, it ought to be $23\frac{1}{2}$ miles.

304. Another great irregularity in the firing of shot is the deflection of the ball to the right or left of the vertical plane passing through the axis of the gun. A deviation of this kind generally takes place when there is considerable windage; for if the ball in its passage should not fit the bore of the gun, it will be deflected from side to side, and will at last quit the piece in a direction inclined to the axis of the bore. In consequence of this, a rapid whirling motion is given to the ball about an axis, the position of which is altogether uncertain. One side of the ball, therefore, is moving in the direction of projection, and the other side is moving in the opposite direction; and, consequently, the velocity and the resistance of the air in the first case are greater than in the latter. Hence it follows, that the ball is continually deflected towards that side where the resistance is least.

In rifle barrels the ball is made to fit the bore exactly; and then, by means of a spiral groove, which makes about $1\frac{1}{4}$ or $1\frac{1}{2}$ turns in the length of the barrel, a whirling motion is communicated which has its axis of rotation in the line of its motion, and, consequently, the ball is equally resisted by the air on all sides.

305. From various experiments which have been made by Dr. Hutton, at Woolwich, he deduces the following conclusions:—

(1) "It appears that the velocity of a ball increases with the increase of charge only to a certain point, which is peculiar to each gun, where it is greatest; and that, by further increasing the charge, the velocity gradually diminishes, till the bore is quite full of powder. That this charge for the greatest velocity is greater as the gun is longer, but yet not greater in so high a proportion as the length of the gun is; so that the part of the bore filled with powder bears a less proportion to the whole bore in the long guns than it does in the shorter ones; the part which is filled being indeed nearly in the inverse ratio of the square root of the empty part.

(2) "It appears that the velocity, with equal charges, always increases as the gun is longer; though the increase in velocity is but very small in comparison to the increase in length; the velocities being in a ratio somewhat less than that of the square roots of the length of the bore, but greater than that of the cube roots of the same, and is indeed nearly in the middle ratio between the two.

(3) "It appears, from the table of ranges, that the range increases in a much lower ratio than the velocity, the gun and elevation being the same. And when this is compared with the proportion of the velocity and length of gun in the last paragraph, it is evident that we gain extremely little in the range by a great increase in the length of the gun, with the same charge of powder. In fact, the range is nearly as the 5th root of the length of the bore; which is so small an increase, as to amount only to about a 7th part more range for a double length of gun. From the same table it also appears, that the time of the ball's flight is nearly as the range, the gun and elevation being the same.

(4) "It has been found, by these experiments, that no difference is caused in the velocity, or range, by varying the weight of the gun, nor

by the use of wads, nor by different degrees of ramming, nor by firing the charge of powder in different parts of it. But that a very great difference in the velocity arises from a small degree in the windage: indeed, with the usual established windage only, viz., about $\frac{1}{20}$ of the calibre, no less than between $\frac{1}{3}$ and $\frac{1}{4}$ of the powder escapes and is lost; and, as the balls are often smaller than the regulated size, it frequently happens that half the powder is lost by unnecessary windage.

(5) " It appears, too, that the resisting force of wood, to balls fired into it, is not constant; and that the depths penetrated by balls, with different velocities or charges, are nearly as the logarithms of the charges, instead of being as the charges themselves, or, which is the same thing, as the square of the velocity. Lastly, these and most other experiments show, that balls are greatly deflected from the direction in which they are projected, and that as much as 300 or 400 yards in a range of a mile, or almost $\frac{1}{4}$th of the range.

(6) " To determine the resistance to the very high velocities, were employed balls of three several sizes, viz., of 2 inches, 2·78 inches, and 3·55 inches in diameter. These were discharged with various degrees of velocity, from 300 feet to 2000 feet in a second of time; and they were also made to strike the pendulum block at several different distances from the guns, in order to obtain the quantity of velocity lost, in passing through those spaces of air; whence the degrees of resistance were obtained, appropriate to the different velocities. These series of resistances, for the three sizes of balls above-mentioned, have been obtained in a state remarkably regular, not only each series in itself, but also in comparison with each other; the terms in every one of them following a certain uniform law, in respect of the velocity, being indeed nearly as the $2\frac{1}{10}$ power of the velocity; and the terms of any one series also, as compared with the corresponding terms of another, with the same velocity, these being in a constant proportion to one another, viz., as the surfaces of the balls moved nearly, or as the squares of their diameters, with about $\frac{1}{20}$ part more in counting from the less ball to the greater, or $\frac{1}{20}$ part less when comparing the greater ball to the less.

(7) " The same laws of resistance were also found to obtain in the slower motions, with the whirling machine, both in respect of the different velocities with the same body, and of the different bodies with the same velocity. From which uniformity of effects it happens, that the numbers resulting from the larger velocities in the one course of experiments, and those derived from the slow motions in the other course, form as it were the terms, in the different parts of one and the same general series of resistances."—(Hutton's Tracts, vol. iii. p. 215.)

306. The following rule, derived entirely from experiment, has been given, to find the velocity of any shot or shell, when the weight of the charge of powder and weight of the shot are known.

Rule.—Divide three times the weight of the powder by the weight of the shot, both in the same denomination. Extract the square root of the quotient. Multiply that root by 1600, and the product will be the velocity in feet.

That is, if p be the weight of the powder, w the weight of the ball, and v the velocity of the ball; then

$$v = 1600\sqrt{\frac{3p}{w}} \text{ feet.}$$

Examples for Practice.

1. If a ball of 1lb. acquire a velocity of 1600 feet per second, when fired with $5\frac{1}{3}$ ounces of powder; it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, viz.

Nature of the shells in inches........	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs.	196	90	48	16	8
Charge of powder in lbs.............	9	4	2	1	$\frac{1}{2}$
Ans. The velocities are.........	594	584	565	693	693

2. If a shell be found to range 1000 yards when discharged at an elevation of 45°; how far will it range when the elevation is 30° 16′, the charge of powder being the same? Ans. 2612 feet, or 871 yards.

3. The range of a shell, at 45° elevation, being found to be 3750 feet; at what elevation must the piece be set to strike an object at the distance of 2810 feet, with the same charge of powder?

Ans. At 24° 16′, or at 65° 44′.

4. With what impetus, velocity, and charge of powder, must a 13-inch shell be fired, at an elevation of 3[illegible]° 12′, to strike an object at the distance of 3250 feet?

Ans. Impetus = 1802; vel. = 340; charge = 2·96 lbs.

5. If, with a charge of 9lbs. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases? Ans. $6\frac{3}{4}$lbs. of powder.

6. What will be the time of flight for any given range, at the elevation of 45°, or for the greatest range?

Ans. The time in secs. is $\frac{1}{4}$ the sq. root of the range in feet nearly.

7. In what time will a shell range 3250 feet, at an elevation of 32°?

Ans. $11\frac{1}{4}$ seconds nearly.

8. How far will a shot range on a plane which ascends 8° 15′, and another which descends 8° 15′; the impetus being 3000 feet, and the elevation of the piece 32° 30′?

Ans. 4244 feet on the ascent, and 6745 feet on the descent.

9. How much powder will throw a 13-inch shell 4244 feet on an inclined plane, which ascends 8° 15′, the elevation of the mortar being 32° 30′? Ans. 4·92535lbs., or 4lbs. 15oz. nearly.

10. At what elevation must a 13-inch mortar be pointed, to range 6745 feet, on a plane which descends 8° 15′; the impetus being 3000 feet? Ans. 32° 30′.

11. Suppose, in Richoret firing, PO = 1200 feet, OH = 10 feet, OR = 50 feet; required the elevation and the velocity, so that the ball shall just clear H and hit R. Ans. Elev. = 11° 46′; vel. = 317·5.

12. In what time will a 13-inch shell strike a plane which rises 8° 30′ when elevated 45°, and discharged with an impetus of 2304 feet?

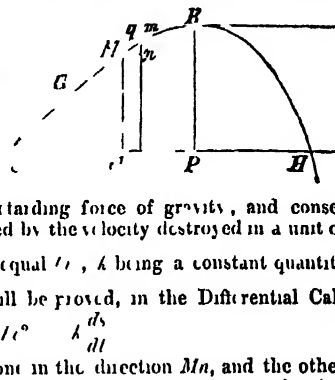
Ans. $14\frac{2}{5}$ seconds nearly.

THE MOTION OF PROJECTILES IN AIR.

307. In consequence of the passage of a body through the atmosphere, the air is displaced or put in motion. Whatever momentum it acquires must be taken from

the ball. This momentum must be in the proportion of the quantity of air displaced, and of the velocity communicated to it. And if the air put in motion be always similarly displaced, the quantity of air displaced will be proportional to the velocity of the ball; also, the velocity communicated to the air from the collision of bodies is as the velocity of the ball. Hence the momentum communicated to the air and lost by the ball must be proportional to the square of the velocity of the ball and the density of the air jointly. But, besides the resistance arising from the inertia of the air which is put in motion by the ball, there must be another cause of resistance arising from the condensation of the air on the anterior surface of the ball, and also a third cause of resistance arising from the hydrostatic pressure of the air on its anterior part, when the motion is so swift that there is a vacuum behind. From all these causes it appears, from the experiments of Robins and Dr Hutton, that the resistance of the air varies in rather a higher ratio than the square of the velocity, until the velocity reaches about 1100 feet per second. From 1100 to 1200 feet the increase of resistance is very great, after this the resistance goes on increasing nearly with its former regularity. Now this velocity is nearly that with which air rushes into a vacuum, and, therefore, it is extremely probable that this great increase of resistance arises from the unequal pressure on the anterior and posterior surfaces of the ball.

308 Prop VI.—*To determine the motion of projectiles in a resisting medium, when the resistance varies as the square of the velocity.*



Let A be the point of projection, M the place of the body after any time t, v the velocity of the ball at the point M, $AP = x$, $PM = y$, arc $AM = s$. Let R represent the resistance of the air; this is frequently estimated in pounds and ounces, in which case R must be considered as having the same proportion to the weight of the body that the retarding force of the air has to the accelerating or retarding force of gravity, and consequently the retarding force of the air, measured by the velocity destroyed in a unit of time, will be equal to $\frac{R \cdot g}{\text{wt. of ball}}$. Put this equal kv^2, k being a constant quantity to be determined from observation, then it will be proved, in the Differential Calculus, that the retarding force of the air $= kv^2 = k\frac{ds^2}{dt^2}$.

Let this force be resolved into two others, one in the direction Mn, and the other in the direction Mq, then the retarding force in the direction Mn is equal to the ultimate value of $k\frac{ds^2}{dt^2} \times \frac{Mn}{Mm} = k\frac{ds^2}{dt^2} \times \frac{dx}{ds} = k\frac{ds}{dt}\frac{dx}{dt}$. Also the retarding force in the direction Mq = ultimate value of $k\frac{ds^2}{dt^2}\frac{Mq}{Mm} = k\frac{ds}{dt}\frac{dy}{dt}$. Hence the whole force in the direction parallel to $AP = -k\frac{ds}{dt}\frac{dx}{dt}$, and the whole force in the direction perpendicular to $AP = -g - k\frac{ds}{dt}\frac{dy}{dt}$. We have, therefore,

$$\frac{d^2x}{dt^2} = -k\frac{ds}{dt}\frac{dx}{dt}, \qquad \frac{d^2y}{dt^2} = -g - k\frac{ds}{dt}\frac{dy}{dt} \text{ ......(1).}$$

In the first equation let $\frac{dx}{dt} = q$, then $\frac{d^2x}{dt^2} = \frac{dq}{dt}$, therefore

$$\frac{dq}{dt} = -k\frac{ds}{dt} \cdot q; \text{ and } \frac{\frac{dq}{dt}}{q} = -k\frac{ds}{dt}$$

Hence, integrating, $\log q = -ks +$ constant. At the point A, $s = 0$, and $q = \frac{dx}{dt}$ = horizontal velocity at $A = V\cos\alpha$, $\therefore$ $\log(V\cos\alpha)$ = constant. Hence

$$-ks = \log q - \log V\cos\alpha = \log\frac{q}{V\cos\alpha};$$

$$\therefore \quad e^{-ks} = \frac{q}{V\cos\alpha}; \quad \text{and} \quad q = \frac{dx}{dt} = V\cos\alpha\, e^{-ks} \quad \ldots. (2).$$

Let $p = \frac{dy}{dx}$, then $\frac{dy}{dt} = p\frac{dx}{dt}$; $\therefore \frac{d^2y}{dt^2} = \frac{dp}{dt}\frac{dx}{dt} + p\frac{d^2x}{dt^2}$; hence we have, from equation (1),

$$-g - k\frac{ds}{dt}\frac{dy}{dt} = \frac{dp}{dt}\frac{dx}{dt} - kp\frac{ds}{dt}\frac{dx}{dt},$$

$$\therefore \quad -g = \frac{dp}{dt}\frac{dx}{dt} = \frac{dp}{dx}\frac{dx^2}{dt^2} \ldots\ldots\ldots\ldots (3)$$

But, from equation (2), we have $\frac{dx^2}{dt^2} = V^2\cos^2\alpha\, e^{-2ks}$, therefore

$$\frac{dp}{dx} = -\frac{g}{V^2\cos^2\alpha}e^{2ks} = -\frac{1}{2h\cos^2\alpha}e^{2ks} \ldots\ldots\ldots\ldots (4).$$

And since $dx\sqrt{(1+p^2)} = ds$, we obtain, by multiplication,

$$dp\sqrt{(1+p^2)} = -\frac{1}{2h\cos^2\alpha}e^{2ks}\,ds, \quad \text{therefore, integrating,}$$

$$p\sqrt{(1+p^2)} + \log(p + \sqrt{1+p^2}) = c - \frac{1}{2hk\cos^2\alpha}e^{2ks},$$

c being the constant quantity necessary to complete the integral. And when $s = 0$, $p = \tan\alpha$, therefore we have, by reduction,

$$c = \tan\alpha\sqrt{(1+\tan^2\alpha)} + \log(\tan\alpha + \sqrt{1+\tan^2\alpha}) + \frac{1}{2hk\cos^2\alpha}.$$

Put $p\sqrt{(1+p^2)} + \log(p + \sqrt{1+p^2}) - c = P$, we then have $P = -\frac{1}{2hk\cos^2\alpha}e^{2ks}$. Divide equation (1) by this equation, and we obtain finally,

$$dx = \frac{dp}{kP}, \quad dy = p\,dx = \frac{p\,dp}{kP} \quad \ldots (5)$$

If these equations could be integrated, we should have x and y in terms of p, and eliminating p, we should obtain the equation to the curve. But as it is impossible to integrate these formulae in finite terms, we must have recourse to the following method of approximation.

When $x = 0$, $p = \tan\alpha$. As x increases the tangent of the angle which the curve makes with the direction of the axis of x becomes continually less and less, until at the vertex it is $= 0$. Afterwards p becomes negative and goes on increasing in magnitude in the descending branch. Let, therefore, $\tan\alpha\,(= b)$ be divided into n equal parts, each equal to β, n being a very great number. If, then, we suppose $dp = \beta$, the differential $\frac{dp}{kP}$ or $\frac{\beta}{kP}$ will be nearly equal to the elementary portion of the integral comprised between two successive values of p, and the integral $\int\frac{dp}{kP}$ will be nearly equal to the sum of all these differentials $\frac{\beta}{kP}$. And the greater n is, the more nearly shall we approximate to the true value of this integral. Make, therefore, successively, $p = \tan\alpha = b$, $p = b - \beta$, $p = b - 2\beta$, &c.; and let the corresponding values of P be P, B_1, B_2, &c., we shall then have the following corresponding values of p, x, and y,

$$p = b; \quad x = 0, \quad y = 0.$$

$$p = b - \beta; \quad x = \frac{\beta}{kB_1}, \quad y = \frac{(b-\beta)\beta}{kB_1}.$$

$$p = b - 2\beta, \quad x = \frac{\beta}{kB_1} + \frac{\beta}{kB_2}, \quad y = \frac{(b-\beta)\beta}{kB_1} + \frac{(b-2\beta)\beta}{kB_2}.$$

$$p = b - 3\beta, \quad x = \frac{\beta}{kB_1} + \frac{\beta}{kB_2} + \frac{\beta}{kB_3}, \quad y = \&c.; \quad p = \&c.$$

309. *Cor.* 1.—The ordinate y, which corresponds to $p = 0$, will be the greatest

height to which the body will rise. Pursuing the calculation to where $y = 0$, the corresponding abscissa will give the horizontal range, and the corresponding value of p will give the angle at which the descending branch cuts the axis of x.

310. *Cor.* 2.—Beyond that point the ordinate y becomes negative, and increases indefinitely; but the value of x never passes beyond a certain limit, from whence it is evident that the descending branch of the curve has a vertical assymptote.

For, when p is negative, if we substitute $-p$ for p, and $-dp$ for dp, we have

$$P = -p\sqrt{1+p^2} + \log(-p + \sqrt{1+p^2}) - c.$$

But $$-p + \sqrt{(1+p^2)} = \frac{1}{p + \sqrt{(1+p^2)}}; \text{ therefore}$$

$$P = -p\sqrt{(1+p^2)} - \log(p + \sqrt{1+p^2}) - c.$$

Now, when p becomes very great, $\sqrt{(1+p^2)} = p$ very nearly; therefore $P = -p^2 - \log 2p - c$, and since the logarithm of a very great number is very small compared with the number itself, $-\log 2p - c$ is very small compared with p^2, and therefore may be neglected, we shall then have

$$dx = \frac{dp}{kp^2}; \quad dy = \frac{dp}{kp}; \text{ therefore, integrating,}$$

$$x = c' - \frac{1}{kp}; \quad y = c'' + \frac{1}{k}\log p,$$

c' and c'' being two constant quantities necessary to complete the integral.

The value of y increases indefinitely with p; but the value of x has c' for its limit; and consequently the curve has a vertical assymptote, to which it approaches as near as we please, without actually ever reaching it.

311. *Cor.* 3.—Since $v^2 = \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = \frac{dx^2}{dt^2}(1+p^2)$ and $\frac{dp}{dx}\frac{dx^2}{dt^2} = -g$, from equation (3), multiplying these equations together, we have

$$-g(1+p^2) = v^2\frac{dp}{dx} = v^2kP, \text{ from equation (5), therefore}$$

$$v^2 = \frac{-g(1+p^2)}{kP},$$

and when p becomes very great, p^2 may be substituted for $1+p^2$, and $-p^2$ for P, therefore v^2 is ultimately $= \frac{g}{k}$.

From this we learn that the velocity gradually approaches to a certain velocity which is constant, and this is equal to the velocity which a body ultimately acquires when falling freely in an atmosphere of uniform density. For since the retarding force of the air continually increases with the velocity, when kv^2 is equal to g, the accelerating force of gravity, these forces will balance each other, and the motion will become uniform. This velocity is called the *terminal velocity*, and it is equal to that which the projectile ultimately acquires.

The terminal velocity of a 2lbs. ball is 295 ft.; of a 9lbs. ball, 380 ft.; and of a 32lbs. ball, 166 feet.

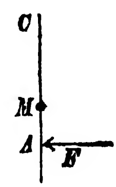
312. The following short table will give us the ranges of a 2lbs. shot, in yards, projected at an elevation of 15°, both in a vacuum and in the air, according to the preceding theory —

Velocity of a 2lbs ball	Range in a vacuum	Range in air	Velocity of a 2lbs ball	Range in a vacuum	Range in air
feet	yards	yards	feet.	yards	yards.
200	414	319	1000	10352	2845
400	1656	1121	1200	14907	3259
600	3727	1812	1600	26501	3950
800	6625	2373	2000	41408	4494

CHAP. VI.—ROTATION OF BODIES.

313. In demonstrating the preceding propositions respecting the motions of bodies, we have either considered the bodies as material points, or we have supposed that every particle of the body was moving with the same velocity as its centre of gravity. We now proceed to consider a system of points so connected with each other that they cannot follow those movements which the forces applied separately would impress upon them, but are compelled to move round some axis in a different manner.

314. Prop. I.—*If* CA *be a rigid line, moveable about the point* C, *and* F *any moving force acting at the point* A; *it is required to determine the weight* N *which, placed at* A, *will have the same effect in resisting the communication of angular motion, as the weight* M *placed at the distance* CM.



From the property of the lever, the effect of the force F on the body at M is $F\dfrac{CA}{CM}$, call this force φ; then the force φ acting at the point M directly, will have the same effect as the force F acting at the point A. But the moving forces, φ and F, acting on the bodies M and N, are proportional to the momenta produced in these bodies; that is,

$$\varphi : F :: M \times \text{vel. of } M : N \times \text{vel. of } N.$$

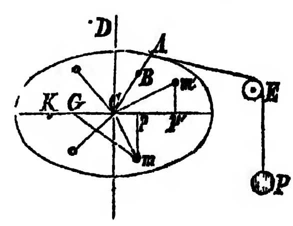
But, since the same angular motion is communicated to the system, the angular velocities of M and N are equal, and therefore their linear velocities are evidently as CM to CA. Hence,

$$\varphi : F :: M \times CM : N \times CA;$$

$$\therefore\ N = M\frac{CM}{CA}\,\frac{F}{\varphi} = M\frac{CM^2}{CA^2}.$$

Hence the inertia by which a body resists the communication of motion round an axis is proportional to the square of its distance from the axis or centre of motion.

315. Prop. II.—*In a rigid system of material points* m, m′, m″, &c., *situated in the same horizontal plane, and moveable about a vertical axis* CD *by means of a string passing over the pulley* E, *and attached to the weight* P; *it is required to determine the accelerating force on the weight* P.



Put $Cm = r$, $Cm' = r'$, &c.; and also, $CA = a$. By the last proposition, the weight $\dfrac{mr^2}{a^2}$ placed at A will have the same effect in resisting the communication of angular motion as the particle m placed at the distance Cm. Similarly, the weight $\dfrac{m'r'^2}{a^2}$ placed at the distance

a from the axis, will have the same effect as m' at the distance r'; and so on. Hence, if we suppose the weights $\frac{mr^2}{a^2}$, $\frac{m'r'^2}{a^2}$, &c. to be placed at the distance a from the axis CD, the same accelerating force will act on the body P.

Now, the moving force, in this case, is P; and the quantity of matter moved at the distance a, from the axis of motion, is equivalent to $P + \frac{mr^2}{a^2} + \frac{m'r'^2}{a^2} +$ &c.; and since the accelerating force f is proportional to the moving force divided by the quantity of matter moved (art. 230); it may be shewn, in the same manner as in art. 264, that

$$f : g :: \frac{P}{P + \frac{mr^2}{a^2} + \frac{m'r'^2}{a^2} + \text{\&c.}} : \frac{P}{P};$$

$$\therefore\ f = \frac{Pa^2g}{Pa^2 + mr^2 + m'r'^2 + \text{\&c.}}.$$

316. *Cor.* 1.—If the points m, m', m'', &c. be not in the same horizontal plane, we may conceive the whole system to be projected upon this plane by lines perpendicular to it. Then, as each particle is thus kept at the same distance from the axis, the effect arising from the rotatory motion will not be changed; therefore the same formulæ will be true.

317. *Cor.* 2.—If $CB = b$, the accelerating force at the point B $= \frac{Pabg}{Pa^2 + mr^2 + m'r'^2 + \text{\&c.}}$. For, from the property of the lever, the effect of the moving force P at the distance b is $\frac{Pa}{b}$; and the quantity of matter placed at B, which has the same effect in resisting the communication of angular motion, is $\frac{Pa^2}{b^2} + \frac{mr^2}{b^2} + \frac{m'r'^2}{b^2} +$ &c. Hence we find the accelerating force in the same manner as before.

318. The denominator of the fraction which expresses the accelerating force on any point of the system is *the sum of each particle multiplied into the square of its distance from the axis.* This sum is called the *Moment of Inertia* with respect to this axis, and is a quantity which continually occurs in considering the rotation of bodies.

319. Prop. III.—*The moment of inertia about any axis is equal to the moment of inertia about an axis parallel to this passing through the centre of gravity, together with the moment of inertia of the body collected in its centre of gravity about the given axis.*

Let m, m', &c. be any system of bodies referred to a plane perpendicular to the axis passing through C, as in art. 316. Let G be their centre of gravity. Draw mp, $m'p'$, &c. perpendicular to CG. Then

$$Cm^2 = CG^2 + Gm^2 + 2CG \times Gp;$$

$$Cm'^2 = CG^2 + Gm'^2 + 2CG \times Gp'; \quad Cm''^2 = \text{\&c.}$$

Hence $\quad m \,.\, Cm^2 + m' \,.\, Cm'^2 + m'' \,.\, Cm''^2 +$ &c.

$$= \left\{ \begin{array}{l} (m + m' + \&c.)\, CG^2 + m \,.\, Gm^2 + m' \,.\, Gm'^2 + m'' \,.\, Gm''^2 \\ \quad + 2CG \times (m \,.\, Gp + m' \,.\, Gp' + m'' \,.\, Gp'' + \&c.); \end{array} \right.$$

but (art. 73), $m \,.\, Gp + m' \,.\, Gp' + m'' \,.\, Gp'' + \&c. = 0$, therefore, putting $m + m' + m'' + \&c. = M$, we have

$$m \,.\, Cm^2 + m' \,.\, Cm'^2 + \&c. = m \,.\, Gm^2 + m' \,.\, Gm'^2 + \&c. + M \,.\, CG^2;$$

which is the proposition to be proved.

320. Defs.—(1). *The centre of gyration* of a system of bodies revolving about an axis is that point in which, if all the matter were collected, the same moving force would produce the same angular velocity in the system.

(2). *The centre of oscillation* is that point in a body revolving about an horizontal axis at which, if the whole mass were collected, it would vibrate through a given angle by the force of gravity in the same time as before.

(3). *The centre of percussion* is that point in a body revolving about an axis which, striking against an immoveable obstacle, the whole motion of the body shall be destroyed; so that if, at the moment of impact, the axis were removed, the body would have no tendency to move in any direction.

321. Prop. IV.—*To find the centre of gyration of any system of material points.* (See fig. to art. 315.)

Let K be the centre of gyration of any number of material points m, m', m'', &c.; and let $CK = k$. Then the accelerating point at the point $A = \dfrac{Pa^2 g}{Pa^2 + mr^2 + m'r'^2 + \&c.}$; and if all the matter be collected at K, and $m + m' + m'' + \&c. = M$, the accelerating force at A will $= \dfrac{Pa^2 g}{Pa^2 + Mk^2}$. And since the same angular velocity is generated in both cases, these accelerating forces must be equal. Hence $Mk^2 = mr^2 + m'r'^2 + m''r''^2 + \&c.$;

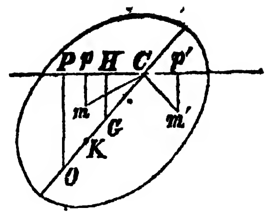
$$\therefore\ k = \sqrt{\left(\frac{mr^2 + m'r'^2 + m''r''^2 + \&c.}{m + m' + m'' + \&c.}\right)}.$$

322. *Scholium.*—In the preceding articles, the bodies m, m', m'', &c. being moved round a vertical axis, are only affected by their own inertia. In the next proposition, we shall suppose the axis of motion to be horizontal, and determine the accelerating force when each of these particles is acted on by the force of gravity.

323. Prop. V.—*A system of material points, moveable about a horizontal axis, has all its parts acted on by gravity; to find the accelerating force at any point* O.

Let m, m', m'' be any number of bodies connected together. Suppose them all to be projected perpendicularly upon a plane which passes through G their centre of gravity, and is perpendicular to the axis of suspension passing through C; then, as each particle is thus kept at the same distance from the axis, the accelerating force at any point O will be the same as before. Now, the moving forces are the weights

m, m', m'', &c.; and the distances at which they act from the axis are Cp, Cp', Cp'', &c. And, from the property of the lever, the effective moving forces at the point O, perpendicular to CO, are $\frac{m \cdot Cp}{CO}, \frac{m' \cdot Cp'}{CO}$, &c. Also, the quantity of matter which, placed at O, will have the same inertia as the system, is



$$\frac{mr^2}{CO^2} + \frac{m'r'^2}{CO^2} + \frac{m''r''^2}{CO^2} + \text{\&c.}$$

Hence it may be shewn, in the same manner as in art. 264, that the

$$\text{accelerating force at } O = \frac{(m \cdot Cp + m' \cdot Cp' + m'' \cdot Cp'' + \text{\&c.})\, CO \times g}{mr^2 + m'r'^2 + m''r''^2 + \text{\&c.}}.$$

And, since $m \cdot Cp + m' \cdot Cp' + m'' \cdot Cp'' + \text{\&c.} = (m + m' + \text{\&c.})\, CH = M \cdot CG \sin\theta$, therefore the

$$\text{accelerating force at } O = \frac{M \cdot CG \cdot CO\, g \sin\theta}{mr^2 + m'r'^2 + m''r''^2 + \text{\&c.}}.$$

324. Prop. VI.—*To find the centre of oscillation of any system of material points moveable about an horizontal axis.* (See the last figure.)

Let O be the centre of oscillation, then, from the last article, the accelerating force of a body M, placed at O, $= \frac{M \cdot CP \cdot CO \cdot g}{M \cdot CO^2} = g \sin\theta$. And since the same angular velocity is generated in both cases, these accelerating forces must be equal. Hence

$$\frac{M \cdot CG \cdot CO \cdot g \sin\theta}{mr^2 + m'r'^2 + m''r''^2 + \text{\&c.}} = g \sin\theta;$$

$$\therefore \quad CO = \frac{mr^2 + m'r'^2 + m''r''^2 + \text{\&c.}}{M \cdot CG}.$$

325. *Cor.* 1.—If K be the centre of gyration of the system, then

$$M \cdot CG \cdot CO = mr^2 + m'r'^2 + m''r''^2 + \text{\&c.} = M \cdot CK^2.$$

Hence $CG \cdot CO = CK^2$; or CK is a mean proportional between CG and CO.

326. *Cor.* 2.—Since the accelerating force of this system at the point O is the same as the accelerating force of a single particle placed at O, the time of oscillation of the system will be the same as that of a simple pendulum whose length is CO. Hence, if $CO = l$, the time of a very small oscillation $= \pi \sqrt{\frac{l}{g}}$.

327. Prop. VII.—*If the centre of oscillation be made the centre of suspension, the former centre of suspension will become the centre of oscillation.*

From art. 324, $CO = \frac{m \cdot Cm^2 + m' \cdot Cm'^2 + \text{\&c.}}{(m + m' + \text{\&c.})\, CG}$; also, from art. 319,

$$m \,.\, Cm^2 + m' \,.\, Cm'^2 + \&c.$$
$$= (m + m' + \&c.)\, CG^2 + m \,.\, Gm^2 + m' \,.\, Gm'^2 + \&c.$$

Put $m + m' + \&c = M$, and $m \,.\, Gm^2 + m' \,.\, Gm'^2 + \&c. = Mk^2$; then

$$CO = \frac{M \,.\, CG^2 + Mk^2}{M \,.\, CG} = CG + \frac{k^2}{CG};$$

$$\therefore \quad GO = \frac{k^2}{CG}, \quad \text{and} \quad CG \times GO = k^2,$$

which is a constant quantity for the same body. Suppose the body to be suspended at any other point C', and O' to be the centre of oscillation in this case, then

$$CG \times GO = k^2 = C'G \times GO';$$

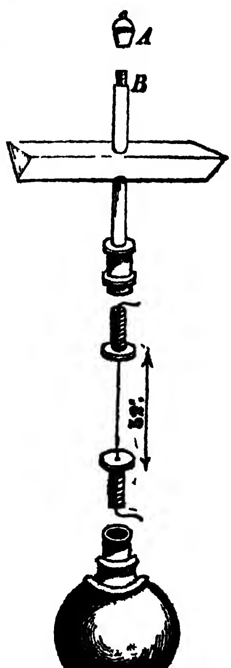
and if C' coincide with O, or $C'G = GO$, then $CG = GO'$, or O' coincides with C.

328. *Cor.*—The time of oscillation is a minimum when CO or $CG + \frac{k^2}{CG}$ is the least. Put $CG = k + x$, then

$$CO = k + x + \frac{k^2}{k + x} = \frac{2k^2 + 2kx + x^2}{k + x} = 2k + \frac{x^2}{CG}.$$

And this is evidently the least when $x = 0$, in which case the length of the pendulum $= 2k$.

Prop. VIII.—*To determine the length of the seconds pendulum experimentally.*

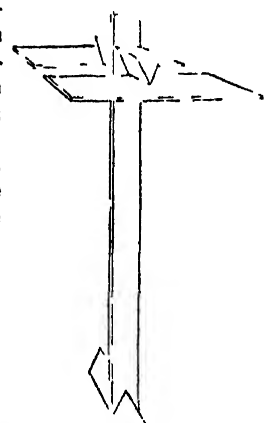


329. *Borda's Method.*—The principle of this method consists in employing a body whose properties approach as near as possible to those of the simple pendulum, and which can be reduced to this ideal case by very easy corrections. The pendulum is formed of a ball of platinum, made exactly spherical, and attached to a fine metallic thread. The lower extremity of the wire is screwed into a spherical cap of copper, of the same radius as the ball, and the cap, being smeared with a little tallow, adheres firmly to the ball. The other end of the wire is attached to a *knife-edge* or prism, which vibrates on a plane of agate, furnished with adjusting screws, by which it can be made perfectly horizontal (the plane of agate is represented in the next figure). The mass of the knife-edge is previously adjusted by means of a small ring of metal A, which screws round a metallic rod B, so that the vibrations of the knife shall be isochronous with those of the pendulum. The wire and ball are then suspended from the knife-edge; and it may easily be proved, either experimentally or by calculation, that the mass of the knife exerts no sensible influence on the vibrations of the pendulum.

The pendulum is now enclosed with the clock, in a glass-case, so that it may not be affected by

the motion of the air. As the length of the wire is regulated in such a manner that the platinum ball and the pendulum of the clock vibrate in times not exactly but nearly equal, one gains upon the other, and after some time they will be seen, through a telescope, to be moving exactly together. They will then separate, and, after an interval of time, they will be together again. Between two consecutive coincidences the pendulum gains or loses two oscillations upon the clock; and, therefore, by a simple proportion, we may obtain the whole gain or loss in a day. The coincidences of the two pendulums must not be very near to each other, for that would multiply unnecessarily the trouble of the observer; but neither must they be made too distant, for, in that case, the precise instant of coincidence would be very difficult to observe.

Let h be the distance from the knife-edge to the centre of the sphere, and a the radius of the sphere: then it will be seen, in the examples afterwards given, that the length of the pendulum is equal to $h + \frac{2a^2}{5h}$. There is a correction to be made on account of the amplitude of the arc of vibration (art. 281), and another on account of the weight of the wire and the copper cap; but, for these and other details, we must refer the student to the third volume of the *Base du Système Métrique Décimal.*



.330 *Captain Kater's Method.*—This method is founded upon the theorem proved in art. 327, that the centres of oscillation and suspension are reciprocal. A pendulum was chosen of such a form that the centre of oscillation could easily be determined by calculation. This consisted of a rectangular bar of plate brass, 1·6 inches wide, and $\frac{1}{8}$ of an inch thick. The length of the bar, from the knife-edge to the extremity, was about five feet; and the weight of the whole pendulum 15lbs. 2oz. Through this bar two triangular holes were made, at the distance of 39·4 inches from each other, to admit the knife-edges that were to serve for the axes of suspension in the two opposite positions of the pendulum. A cylindrical weight, $3\frac{1}{2}$ inches in diameter, and weighing about 2lbs. 7oz., was firmly fixed a little nearer to one end of the pendulum than the knife-edge. A second weight, of about $7\frac{1}{2}$ ounces, was made to slide on the bar at the opposite end; and it was moved to different positions, until the oscillations about the two knife-edges were nearly equal, and then it was firmly fixed. A third weight, or slider, of only four ounces, was moveable along the bar, and it was placed near the middle of the rod, where any change in its position produces the least effect upon the oscillations. It was capable of nice adjustment, by means of a screw and clasp, and it was shifted until the times of oscillation were exactly equal.

331. *Improvement of the pendulum.*—The construction of this pendulum has been simplified in some of the later experiments. In two of those taken out by Captain Forster, we are told by Mr. Bailey, that "the pendulum consisted merely of a plain straight bar, 2 inches wide, $\frac{1}{4}$ an inch thick, and about $62\frac{1}{4}$ inches long. The property of converti-

bility in these pendulums, instead of being effected by moveable or sliding weights, is produced by *filing away* one of the ends of the pendulum until the number of vibrations on the two knife-edges are in a given time equal to each other, or nearly so, after the proper corrections are made. At the distance of five inches from one end of the bar is placed the apex of one of the knife-edges *A*; and at the distance of 39·4 inches therefrom is placed the apex of the other knife-edge *B*.

"If it should be found (as probably will be the case) that the knife-edge *B* makes a less number of vibrations in a day than the knife-edge *A*, we must file away a portion of the bar at the end *B*, until the vibrations on the two knife-edges are synchronous, or nearly so." To make the adjustment perfect, a small hole *C* was made in the bar, about $1\frac{1}{2}$ inch from the end *B*, and about $\frac{1}{2}$ an inch in diameter. This hole is fitted by two screws, drawing towards each other, leaving a small space in the centre, which enables the experimentalist to place between them a small piece of sheet-lead, or other substance; by means of which contrivance the adjustment may be carried to any required degree of accuracy.

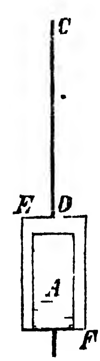
The form and construction of this pendulum appear to possess advantages which do not attach to the pendulum as usually constructed; whether we consider it as a convertible pendulum for determining the important problem first practically illustrated by Captain Kater; or whether it be viewed only as a travelling instrument for ascertaining the figure of the earth. In the former case, it is preferable, on account of its simplicity and compactness: none of the parts slide over another; there are no sliding weights, no moveable screws, no wooden tail-pieces. In the latter case there are several advantages, which Mr. Bailey enumerates; he then concludes thus: "But were I to construct another pendulum of this kind, I should not make the distance between the knife-edges more than 39 inches. This would be a very convenient distance; since we need not employ any other fractional part of the inch than what would be comprised within the run of the microscope. A brass pendulum of this sort would weigh about 25 pounds Troy; and its vibrations would continue 3 hours, commencing with an arc of 1° and terminating with an arc of 0°·10.

332. PROP. IX.—*To explain the principle of compensation pendulums.*

Since all bodies vary their dimensions by an increase or diminution of temperature, there is a continual change in the position of the centre of oscillation of any pendulum made of simple materials, and, consequently in the time of its vibration. To obviate this inconvenience, several methods have been devised; we shall explain two of them which are most in use.

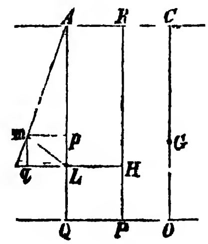
1. *Harrison's Gridiron Pendulum.*—The principle of this will easily be understood, from the accompanying diagram. Different metals expand in different degrees by an increase of temperature. *AF* represents one-half of a pendulum, suspended at *A*, the other half being omitted, to render the explanation more distinct. *Aa* is a rod of steel; at the bottom is the bar *aB*, to which is affixed the rod of copper

Bb. Again, Cc is a rod of steel, affixed to the bar bC; Dd a rod of copper; and, lastly, Ee is a rod of steel which carries the bob F. Now, suppose there to be an increase of temperature; the rod Aa will expand downwards, because it is fixed at A. But, for the same reason, Bb will expand upwards, because it is fixed to the bar aB. In like manner, Cc will expand downwards, Dd upwards; and, finally, Ee, which carries the bob F, will have its expansion downwards. Let α, β, γ, δ, ϵ be the respective expansions of these rods; then the whole expansion of the steel rods downwards will evidently be $\alpha + \gamma + \epsilon$; and the whole expansion of the copper rods upwards will be $\beta + \delta$. It is manifest, also, that the sum of the lengths of the steel rods exceeds the lengths of the copper rods by the length of the pendulum. But, as copper expands in a greater degree than steel, their lengths may be so adjusted that the centre of oscillation shall neither rise nor fall; in which case, the times of vibration will all be equal to each other.



2. *Graham's mercurial pendulum.*—The rod CD of this pendulum is usually made of steel, and about 32 inches long. This carries a frame or stirrup EF, on which is supported a glass cylinder A containing mercury. The cylinder is generally about 2 inches wide and 8 deep, having 6·8 inches of this depth filled with the fluid, but the quantity of mercury is dependent on the weight and expansibility of the other materials. When the temperature increases, the steel rod is lengthened, whilst the mercury rises in the cylinder from the same cause. By a proper adjustment, therefore, of the quantity of mercury, these two effects may be made to neutralise each other, and the centre of oscillation be kept in an invariable position. As the great mass of mercury is not affected by the changes of temperature so rapidly as the metallic rod, Capt. Kater had the cylinder and rod made in one piece of glass, without any other frame. On the other hand, the cylinder is sometimes made of iron or brass, and varnished inside to prevent the action of the mercury.

333. Prop. X.—*To find the centre of percussion of any system of material points.*



Let ACG be a plane passing through AC, the axis of rotation, and G the centre of gravity of the system; draw GC perpendicular to AC. Let m be any particle without this plane; draw mA perpendicular to AC, and AQ parallel to CG. Also draw mL perpendicular to Am in the plane mAQ, and mp perpendicular to AQ. Let $Am = r$, and let ω be the angular velocity of the body at the instant of impact, or the velocity of a point whose distance from the axis AC is 1. Then the velocity of $m = r\omega$, and as it is evidently moving in the direction mL, its momentum or force in that direction $= mr\omega$. We may suppose this force to act at the point L. Now, if the effect of all these forces to turn the body about a fixed obstacle P, when the body strikes against it, be nothing, their effect in turning the body about any fixed line PQ, or PR, passing through P, will evidently also be nothing. We will first estimate their effect about the line PQ, parallel to AC.

(1). Draw mp, Lq perpendicular to AQ. Because CA is perpendicular to Am and AQ, it is perpendicular to the plane mAQ; therefore, the planes ACG and mAQ are at right angles to each other (Geom. prop. 100). Hence mp and Lq are perpendicular to the plane ACG. Let mL represent the force $mr\omega$, and let it be resolved into the forces pL, qL. The force pL has no effect in turning the body about PQ; but the force qL is entirely effective.

Now, the whole force at L in the direction mL is to the effective force in the direction qL, as

$$mL : qL :: Am : Ap :: mr\omega : m\omega \,.\, Ap;$$

and since the force mL is equivalent to $mr\omega$, therefore the effective force qL is equivalent to $m\omega \,.\, Ap$. Also, the energy of this force to turn the body about the line PQ is equal to

$$m\omega \,.\, Ap \,.\, QL = m\omega \,.\, Ap \,.\, (AQ - AL) = m\omega \times Ap \times PR - m\omega r^2,$$

since $Ap \times AL = Am^2 = r^2$. Hence, the energy of the whole system to turn the body round the line PQ is equal to

$$\omega \,.\, PR\,(m \,.\, Ap + m' \,.\, Ap' + \&c.) - \omega\,(mr^2 + m'r'^2 + \&c.);$$

and when P is the centre of percussion, the sum of all these forces $= 0$; therefore,

$$PR = \frac{mr^2 + m'r'^2 + m''r''^2 + \&c.}{m \,.\, Ap + m' \,.\, Ap' + \&c.} = \frac{mr^2 + m'r'^2 + \&c.}{(m + m' + \&c.)\, CG}.$$

(2). Again, the energy of the force qL to turn the body about the line RP, parallel to CG, is equal to

$$m\omega \times Ap \times HL = m\omega \times Ap \times (CA - CR).$$

Hence the energy of the whole system to turn the body about the line RP is equal to

$$\omega(m \,.\, Ap \,.\, CA + m' \,.\, Ap' \,.\, CA' + \&c.) - \omega \,.\, CR(m \,.\, Ap + m' \,.\, Ap' + \&c.)$$

and since the sum of all these forces $= 0$, we have

$$CR = \frac{m \,.\, Ap \,.\, CA + m' \,.\, Ap' \,.\, CA' + \&c.}{m \,.\, Ap + m' \,.\, Ap' + \&c.}.$$

334. *Cor.* 1.—The distance of the centre of percussion from the axis AC is equal to the distance CO of the centre of oscillation from this axis. And if the body be symmetrical with respect to the line CG, the point P will evidently coincide with O

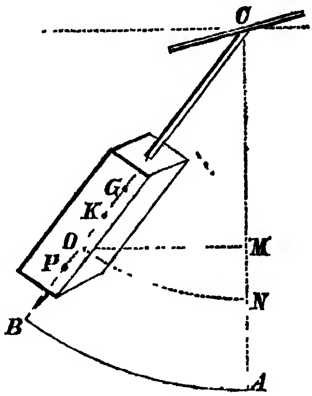
335. *Cor.* 2.—The rectangle $CG \times GO$ (from art. 327) is a constant quantity, and therefore the point O is always farther from the axis of rotation than G.

336. *Cor.* 3.—The energy of the force qL to turn the body about the axis AC is $m\omega \,.\, Ap \,.\, AL = m\omega \,.\, Am^2$. Therefore the energy of the whole system to turn the body about $AC = \omega\,(mr^2 + m'r'^2 + \&c.)$

337. *Scholium.*—Since the reaction of the obstacle P destroys all the motion of the body, if we conceive a force at P equal to this reaction to act upon the body at rest, it is manifest that it would produce a motion in the body equal to that which was destroyed; and, therefore, the body would begin to move precisely in the same manner, and with the same velocity, as that with which its motion was stopped; that is, it would begin to move about the axis AC. In this case, AC is called the *axis of spontaneous rotation.*

THE BALLISTIC PENDULUM AND THE EPROUVETTE.

338. Prop. XI.—*To determine the velocity with which a ball strikes ballistic pendulum, when the pendulum is made to vibrate through a given angle.*



The *ballistic pendulum* consists of a heavy block of wood suspended vertically by a strong horizontal iron axis, to which t is connected by a firm iron stem. When this is at rest, a ball is fired in the block, which causes the pendulum to vibrate through an angle ACB. The extent of the vibration is known by means of a sharp needle, which, proceeding from the bottom of the pendulum, just enters and scratches a soft composition laid in a groove of the arc AB.

The centre of gravity of the body was found, by Dr. Hutton, either by balancing it on the edge of a triangular prism laid parallel to the axis of rotation, or by supporting the extremities of the axis on fixed uprights, and attaching the lower end of the block to a cord passing over a fixed pulley, and fastened to such a weight as made the pendulum perfectly horizontal. A slight calculation will then give the place of the centre of gravity. The centre of oscillation was found by making the pendulum vibrate in small arcs for 5 or 10 minutes, and observing the number of vibrations in that time. The distance of the centre of oscillation would then be found from art. 326. The centre of gyration is determined from art. 325, its distance from the axis of suspension being a mean proportional between the distances of the centre of gravity and the centre of oscillation.

Let K be the centre of gyration of the pendulum, P the point where the ball strikes the pendulum. Let the whole weight of the pendulum $= M$, the weight of the ball $= P$, $CK = k$, and $CP = a$.

Since K is the centre of gyration, the same motion will be communicated to the pendulum as if all the matter were collected at that point. Also, if the mass $\frac{Mk^2}{a^2}$ be placed at P, the pendulum would receive the same motion as when the mass M is placed at K (art. 314). We may, therefore, consider the body P to impinge directly with a velocity V upon the body $\frac{Mk^2}{a^2}$ at rest, and the two bodies will move on together after impact in the same manner as if they were inelastic. Let x be their common velocity after the stroke; then, since the momentum is the same before and after impact (art. 231), we have

$$x\left(P + \frac{Mk^2}{a^2}\right) = PV; \quad \therefore \quad x = \frac{PVa^2}{Pa^2 + Mk^2}.$$

Let G and O be the centres of gravity and oscillation of the pendulum, including the ball; then the motion of the pendulum afte ct,

when left to itself, will be the same as if all the matter were collected at O. And the arc NO, through which O will ascend, will be the arc down which it will acquire the same velocity as that with which it began to ascend; and this velocity is equal to that which a body would acquire in falling freely by the force of gravity through MN, the versed sine of this arc. Let $CO = l$, then $MN = l \text{ vers } \theta = 2l\sin^2\frac{1}{2}\theta$; and

$$\text{vel. at } O = \sqrt{(2g \,.\, MN)} = 2\sin\tfrac{1}{2}\theta\,\sqrt{(gl)}.$$

But $\quad \text{vel. at } O : \text{vel. at } P\,(x) :: CO : CP$;

or, $\quad 2\sin\dfrac{\theta}{2}\sqrt{gl} : \dfrac{PVa^2}{Pa^2 + Mk^2} :: l : a.$

Hence
$$PVal = 2\sin\tfrac{1}{2}\theta\sqrt{gl}\,(Pa^2 + Mk^2)$$
$$= 2\sin\tfrac{1}{2}\theta\sqrt{gl}\,(P + M)hl \quad \text{(art. 324)};$$
$$\therefore \quad V = 2\sin\frac{\theta}{2}\,\frac{P+M}{P}\,\frac{h}{a}\,\sqrt{(gl)}.$$

The value of $2\sin\frac{1}{2}\theta$ is found by dividing the chord AB by the radius CA.

339. *Cor.* 1—If the pendulum, after being struck by the ball, makes n oscillations in a minute, we have

$$\frac{60}{n} = \text{time of oscillation} = \pi\sqrt{\frac{l}{g}}; \qquad \therefore \quad \sqrt{gl} = \frac{60g}{\pi n},$$
$$\therefore \quad V = 2\sin\frac{\theta}{2}\,\frac{P+M}{P}\,\frac{60gh}{\pi na}.$$

340. *Scholium.*—There are several circumstances that have been neglected in this solution, which Dr. Hutton has noticed in the second volume of his Tracts: 1st, The resistance of the air. 2nd, The friction on the axis. 3rd, The time employed by the ball in communicating its motion to the pendulum. All these errors, however, have been shown by him to be of such a trifling nature, that they do not affect the result more than half a foot. The quantities g and l may also be taken for the centre of gravity and oscillation of the pendulum alone, without any sensible error.

341. *The Eprouvette.*—In many of his experiments, Dr. Hutton suspended a small brass gun, about $2\frac{1}{2}$ feet long, and then observed the arc through which it was driven by the recoil. The same formula is likewise applicable to this case, M now representing the weight of the cannon and its apendages. Also, the momentum communicated at the axis of the cannon will be PV, because it is equal to the momentum communicated by the ball in the opposite direction. A cannon suspended in this manner is called an *eprouvette;* it has an index attached to it, which moves along a graduated arc, to indicate the angle of the recoil.

Examples.

1. Let the weight of the pendulum = 570lbs., the weight of the ball = 1·131 lbs., h = 78·5 in., l = 84·78 in., a = 94·3 in., and chord described by O = 18·73 in.; to find the velocity of the ball.

Ans. 1401 feet.

2. Let the weight of the pendulum $= 740 \cdot 8$ lbs., the weight of the ball $= 6 \cdot 1$ lbs., $a = l = 11 \cdot 6$ feet, $h = 10$ feet, and the angle $\theta = 3^\circ\ 34'$; required the velocity of the ball. Ans. $126 \cdot 97$ feet.

3. The weight of the pendulum $= 56\frac{3}{16}$ lbs., the weight of the ball $= \frac{1}{12}$ lb., the chord described by the steel point $= 17\frac{1}{4}$ in., and the radius of this arc $= 71\frac{1}{8}$ in., $l = 62\frac{2}{3}$ in., $h = 52$ in., $a = 66$ in.; required the velocity of the ball. Ans. $1673 \cdot 2$ feet.

THE MOMENT OF INERTIA.

342. As the expressions for the moment of inertia continually occur in problems on rotatory motion, we insert the following results in this place. The method for determining them will be given in the Integral Calculus.

1. The moment of inertia of a straight line $CA\ (= a)$, revolving about an axis perpendicular to it at $C = k^2 M = \frac{1}{3} a^2 M$.

2. In a circle revolving about its centre in its own plane (radius $= a$), the moment of inertia $= k^2 M = \frac{1}{2} a^2 M$.

3. In a circle moving about an axis lying in its own plane, at a distance c from its centre, $k^2 M = (a^2 + \frac{1}{4} c^2)\, M$.

4. In a cylinder revolving about its axis, $k^2 M = \frac{1}{2} a^2 M$.

5. In a cone revolving about its axis (rad. of base $= a$), $k^2 M = \frac{3}{10} a^2 M$.

6. In a cone revolving about its vertex, $k^2 M = \frac{3}{5} M (\frac{1}{4} a^2 + b^2)$.

7. In a sphere revolving about a diameter, $k^2 M = \frac{2}{5} a^2 M$.

8. In a hollow sphere, $k^2 M = \dfrac{2}{5}\, \dfrac{a^5 - b^5}{a^3 - b^3}\, M$.

9. In a paraboloid about its axis, $k^2 M = \frac{1}{3} a^2 M$.

PROBLEMS ON ROTATORY MOTION.

1. Suppose a cylinder that weighs 100lbs. to turn upon a horizontal axis, and suppose motion to be communicated by a weight of 10lbs. attached to a cord which coils upon the surface of the cylinder; how far will that weight descend in 10 seconds? Ans. $268\frac{1}{3}$ feet.

2. Required the actuating weight, such that, when attached in the same way to the same cylinder, it shall descend $16 \cdot 1$ feet in 3 seconds. Ans. Weight $= 6\frac{1}{4}$ lbs.

3. Another cylinder, which weighs 200lbs., is actuated in like manner by a weight of 30lbs.; how far will the weight descend in 6 seconds? Ans. $133 \cdot 75$ feet.

4. Suppose the actuating weight to be 30lbs., and that it descends through 48 feet in 2 seconds; what is the weight of the cylinder? Ans. $20\frac{1}{2}$ lbs.

5. Suppose a cylinder that weighs 20lbs. to have a weight of 30lbs. actuating it, by means of a cord coiled about the surface of the cylinder; what velocity will the descending weight have acquired at the end of the first second? Ans. $24 \cdot 15$ feet.

6. A sphere W, whose radius is 3 feet, and weight 500lbs., turns upon a horizontal axis, and is put in motion by a weight of 20lbs. acting by means of a string that goes over a wheel whose radius is half a foot; how long will the weight be in descending 50 feet? Ans. 33·483″.

7. If, in the last example, the radius of the wheel be equal to that of the sphere, what ratio will the accelerating force bear to that of gravity?

8. A paraboloid W, whose weight is 200lbs. and radius of base 20 inches, is put in motion upon a horizontal axis by a weight P of 15lbs., acting by a cord that passes over a wheel whose radius is 6 inches. After P has descended for 10 seconds, suppose it to reach a horizontal plane and cease to act, then how many revolutions would the paraboloid make in a minute? Ans. 122·3.

PART III.—HYDROSTATICS.

343. Hydrostatics is that branch of the science of mechanics which treats of the equilibrium of fluids.

344. A *fluid body* is a collection of extremely minute particles, which yield without resistance to the smallest force impressed. In treating of fluids we shall suppose them to possess this state of *perfect* fluidity, although there are many which are endowed with a certain quality of coherence, termed *viscocity* or *tenacity*, and which, therefore, have been called *imperfect* fluids.

345. Fluids are divided into *elastic* and *inelastic* fluids. An *elastic* fluid is one whose dimensions are diminished by increasing the pressure, and increased by diminishing the pressure; of which description are common air, and the different kinds of vapours and gases. An *inelastic* fluid is one whose dimensions are but very slightly affected by any pressure, however great, such as water, mercury, alcohol, &c. These are also called *liquids*.

346. It was generally believed, for some time, that water, mercury, and other liquids, could not be made to occupy a smaller space by the application of any external force. This opinion was principally founded on an experiment made by the Florence Academy, in which a sphere of gold was filled with water, and being then closed up, was subjected to very great pressure. The water, however, passed through the pores of the metal, and stood on its surface like dew. Now, as the content of a sphere is greater than that of any other figure of equal surface, it was inferred that, when the figure of the globe was changed, the water, being incompressible, forced its way through the pores of the gold. But, unless a comparison could be made between the diminution of the content

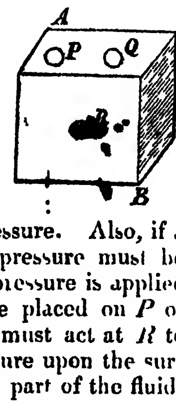
of the sphere and the quantity of water forced through the pores, it is evident that no conclusion could be drawn from this experiment.

347. Canton proved the incorrectness of this inference by a very simple experiment. He took a glass tube, of small diameter, with a large bulb at the bottom, and filled it to a certain height with water. When this was placed under the receiver of an air-pump, and the air exhausted, the water rose in the graduated tube, and it sunk again to its original position when the air was re-admitted. Also, when it was placed in the receiver of a condenser, and subjected to an increased pressure, the water fell in the tube. From the experiments of Colladon and Sturm, it appears that the amount of compression due to a pressure equivalent to that of one atmosphere is 0·00004965 in water, 0·00009165 in alcohol, and 0·00000338 in mercury.

CHAP. I.—PRESSURE OF FLUIDS

348. Prop. I.—*Any pressure communicated to a fluid in equilibrium is equally transmitted through the whole fluid.*

This proposition, which is generally made the foundation of the doctrine of hydrostatics, is proved from experiment.

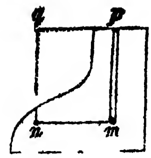


Let AB represent a closed box full of water; P, Q two vertical pistons, of equal transverse section, fitted into the upper face of the box, and allowed to move as freely as possible. Then, if a weight be placed on P, an equal weight must be placed on Q, to preserve the equilibrium, showing that the weight on P is transmitted through the fluid to the under surface of Q, and also with equal force, since it requires an equal weight on Q to balance this pressure. Also, if a piston equal to P be fitted at R, it is found that a pressure must be exerted at R to preserve the equilibrium, before any pressure is applied at P; and when the equilibrium exists, if a weight be placed on P or Q, an additional pressure, equal to the weight P, must act at R to preserve the equilibrium: which proves that the pressure upon the surface at P is transmitted with equal force through every part of the fluid.

349. *Scholium.*—That any pressure should be transmitted in every direction is one of the most extraordinary properties of fluids, and can be conceived to arise only from the perfect freedom with which the particles move amongst each other. This is the characteristic distinction between them and solids: a solid transmits pressure only in the direction in which the force is exerted; a fluid transmits pressure in all directions.

350. Prop. II.—*The pressure at any point* m *in the interior of a fluid whose density is uniform, and which is acted upon by no forces but gravity, is equal to the weight of the vertical column* mp.

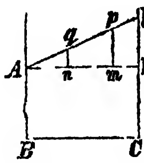
Let us suppose all the fluid in the vessel, except the column mp, to become solid; then, since no new forces are introduced, by congealing



the fluid, it is obvious that the particle at m will be precisely in the same state as before. But, in this case, it is evident that the pressure upon the particle m is equal to the pressure of the column above it mp. Hence, when the whole is fluid, the particle is equally pressed in all directions, by a force equal to the weight of the vertical column above it.

If the point n be not directly under the surface of the fluid, draw nm parallel and mp perpendicular to the surface. Now, by the last proposition, the pressure at the point m is transmitted along the line mn, and, therefore, the pressure at n must be equal to the pressure at m, otherwise the equilibrium would be destroyed. Hence the pressure at n is equal to the weight of the vertical column mp.

351. PROP. III.—*The surface of every fluid at rest is horizontal, or perpendicular to the direction of gravity.*



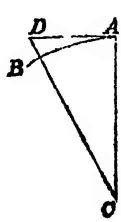
If the surface AD be not horizontal, draw AE parallel to the horizon, and any two lines mp, nq perpendicular to it. Now, by the last proposition, the pressures at m and n are as mp and nq; and these pressures are transmitted in the directions mn and nm. And since the pressure in the direction mn is greater than that in the direction nm, the particles in the line mn will be driven towards A, and the fluid will not be at rest. But, when the surface AD becomes horizontal, these pressures will be equal, and the fluid will be in equilibrium.

352. *Cor.* 1.—In like manner it appears that, if there be two fluids of different densities in the same vessel which do not mix, their common surface will be horizontal.

353. *Cor.* 2.—Since the surface of a fluid at rest is horizontal, it follows that a fluid in a system of vessels in free communication with each other cannot be at rest unless the surface of the fluid in these different vessels be on the same level.

354. *Cor.* 3.—In surfaces of small extent, gravity may be supposed to act in parallel lines; and, in this case, the surface of every fluid at rest is a *plane* perpendicular to the direction of gravity. But, in surfaces of great extent, such as the surface of a lake, or a sea, the directions of gravity converge to a point near the centre of the earth, and, in this case, the surface of the fluid is a portion of a spherical surface having that point for a centre.

355. PROP. IV.—*To explain the principles of levelling.*



Levelling may be defined to be the art of drawing a line on the surface of the earth, to cut the directions of gravity everywhere at right angles. And since the figure of the earth is nearly a sphere, this line may be considered as the arc of a circle, having the centre of the earth for its centre. Let AB be a small portion of the surface of the earth, C the centre, and CA the radius. Let AD be a horizontal line perpendicular to CA; join CB and produce it to D. Now, the point D is apparently on the same level with A, but B is in reality

on a true level with A. The line BD is called the *depression,* and may be calculated as follows :—

Put $CA = r$, arc $AB = d =$ tangent AD very nearly ; since AB never exceeds a few miles in length, or a few minutes of a degree. Also, put the depression $BD = h$; then (Geom. prop. 79),

$$d^2 = h(2r + h) = 2rh, \text{ very nearly};$$

and, since the mean value of $2r = 7912$ miles,

$$h = \frac{d^2}{7912} \text{ miles} = \frac{5280d^2}{7912} \text{ feet} = \frac{2 \times 261}{3 \times 263\frac{2}{30}} d^2;$$

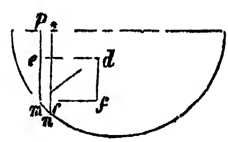
$\therefore \quad h = \frac{2}{3}d^2$, nearly, $\quad h$ being measured in feet, and d in miles;

or, $\quad h = 8d^2$, $\quad h$ being measured in inches, and d in miles.

356. *Cor.*—Let $d = 1$ mile, then the depression $= \frac{2}{3}$ of a foot $= 8$ inches.

Let $d = \frac{1}{8}$ of a mile, the depression $= \frac{2}{3}$ of $\frac{1}{64}$ of a foot $= \frac{1}{8}$ of an inch ; that is, the depression for $\frac{1}{8}$ of a mile is $\frac{1}{8}$ of an inch.

357. PROP. V.—*If the fluid contained in any vessel be at rest, and subjected to no forces but gravity, the pressure on an indefinitely small area* mn, *at any point of the bottom or sides, is perpendicular to the plane of that area, and equal to the weight of a vertical column, whose base is* mn, *and altitude* mp.



For the number of particles in contact with mn is proportional to the area mn. Also, fluids press equally in all directions and in proportion to their depths ; therefore, the pressure of each particle perpendicular to mn is equal to a column of fluid whose height is mp. Hence the whole pressure perpendicular to mn is equal to the weight of a column of fluid whose base is mn and altitude mp.

358. *Cor.*—The pressure exerted upon mn in the *direction of gravity* is equal to the weight of the fluid pn. Let p represent the pressure upon mn in the direction of gravity, and w the weight of the fluid pn; also, take $md = mp$, to represent the perpendicular pressure of any particle against mn. Then this pressure may be resolved into the two, em, fm; and em is that part which acts in the direction of gravity. Now, since mn is indefinitely small, we have, by this proposition,

$$p : w :: \text{area } mn \times me : \text{area } mr \times mp.$$

But $\quad$ area mn : area mr :: md : me; therefore,

area $mn \times me =$ area $mr \times mp$; and, consequently $p = w$.

359. PROP. VI.—*The pressure of a fluid on any surface is equal to the weight of a column of the fluid, whose base is the surface pressed, and height equal to the depth of the centre of gravity of that surface below the surface of the fluid.* (See the last figure.)

Let the surface M be divided into an indefinite number of elementary portions m, m', &c. whose distances from the surface of the fluid are x, x', &c. ; then the pressure of the fluid against the indefinitely small

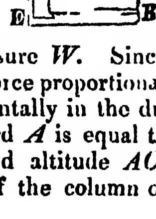
area m is equal to the weight of a column of fluid, whose base is m and height x, (art. 357); and if ρ be the density of this fluid, or the weight of each unit of bulk, the pressure on m will be $mx \times \rho$; and the sum of all the pressures will be $(mx + m'x' + m''x'' + \&c.) \times \rho$. But, by art. 71, $mx + m'x' + \&c. = Mh$, M being equal to $m + m' + \&c.$, and h equal to the distance of the centre of gravity of M from the surface of the fluid. Hence the whole pressure against the surface $M = Mh\rho$; that is, it is equal to a column of the fluid whose base is M and altitude h.

360. *Cor.* 1.—Hence the pressure against the side of a cubical vessel filled with fluid is equal to half the pressure against the bottom, or is equal to half the weight of the fluid.

361. *Cor.* 2.—If a be the altitude of a cylinder, and r the radius of the base, the pressure against the base $= \pi r^2 \times a \times \rho = \pi a r^2 \rho$. Also, the pressure against the upright surface $= 2\pi a r \times \frac{1}{2}a \times \rho = \pi a^2 r \rho$. Therefore, the pressure against the base is to the pressure against the upright surface as r to a.

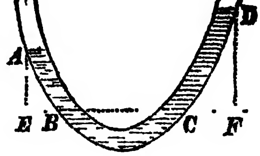
362. *Cor.* 3.—The pressure against the base of a cone filled with fluid is equal to three times the weight of the fluid. In this case there is a pressure downwards against the base equal to three times the weight of the fluid, and a pressure upwards against the sides of the cone equal to twice the weight of the fluid, so that the resultant of the pressures downwards is equal to the weight of the fluid.

363. *Scholium.*—It is upon this principle that the experiment called the *hydrostatical paradox* may be explained. *AF* is an apparatus called the hydrostatic bellows; *A*, *B* are two flat boards united by leather; *EF* is a tube communicating with the interior of the bellows. The upper board *A* is loaded with heavy weights and pressed against the lower board *B*. If water be now poured into the funnel *F*, it will enter between the boards and raise the weights, the small weight of water in the tube balancing the enormous pressure *W*. Since the fluid at *E*, the bottom of the tube, is pressed with a force proportional to the altitude *EF*, this pressure is transmitted horizontally in the direction *EB*, and the pressure upward against the board *A* is equal to the weight of a column of fluid whose base is *A* and altitude *AC*. Hence the weight *W* upon *A* is equal to the weight of the column of water *AC*.



364. Prop. VII.—*If two fluids communicate in a bent tube, they will be in equilibrium when their perpendicular altitudes above the horizontal plane where they meet are inversely as their densities.*

Let *ABCD* be the tube, *B* the place where the fluids meet, *EF* a horizontal plane passing through *B*. Let ρ be the density of the fluid in the tube *AB*, ρ' the density of the fluid in *BCD*, and *B* the area of the transverse section at *B*. Then the pressure of the fluid in *AB* downwards upon the section $B = B \times AE \times \rho$, and the pressure of the fluid in

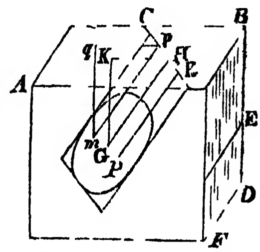


BD upwards upon this section $= B \times DF \times \rho'$; and since this fluid is at rest, $B \times AE \times \rho = B \times DF \times \rho'$, therefore,

$$AE : DF :: \rho' : \rho.$$

365. Prop. VIII.—*To find the centre of pressure of a plane surface.*

The centre of pressure is that point of a surface pressed by any fluid, to which, if the whole pressure were applied, the effect would be the same as when the pressure was distributed over the whole surface. Hence, if a force equal to the total pressure be applied at this point in the contrary direction, it would keep the surface at rest.



Let AB be the horizontal surface of the fluid which presses upon the plane GP; CR the common intersection of these planes, and P the centre of pressure. Conceive the whole area GP to be divided into an indefinite number of elementary portions m, m', &c. Draw mp, pq perpendicular to CR, and mq perpendicular to pq. Because CR is perpendicular to pm, pq, it is perpendicular to the plane mpq, therefore the planes AB, mpq are at right angles to each other. Hence the line mq is vertical (Geom. prop. 103). Let $\theta =$ the angle $mpq =$ the inclination of the planes AB, GP. Now, the pressure on the indefinitely small area m is proportional to

$$m \,.\, mq = m \,.\, mp \sin\theta = mr \sin\theta; \text{ putting } mp = r.$$

And the effect of this force to turn the plane about the line CR is as $mr \sin\theta \times r = mr^2 \sin\theta$. Hence the effect of all the pressures to turn the plane about CR is proportional to

$$(mr^2 + m'r'^2 + m''r''^2 + \&c.) \sin\theta.$$

Also, if $M = m + m' + m'' + \&c. =$ area GP, the pressure on GP is as $M \,.\, GK = Mh \sin\theta$, putting $GH = h$. And the effect of this pressure at P to turn the plane about CR is proportional to $Mh \sin\theta \times PR$. Hence

$$Mh \sin\theta \times PR = (mr^2 + m'r'^2 + m''r''^2 + \&c.) \sin\theta;$$

$$\therefore \quad PR = \frac{mr^2 + m'r'^2 + m''r''^2 + \&c.}{Mh}.$$

Again, the effect of the force $mr \sin\theta$ to turn the plane about GH is as $mr \sin\theta \,.\, Hp$. And the effect of the force $Mh \sin\theta$ at P to turn the plane about GH is as $Mh \sin\theta \times HR$. Hence, as before,

$$Mh \sin\theta \times HR = mr \sin\theta \,.\, Hp + m'r' \sin\theta \,.\, Hp' + \&c.;$$

$$\therefore \quad HR = \frac{mr \,.\, Hp + m'r' \,.\, Hp' + m''r'' \,.\, Hp'' + \&c.}{Mh}.$$

Hence it appears that the centre of pressure of the plane GP coincides with the centre of percussion of this plane when it moves about the axis CR (art. 333).

366. Cor. 1.—The centre of pressure against the rectangular side BF is at $\frac{2}{3}$ds of the depth from the surface. Let $BD = a$, $DF = b$; and let BD be divided into n equal parts, each equal to α, so that $a = n\alpha$; and suppose lines to be drawn through these divisions parallel

to DF, the area BF will be divided into n rectangles, each equal to $b\alpha$. If now we suppose every point in each of these rectangles to be at the same distance from the surface of the fluid as the base of this rectangle, we shall manifestly have

$$\begin{aligned} mr^2 + m'r'^2 + \&c. &= b\alpha \times \alpha^2 + b\alpha \times (2\alpha)^2 \ldots\ldots + b\alpha \times (n\alpha)^2 \\ &= b\alpha^3 (1^2 + 2^2 + 3^2 \ \ldots\ldots\ldots\ + n^2) \\ &= b\alpha^3 \frac{n(n+1)(2n+1)}{6} = \frac{bn^3\alpha^3}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right) \\ &= \frac{a^3b}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right). \end{aligned}$$

Let each of these parts α be diminished indefinitely, or the number of divisions n be increased without limit; then the pressure, in this case, will be the same as the pressure of the fluid upon BF. And since $\frac{1}{n}$, $\frac{1}{2n}$ are indefinitely small, the expression above becomes

$$mr^2 + m'r'^2 + m''r''^2 + \&c. = \tfrac{1}{3}a^3b.$$

Hence the value of PR is, in this case, $= \dfrac{\frac{1}{3}a^3b}{ab \times \frac{1}{2}a} = \dfrac{2a}{3}$.

367. *Cor.* 2.—If $BD = a$, $BE = a'$, the distance of the centre of pressure of the rectangle EF from the surface is equal to

$$\frac{\frac{1}{3}a^3b - \frac{1}{3}a'^3b}{(a-a')b \times \frac{1}{2}(a+a')} = \frac{2}{3}\,\frac{a^3 - a'^3}{a^2 - a'^2}.$$

Scholium.

368. The laws of equilibrium of fluids, which we have explained in this chapter, are subject to a remarkable exception, in the case of tubes of a very small bore. When the internal diameter does not exceed $\frac{1}{10}$th of an inch, the water within the tube will rise to a greater height than its level on the outside, and this height is nearly in the inverse ratio of the diameter of the tube.

Tubes of this description are called *capillary* tubes, and the power by which the water is elevated is called *capillary* attraction, from the Latin word *capillus,* a hair. Many attempts have been made to account for the elevation of the water within the tube; but that which appears to be most successful in explaining all the different phenomena, is the theory of Laplace. When a capillary tube is immersed in the water, the surface of the water is concave upwards, and if, by taking the tube out of the water and inclining it, the fluid be made to move along it, the concavity appears at both ends of the column, and has the same figure, whatever be the position in which the tube is held. From this fact Laplace infers, that a narrow ring or zone of glass immediately above the water exerts its force on the water, while the water exerts its own attraction on the particles of the column immediately underneath, by which means the gravity of those particles is diminished, and the water rises in the tube above its level on the outside, to supply their deficiency of weight.

If a capillary tube be immersed in mercury, the mercury is *depressed*

within the tube below its level on the outside, and its surface within is *convex*, as it is likewise without all round the tube. In this case there is an attraction between mercury and glass, but the mutual attraction between the particles of mercury is considerably greater.

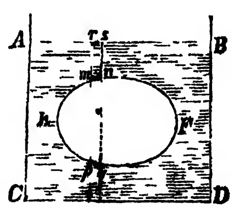
If two plates of glass be kept parallel and near to one another, and if their ends be immersed in water, the water will ascend between them to half the height to which it would rise in a tube, having its diameter equal to the distance of the plates; for, in this case, the quantity of the attracting zone is diminished nearly in this proportion. When the plates of glass make an angle with one another, and are immersed with the line of their intersection vertical, the water will ascend between them, and form at its surface a hyperbolic curve. We can merely allude to this subject, and refer the student for an explanation of the various phenomena to Laplace, and also to Poisson, *Nouvelle Théorie de l'Action Capillaire.*

CHAP. II. — SPECIFIC GRAVITIES, AND THE EQUILIBRIUM OF FLOATING BODIES.

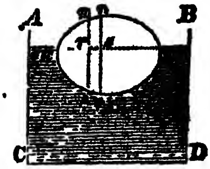
369. Def.—The *specific gravity* of a body is the relation of its weight compared with the weight of some other body of the same magnitude. Thus, brass is said to have 8 times the specific gravity of water, because a cubic inch of brass contains 8 times the quantity of matter, or is 8 times heavier than a cubic inch of water. Hence the specific gravity of a body is proportional to its density.

It is usual to consider the specific gravity of distilled water at a temperature of 60° as the unit of comparison, or 1, for all solids and liquids; and the specific gravity of dry atmospheric air, when the thermometer is at 60° and the barometer at 30 inches, as the unit of comparison for all vapours and gases.

370. Prop. I.—*When a body is either partly or wholly immersed in a fluid, it is pressed upwards by a force equal to the weight of the fluid displaced.*



Let AB be the horizontal surface of a fluid in the vessel AD; and EF a heavy body suspended in it. Any indefinitely small portion mn of the upper surface of this body is pressed downwards by the column of fluid $mrsn$, and the similar portion pq of the surface immediately under it is pressed upwards with a force equal to a column of fluid $prsq$ (art 358). Therefore, the difference of these forces, or the resulting force, which presses upwards against pq, is the weight of a column of fluid equal in bulk to $mpqn$. In the same way it may be shown that the pressures *upwards* against the whole body EF exceed the pressures *downwards* by the weight of fluid equal in bulk to the body EF, and, consequently,



the body is pressed upwards by a force equal to the weight of the fluid displaced.

When the body floats in the fluid, the pressure upwards against pq is equal to the column of fluid $prsq$, and, therefore, the sum of all the pressures upwards is the weight of fluid equal in bulk to EF; that is, the body is pressed upwards also in this case by a force equal to the weight of the fluid displaced.

371. *Cor.* 1.—*The weight which a body loses when wholly immersed in a fluid is equal to the weight of an equal bulk of fluid.* For, since the body is pressed upwards with a force equal to the weight of the fluid displaced, the weight of the body must be diminished by this quantity. The weight lost is communicated to the fluid.

372. *Cor.* 2.—*When a body floats in a fluid, the weight of the quantity of fluid displaced is equal to the weight of the floating body.* For, since the body is at rest, the pressure upwards must be equal to the pressure downwards, but the pressure upwards is equal to the weight of the fluid displaced, and the pressure downwards is equal to the weight of the body, and therefore these weights are equal to each other.

373. *Cor.* 3. —A solid immersed in a fluid will sink, if its specific gravity exceed that of the fluid; it will float on the surface, partly immersed, if its specific gravity be less than that of the fluid; and it will remain wholly immersed, in any position if the specific gravities of the solid and fluid are equal. In the first case, the weight of the solid exceeds the pressure upwards, and, therefore, it must sink. In the second case, the pressure upwards exceeds the weight of the body, and therefore, the body must rise and float on the surface. In the third case, these forces being equal to each other, the solid will remain at rest in any position.

374. Prop. II.—*To determine the specific gravities of bodies.*

(1). *A solid heavier than its bulk of water.* The specific gravity of a solid body is very easily found by the *hydrostatic* balance, which is a common pair of scales, with a fine silver thread attached to the under surface of one of the scales. The substance whose specific gravity is [illegible] first weighed in air, and then being attached to the thread is [illegible] in distilled water, at the temperature of 60°, and again weighed.

Let W be the weight of the body in air, and w its weight in water; then $W - w$ is the weight lost, which is equal to the weight of the fluid displaced (art. 371); therefore, $W - w$ is the weight of water equal in bulk to the body whose weight is W. Hence

$W : W - w$ :: weight of the body : weight of an equal bulk of water
:: specific gravity of the body : ditto of water.

And since the specific gravity of water at the standard temperature is 1 (art. 369); therefore, the

$$\text{specific gravity of the body} = \frac{W}{W - w}.$$

If the body is soluble in water, or so porous as to absorb water, it should be well covered with varnish.

(2). *A solid lighter than its bulk of water.*—Fasten to it another solid heavier than water, so that they may sink together. Let H be the weight of the heavy body in air, h its weight in water; also, let C be the weight of the compound body in air, and c its weight in water. Then the

weight of water equal in bulk to the compound body $= C - c$,

weight of water equal in bulk to the heavy body $= H - h$,

∴ weight of water equal in bulk to the given body $= (C-c)-(H-h)$.

Hence $W : (C-c)-(H-h)$ :: spec. grav. of the body : do. of water; and, therefore, we have,

$$\text{specific gravity of the body} = \frac{W}{(C-c)-(H-h)} = \frac{W}{W+h-c}.$$

(3). *A powder or liquid.*—Weigh a vial when empty,—when filled with the powder,—and when filled with distilled water. The weight of the powder divided by the weight of the water will be the specific gravity required. The same method is used in finding the specific gravities of fluids.

(4). *Air, or any gas.*—A large flask, containing about two quarts, is formed of thin copper, with a narrow neck, in which is placed a stop-cock, that may be opened or closed at pleasure. The air is then exhausted, as far as can be done, by means of the air-pump, and the vessel weighed. Now, let the vessel communicate with a gasometer or bladder containing the gas whose specific gravity we wish to find. After having filled the flask, and turned the stop-cock to prevent any communication with the atmosphere, the vessel is again weighed. The difference of these weights will be the weight of the gas. And, as the content of the bottle is known, the specific gravity of the gas is found as before.

THE HYDROMETER.

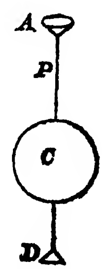
375. *The hydrometer* is a simple instrument, invented for the purpose of determining the specific gravities of spirituous liquors with great expedition. AB is a graduated stem, fixed to the hollow globe C; D is a ball loaded with weights. In order to graduate the scale, immerse the hydrometer in distilled water of the temperature of 60°, and load the ball so that it may stand at the point B near the bottom of the stem. At B place the number 1·000, the specific gravity of water. The hydrometer is then to be plunged in another fluid less dense than water; suppose oil, whose specific gravity may be ·900, and the point A marked to which it sinks. If, then, the scale AB be divided into 100 equal parts, and marked from ·900 to 1·000, the hydrometer will at once give the specific gravity of any fluid between these limits. And in the same manner it may be extended to determine the specific gravities beyond these limits.

376. *Sykes' hydrometer* is another form of this instrument, used by the government for collecting the revenue on ardent spirits. The stem

AB is made of brass, and 3·4 inches long. It is marked by 11 divisions, at equal distances, numbered 0, 1, 2,......9, 10; and each of these parts is subdivided into two. There are several additional weights which accompany this instrument, numbered 10, 20, 30, &c. By means of these weights the specific gravities of all spirituous liquors can easily be found.

The specific gravity of alcohol, or pure spirit, varies, according to different chemists, from 0·791 to 0·798, at the temperature of 60°. *Proof spirit* consists of nearly equal bulks of alcohol and water, and its specific gravity, according to the recommendation of the committee of the Royal Society, is 0·92, at the temperature of 62°.

377. *Nicholson's hydrometer* is an improvement of this instrument, and serves to determine the specific gravity both of solids and fluids. C is a hollow globe, attached to the cup A by a stem of hardened steel. To the lower extremity of the ball is affixed another stem, carrying the cup D. This instrument is so adjusted, that when the lower cup D is empty, and the upper cup A contains 1000 grains, it will sink in distilled water to a given point P. Let the hydrometer be immersed to the point P in any other fluid of the same temperature, which may be done by increasing or diminishing the weights in the cup A. Let W be the weight of the instrument, and w the weight placed in the upper cup requisite to sink it to the point P in the fluid F. Then, since 1000 grains will sink it to P in distilled water,



the weight of the water displaced $= W + 1000$;

also, the weight of the fluid F displaced $= W + w$,

and the bulk is the same in each case, since the instrument is sunk to the same point. Hence

$$W + w : W + 1000 :: \text{specific gravity of } F : \text{do. of water};$$

$$\therefore \text{ specific gravity of the fluid } F = \frac{W + w}{W + 1000}.$$

378. *To determine the specific gravity of a solid body by this instrument.*—Immerse the instrument in distilled water, and load the cup A until the surface of the water is at P; then, if the temperature of the water be 60°, the weight in the cup will be 1000 grains. Remove now this weight, and place the solid there whose specific gravity is required, and load the cup A again until the instrument sinks to P. Let W be the weight of the body, and a the weight necessary to sink the instrument to the point P; then $W + a = 1000$, and $W = 1000 - a$. Place the solid in the cup D, and add weights in the upper cup until the hydrometer again sinks to P. Let w be the weight of the solid in water, and b the weight added in the cup A, then $w + b = 1000$, and $w = 1000 - b$. Having now determined the weight of the body, both in air and water, the specific gravity will be found, as before, to be $\frac{1000 - a}{b - a}$.

379. In the preceding methods of determining the specific gravities of bodies, we have supposed that the weight of the body *in air* was the

true weight of the body, or the same as its weight in a vacuum; but, as air itself is a fluid, the body loses a portion of its weight in air, in the same manner as when it is weighed in water. A small correction is, therefore, necessary on this account.

380. PROP. III.—*The specific gravity* s *of a body being given, as determined by weighing it in air and water; it is required to find its true specific gravity.*

Let W be the weight of the body in air, w its weight in water, and let x be its true weight, or its weight in a vacuum. Also, let α be the specific gravity of air, as compared with water. Now, the weight of an equal bulk of water $= x - w$, and the weight of an equal bulk of air $= x - W$; consequently

$$x - w : x - W :: 1 : \alpha; \qquad \therefore \; x - W = \alpha(x - w);$$

hence $$x = \frac{W - \alpha w}{1 - \alpha}, \quad \text{and} \quad x - w = \frac{W - w}{1 - \alpha}.$$

Now, the true specific gravity of the body is

$\dfrac{x}{x - w}$, which, from the two last equations, is equal to $\dfrac{W - \alpha w}{W - w}$.

But $$\frac{W}{W - w} = s, \quad \text{and} \quad \frac{w}{W - w} = s - 1;$$

$$\therefore \text{ true specific gravity } = s - \alpha(s - 1).$$

381. *Cor.*—The true weight of the body $= \dfrac{W - \alpha w}{1 - \alpha}$

$$= W + \alpha(W - w), \text{ nearly.}$$

A table of specific gravities will be given at the end of Hydrodynamics.

Examples.

1. A piece of copper weighs 31 grains in air, and $27\frac{1}{2}$ grains in water; required its specific gravity. Ans. 8·857.

2. Suppose that a piece of elm weighs 15lbs. in air, and that when a piece of copper, which weighs 16lbs. in water, is affixed to it, the compound weighs 6lbs. in water; required the specific gravity of the elm. Ans. ·6.

3. A Nicholson's hydrometer, weighing 250 grains, requires 72 grains to sink it to the required depth in water, and 9 grains in alcohol; what is the specific gravity of alcohol? Ans. ·804.

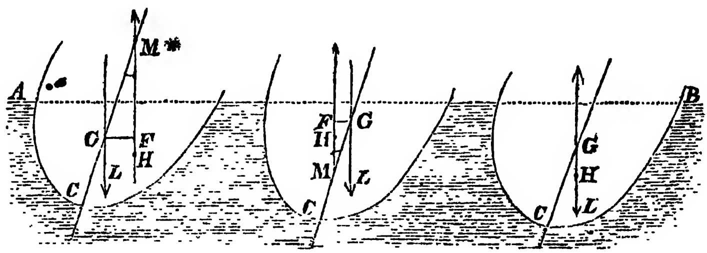
EQUILIBRIUM OF FLOATING BODIES.

382. PROP. IV.—*If a body float in a fluid, the centres of gravity of the body and of the fluid displaced are in the same vertical line.*

The pressure of the body downwards is its weight, which may be supposed to be applied at its centre of gravity (art. 62). Also, the pressure of the fluid upwards is precisely the same as the weight of the

fluid downwards, acting in the opposite direction; and as the resultant of all the forces downwards passes through the centre of gravity of the fluid, the resultant of all the opposite forces upwards must also pass through its centre of gravity. And since the body is at rest, the weight of the body downwards, and the pressure of the fluid upwards, must be opposite and equal. Hence the centres of gravity are in the same vertical line.

383. PROP. V.—*To determine when the equilibrium is stable, unstable, or indifferent.*



Let G be the centre of gravity of a body floating in a fluid whose surface is AB; let the centre of gravity of the fluid displaced be in the line GC when the body is in equilibrium, and let H be the centre of gravity of the displacement when the body is moved through a small angle θ. Let the vertical through H meet GC in M. Now, the weight of the floating body acts downwards in the direction GL, and the pressure of the fluid acts upwards in the direction HF. And, in the first figure, where M is situated above G, these two forces evidently tend to bring the body back to its original position. But, if M be below G, as in the second figure, the weight of the body downwards, and the pressure of the fluid upwards, tend to move the body farther from its original position, and the equilibrium is therefore *unstable*. Lastly, if G and H be in the same vertical line, or M coincides with G, the forces being equal, and being applied at the same point in opposite directions, the body is at rest in every position, or the equilibrium is indifferent. Hence the equilibrium of a floating body is stable, unstable, or indifferent, according as the point M is above, below, or coincident with the centre of gravity of the body.

384. *Cor.* 1.—The moment of force tending to bring the body back to its original position, or to move the body farther from it, is the weight of the body $W \times$ the arm of the lever $FG = W \times MG \times \sin\theta$ putting the angle $GMH = \theta$. Hence, if W and θ be given, the stability of the body is proportional to GM.

385. *Cor.* 2.—When a floating body revolves about a given axis, the positions of equilibrium through which it passes are alternately those of stability and instability. For, between a state in which the body has a tendency to remain, and another in which it has also a tendency to remain, as these tendencies are opposite to one another, there must be an intermediate position, in which this tendency changes its direction. Suppose the body to be in this position, then, whichever way it is disturbed, there is a tendency to move from this position; and, therefore, the equilibrium is unstable.

386. Def.—The point of intersection of a vertical line passing through the centre of gravity of the fluid displaced, when the body has been disturbed through a very small angle, and a vertical line passing through the centre of gravity of the body when it was at rest, is called the *metacentre* of the body.

387. Prop. VI.—*A body immersed in a fluid ascends or descends with a force equal to the difference between its own weight and the weight of an equal bulk of fluid; the resistance of the fluid being neglected.*

Let W be the weight of the body, w the weight of an equal bulk of the fluid. Now, the pressure downwards is W, and the pressure upwards is w (art. 370); therefore $W - w$ is the difference of pressures which causes the body to descend. Also, W is the weight moved; hence it may be shown, in the same manner as in article 264, that the accelerating force downwards is $\frac{W-w}{W}g$.

Let $W : w :: \rho' : \rho$; then $\frac{W-w}{W} = 1 - \frac{\rho}{\rho'}$,

$$\therefore \text{ accelerating force downwards} = \left(1 - \frac{\rho}{\rho'}\right)g.$$

If W is less than w, the body will ascend, and the accelerating force upwards is $\left(\frac{\rho}{\rho'} - 1\right)g$.

PONTOONS.

388. *Pontoons* are portable boats carried in the train of an army, with light anchors and cables for mooring them, and with a superstructure of baulks and planks, or chesses, to form floating military bridges over rivers, for the passage of cavalry, infantry, and artillery, but not usually including battering guns. At the end of the last century, they were generally made of copper or tin, with sloping ends, flat bottoms, and perpendicular sides; but, in the late wars, this form having been found inefficient and unsafe for large and rapid rivers, most of the other European powers adopted wooden boats, larger, and of a more appropriate form than the punt-shaped metal pontoons before alluded to; whilst in the British service the open form has been altogether reprobated, and improvements recommended, first by Lieut.-Colonel Sir James Colleton, Bart., late of the Royal Staff Corps, and afterwards by Major-General Pasley and Lieut.-Colonel Blanshard, of the Royal Engineers. Sir James Colleton proposed cylindrical wooden buoys, with pointed ends, gradually tapering from the cylinder to the cone. Major-General Pasley proposed fine-shaped copper canoes, curved at the bottom and at both ends, with flat wooden decks; whilst Lieut.-Colonel Blanshard, adopting the cylindrical form, proposed very light tin cylinders with conical ends, with a new sort of superstructure and exercise, of a very ingenious nature, in which he facilitated the operations of launching and landing the pontoons, which had been very troublesome before, by rolling his cylinders like casks. All these plans were allowed to be efficient; but

Lieut.-Colonel Blanshard's arrangements being approved by a committee, held by authority of the Master-General of the Ordnance, have been adopted in the British service, as the new pattern pontoon equipage; but hemispherical ends have been substituted in preference to his original cones.

Each of these new pontoons is of tin, with a number of water-tight circular partitions, and measures 22 feet in extreme length, and 2 feet 8 inches in diameter, weighing 452lbs. The portion of superstructure belonging to each pontoon, which is capable of forming 12 feet in length of bridge, weighs 1098lbs.; therefore, the total weight of pontoon and superstructure for 12 feet of bridge amounts to 1550lbs. The spaces between adjacent pontoons are called bays, of which there is one more in every bridge than the number of pontoons. Hence spare superstructure is necessary in a pontoon train.

389. Prop. VII.—*To calculate the buoyancy of open pontoons.*

The original open copper or tin pontoons, and all boats used as such, must not be depressed too near to the surface, lest they should fill with water. Their whole buoyancy cannot, therefore, be available, as in the cylindrical or decked pontoons. Generally, in calculating the buoyancy of boats for military bridges, about 1 foot of the sides should stand out of water in the passage of troops, guns, &c. Even in water-tight pontoons it is proper to allow some little excess of this sort, which, in the new cylindrical pontoons, may be from 4 to 6 inches, but in the decked canoes less than one-half of the above will suffice.

To calculate the buoyancy of a boat used as a portion of a military bridge, allow 1 foot out of the water, and compute the capacity of the lower or immersed portion of the boat only in cubic feet, from whence estimate the weight of water displaced at $62\frac{1}{2}$lbs. to the cubic foot. From this deduct the known weight of the whole boat, and of its portion of superstructure, and the remainder will be the effective buoyancy or weight of men, horses, or guns, that each one-boat portion of the bridge can bear with safety. The capacity of the old punt-shaped boat is easily calculated; for its figure is a prism, whose two sides are the bases or ends of the prism, and the breadth of the pontoon is the length or height of the prism. Hence, the content of the part immersed may be found from Mensuration, prob. 23.

390. Prop. VIII.—*To calculate the buoyancy of vessels or boats.*

The curves of boats render it difficult to calculate their capacity with accuracy; but a rough estimate, or approximation to the content of the part immersed will, in general, be sufficient for practical purposes. The following rule is used by eminent surveyors as a tolerably near approximation to the truth. If l represent the length, b the breadth, and d the depth of the boat in feet, then,

For a full body $\frac{2}{3} \times 62\frac{1}{2} \times l \times b \times d =$ displacement in lbs.

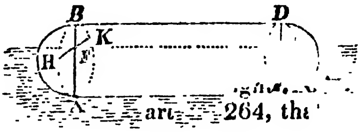
$$\frac{l \times b \times d}{54} = \text{displacement in tons.}$$

For a moderately full body $\frac{7}{12} \times 62\frac{1}{2} \times l \times b \times d =$ displacement in lbs.

391. **Prop. IX.**—*To calculate the buoyancy of the new cylindrical pontoon.*

The whole capacity is equal to that of a cylinder of 19 feet 4 inches in length, and 2 feet 8 inches in diameter, added to that of two hemispheres, or of one sphere of the same diameter; the pontoon in this case being supposed to be entirely immersed.

Let $2r$ be the diameter, and A the area of the common circular section of the hemispheres and cylinder, and let l be the length of the latter; then $A(l + \frac{4}{3}r)$ will give the whole capacity, by mensuration, prob. 23, and prob. 26.



But if it be desired to estimate the capacity of the immersed part only, let BC be the pontoon, $AHBK$ the common circular section of the cylinder and hemisphere. Also, let $BC = l$, $AB = 2r$, $AF = x$, and A = the area of the immersed part AHK.

Now, the portion of the pontoon immersed in the nsists of the segment of a cylinder cut off by a horizontal plane p allel to the axis, and the segments of the two hemispheres The segment of the cylinder may be considered as a prism, having the segment of the circle $HFKA$ for each of its ends, and its length equal to that of the cylinder. The segments of the two hemispheres evidently form together the segment of a sphere, whose height is AF and diameter AB.

Since the diameter AB, and height AF, are supposed to be given, the area AHK of the segment of the circle may be found by mensuration, prob. 13. Then,

$$\text{content of the cylindrical segment} = A \times l.$$

Also, by mensuration, prob. 27, the content of the spherical segment

$$= \tfrac{1}{6}\pi(6r - 2x)x^2 = \tfrac{1}{3}\pi x^2(3r - x);$$

$$\therefore \text{ whole displacement of water} = Al + \tfrac{1}{3}\pi x^2(3r - x).$$

And since the weight of a cubic foot of water = 1000 ounces = 62·5lbs. nearly, we have

$$1728 \text{ in.} : Al + \tfrac{1}{3}\pi x^2(2r - x) :: 62{\cdot}5\text{lbs.} : \text{wt. of water displaced.}$$

Hence the weight of the displacement of water will be equal to

$$\frac{62{\cdot}5}{1728} + \left\{Al + \tfrac{1}{3}\pi x^2(3r - x)\right\} \text{ in pounds avoirdupois}$$

$$= Al \times {\cdot}036169 + x^2(3r - x) \times {\cdot}037876;$$

each of the dimensions x, r, l, &c. being taken in inches.

Examples.

Ex. 1.—Supposing the new cylindrical pontoon to be of the dimensions and weight before stated, what additional weight will be required to force it under water? Ans. 5819lbs.

Ex. 2.—The dimensions and weight of the new cylindrical pontoon, and the weight of superstructure resting upon it when supporting 12 feet of bridge, having already been stated, what additional weight of cavalry, infantry, or artillery, will depress it to within 4 inches of the surface of the water? Ans. 5305½lbs.

The following table will be found useful in finding the depth to which any given weight will sink the pontoon :—

Depth sunk.	Weight of water displaced.	Depth sunk.	Weight of water displaced.	Depth sunk.	Weight of water displaced.	Depth sunk.	Weight of water displaced.
Inches	lbs.	Inches.	lbs.	Inches.	lbs.	Inches.	lbs.
9	1676·1	15	3387·2	21	5145·7	27	6655·0
10	1945·7	16	3684·6	22	5423·4	28	6855·6
11	2223·6	17	3982·0	23	5693·0	29	7034·4
12	2507·9	18	4275·1	24	5932·8	30	7186·4
13	2797·5	19	4571·6	25	6201·1	31	7304·7
14	3091·1	20	4861·3	26	6435·9	32	7369·2

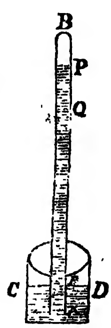
CHAP. III.—ELASTIC FLUIDS.

392. Most of the preceding propositions apply equally to all fluids, both liquids and gases, since the properties on which they depend are common to all. But air and other elastic fluids have some properties peculiar to themselves, which are the necessary consequences of the repulsive forces that exist between the particles. As atmospheric air is the best known of all elastic fluids, we shall generally take it in the following chapter for the subject of our investigation.

393. Prop. I.—*Air has weight.*

We have already seen, that if a vessel be exhausted of air, it will be lighter than when it was full. And we have shown how the weight and specific gravity of air may be actually determined by means of the hydrostatic balance. Air, therefore, has weight.

THE BAROMETER.



394. The weight of a column of the atmosphere is known by the barometer.* Let *AB* be a glass tube, 32 inches long or upwards, open at *A* and closed at *B*. Fill the tube with mercury, and placing the finger firmly on the end *A*, so as to prevent the mercury from escaping out of the tube, invert it, and plunge the end *A* into a vessel of mercury. If the finger be now removed, it will be found that the mercury will stand at about 29 or 30 inches in the tube above the level of the mercury in the basin.

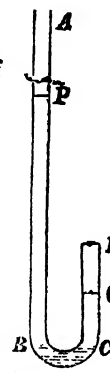
That the mercury is sustained by the pressure of the air upon the surface of the mercury in the basin, is clearly proved by placing the barometer under the receiver of an air-pump. As the air is exhausted the mercury sinks in the tube, and when the exhaustion is carried as far as possible, so that very little pressure is exerted on the surface of the mercury in the basin, the

* From Βάρος, weight, and μέτρον, measure.

mercury in the tube and in the basin are nearly on the same level. When the air is again admitted into the receiver, the mercury rises in the tube to its former height.

Since the pressure of a fluid on any portion of the surface is the weight of the superincumbent column of the fluid, the pressure of the mercury upwards against the surface CD is the weight of a column of mercury whose base is CD and altitude PE; and this pressure is balanced by the pressure of the air downwards on the surface CD.

395. Prop. II.—*The density of the air is proportional to the force that compresses it.*



Let $ABCD$ be a bent cylindrical glass tube, having one end open and the other closed. Let the communication between the two branches be cut off by a small quantity of mercury poured in at A until it just fills the base BC. Then the air in CD will be of the same density as the air in AB. Pour in mercury at A, and it will force the mercury to rise in DC, and continue this until the mercury stands at P, as high above the point Q, to which it has risen in DC, as the altitude of the mercury in the barometer; then that column of mercury (art. 394) is equivalent to the weight of the column of air incumbent upon it. Hence the pressure against the air in DQ, arising from the pressure of the atmosphere and the mercury in PB, is twice as great as it was against the air in CD; and it is observed that $DQ = \frac{1}{2}CD$; therefore, the air being compressed into half the space, the density is doubled. In like manner, if another column of mercury be added, so that the altitude of the mercury in AB, above the mercury in CD, shall be twice the altitude of the mercury in the barometer, the pressure on the air in DQ, arising from the weight of the atmosphere and the mercury in Pq, will now be three times as great as it was against the air in CF. Also, DQ will be found to be $= \frac{1}{3}CD$, and, therefore, the density in CQ is $=$ 3 times the density of the atmosphere. In this manner the density is found in all cases, within a moderate extent, to be proportional to the compressing force.

Cor.—Since the force which compresses the air is balanced by the air's elasticity, the elastic force of the air is equal to the compressing force, and may be measured by it. Hence, also, the air's elasticity is proportional to its density.

396. Prop. III.—*When the density of any gas remains the same, its elastic force increases in proportion to the increase of temperature.*

From the experiments of Dalton and Gay Lussac, it appears: 1st. That all gases, under the same pressure, expand uniformly for equal increments of temperature, at least this is true from the freezing to the boiling point of the thermometer; 2d. That the expansion due to the same increase of temperature is exactly the same for all gases; and 3d. That this expansion for a unit of bulk is 0·375, or $\frac{3}{8}$ from the freezing to the boiling points; and, therefore, the expansion for each degree of Fahrenheit is $\frac{1}{180}$ of $\frac{3}{8}$, or $\frac{1}{480}$.

Hence, if V be the volume or solid content of any gas at the temperature of 32°, and $\alpha = \frac{1}{480}$; its volume at the temperature at t degrees will be

$$V[1 + \alpha(t - 32^\circ)] = V(1 + \alpha\tau),$$

putting $\tau = t - 32^\circ$. Let P be the pressure on a unit of surface, when the gas occupies the space $V(1 + \alpha\tau)$, and p its pressure when it is reduced to the volume V; we have then, from art. 395,

$$p : P :: V(1 + \alpha\tau) : V;$$

$$\therefore\ p = P(1 + \alpha\tau),$$

consequently the increase of pressure $p - P$, is proportional to τ, the increase of temperature, when the density is constant.

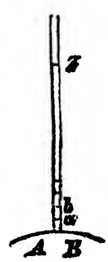
397. PROP. IV.—*The density and elastic force of any gas at a temperature of 32° being given; to find the elastic force for any other temperature and density.*

Let P be the pressure on a unit of surface for a given density R, at the temperature of 32°; and p the pressure when the density is ϱ, and the temperature is t degrees. Also, let p' be the pressure when the density is ϱ, and the temperature 32°. Now, since (art. 395) the pressure is proportional to the density when the temperature is the same, therefore $p' = P\dfrac{\varrho}{R}$. And because the increase of pressure is proportional to the increase of temperature, when the density is constant, $p = p'(1 + \alpha\tau)$. Hence, eliminating p' from these two equations, we get

$$p = \frac{P}{R}\varrho(1 + \alpha\tau) = k\varrho(1 + \alpha\tau),$$

putting the constant quantity $\dfrac{P}{R} = k$.

398. PROP. V.—*To determine the law by which the density and elasticity of the air diminish as the height above the surface of the earth increases; supposing the temperature of the atmosphere to be constant.*



Suppose AZ to be a column of the atmosphere resting on the base AB, whose area is 1; and suppose this column to be divided into an indefinite number of strata, of equal thickness, parallel to the horizon. Then, since each stratum of air is compressed by the weight of those above it, the lower strata will be more compressed, and, therefore, will be denser than those above them. Let $AZ = z$, $n =$ the number of strata in AZ, and $\zeta =$ thickness of each of these strata, then $z = n\zeta$. Let $p =$ pressure on the base AB, $p_1 =$ the pressure at a, $p_2 =$ pressure at b, ... $p_n =$ pressure at Z; and put $p - p_1 = \varpi$. Also, let $\varrho =$ density of the air at A, $\varrho_1 =$ density at a, ... $\varrho_n =$ density at Z.

$$\text{Now } \varpi = p - p_1 = \text{weight of the column } Ba.$$

And, since Ba or ζ is indefinitely small, the density of the air in Ba may be supposed to be uniform and equal to ϱ, the density at A, therefore, $\varpi = 1 \times \zeta \times \varrho$. In this expression ϱ is measured by the weight of a unit of bulk. But, if the force of gravity varies, the weight ϖ will vary in the same proportion, when the bulk and the density ϱ continue

the same. It is necessary, therefore, to take this into consideration. Let g be the ratio of the force of gravity at A to the force of gravity at some given place, which may be considered as the unit of force. We shall then have

$$\varpi = 1 \times \zeta \times \varrho \times g = \zeta \frac{pg}{k(1 + \alpha\tau)} \text{ (art. 397);}$$

$$\therefore \; p_1 = p - \varpi = p\left[1 - \zeta \frac{g}{k(1 + \alpha\tau)}\right] = p(1 - K\zeta),$$

putting $\frac{g}{k(1 + \alpha\tau)} = K$. In like manner, if g be supposed constant in the column AZ, we have

$p_2 = p_1(1 - K\zeta) = p(1 - K\zeta)^2 \dots$ and $p_n = p(1 - K\zeta)^n$.

Let H be the altitude of the barometer at A, and h its altitude at z, then $p : p_n :: H : h$; hence

$$\frac{h}{H} = \frac{p_n}{p} = (1 - K\zeta)^n$$

$$= 1 - nK\zeta + \frac{n(n-1)}{1 \cdot 2} K^2\zeta^2 - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} K^3\zeta^3 + \&c.$$

$$= 1 - \frac{nK\zeta}{1} + \frac{n^2K^2\zeta^2}{1 \cdot 2}\left(1 - \frac{1}{n}\right) - \frac{n^3K^3\zeta^3}{1 \cdot 2 \cdot 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \&c.$$

And since n is indefinitely great, and $n\zeta = z$, this series becomes

$$\frac{h}{H} = 1 - \frac{Kz}{1} + \frac{K^2z^2}{1 \cdot 2} - \frac{K^3z^3}{1 \cdot 2 \cdot 3} + \&c. = e^{-Kz} \text{ (Alg. art. 399);}$$

$$\therefore -Kz = \text{Nap. log} \frac{h}{H}; \quad \text{or, } Kz = \text{Nap. log} \frac{H}{h} = \frac{1}{M} \log \frac{H}{h};$$

these logarithms being taken from the ordinary tables, and M being the modulus of the system. Hence, substituting for K its value, we finally get*

$$z = \frac{k(1 + \alpha\tau)}{Mg} \log \frac{H}{h}.$$

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* This expression may more easily be found by means of the Integral Calculus. Let z be the height of any thin stratum of air above the surface of the earth, ζ its thickness, ϱ its density, and p, $p - \varpi$ the pressures of a column of atmosphere at the altitudes z and $z - \zeta$. Then the weight of this stratum is evidently ϖ. But its weight is also $1 \times \zeta \times \varrho \times g$, g being the ratio of the force of gravity in Az, which is supposed to be constant, to this force at some given place taken as a unit of force. Hence

$$\varpi = \zeta \times \varrho \times g = \zeta \frac{pg}{k(1 + \alpha\tau)} \text{ (art. 397);}$$

and, therefore, from the principles of the differential calculus,

$$-dp = dz \frac{pg}{k(1 + \alpha\tau)}; \quad \text{and} \quad dz = \frac{k(1 + \alpha\tau)}{g} \frac{-dp}{p}$$

$$\therefore \; z = \frac{k(1 + \quad)}{Mg} \log \frac{1}{p} + C.$$

399. Prop. VI.—*To determine the heights of mountains by the barometer.*

In the last proposition we have found a general expression for determining the relation between the elasticity of the air and the height above the surface of the earth; but, in the application of this formula, there are certain corrections necessary, which we will now explain.

(1). We have supposed the temperature of the air to be constant, whereas it always decreases as we ascend higher from the surface of the earth. But as we are ignorant of the law of this change, and as the correction is always small, we may approximate to the true value of z by taking τ a mean between the values at the two stations, and considering it constant. Let t, t' be the temperatures at the two stations by Fahrenheit's thermometer, then the two values of τ are $t - 32^\circ$, $t' - 32^\circ$, and, therefore, the mean value of τ is $\frac{1}{2}(t + t' - 64^\circ)$.

(2). The value of α is $\frac{1}{480}$, when the composition of the air is uniform, whether it be perfectly dry, or it contain a certain proportion of vapour. But when the temperature increases, there is generally a greater quantity of moisture in the atmosphere; and, since the density of vapour is to that of air (the barometer standing at 30 in.) as 10 to 16, it follows that the air will become rarer on this account, or it will expand in a higher ratio than $\frac{1}{480}$ to 1. Laplace, therefore, proposes that α should be increased to $\frac{4}{10}$ of $\frac{1}{100}$ in the centigrade thermometer, which, in that scale, is a very convenient number in practice; or to $\frac{4}{10}$ of $\frac{1}{180}$ in Fahrenheit's thermometer.

(3). The temperature of the mercury has been supposed to be the

---

Let $p = P$, when $z = 0$, then

$$0 = \frac{k(1+\alpha\tau)}{Mg}\log\frac{1}{P} + C.$$

Subtracting this equation from the preceding one, we get the expression in the text,

$$z = \frac{k(1+\alpha\tau)}{Mg}\log\frac{P}{p} = \frac{k(1+\alpha\tau)}{Mg}\log\frac{H}{h}.$$

If g be not constant, but vary inversely as the square of the distance; let g' be the ratio of the force of gravity at the altitude z, to the unit of force, and g this ratio at A, then

$$g' = g\frac{r^2}{(r+z)^2},\ r \text{ being the radius of the earth. Hence, in this case,}$$

$$\varpi = \zeta\varrho\frac{gr^2}{(r+z)^2} = \zeta\frac{pg}{k(1+\alpha\tau)}\frac{r^2}{(r+z)^2};$$

and, as above, $$\int\frac{r^2dz}{(r+z)^2} = \frac{k(1+\alpha\tau)}{g}\int\frac{-dp}{p}.$$

Taking these integrals from $z = 0$, and $p = P$,

$$r^2\left(\frac{1}{r} - \frac{1}{r+z}\right) = \frac{k(1+\alpha\tau)}{Mg}\log\frac{P}{p}.$$

And since the pressures are evidently proportional to the height of the mercury in the barometer multiplied by its gravitation,

$$P : p :: H \times g : h \times g' :: \frac{H}{r^2} : \frac{h}{(r+z)^2}.$$

Hence we get, by substitution in the last equation and reduction,

$$z = \frac{k(1+\alpha\tau)}{Mg}\left(1+\frac{z}{r}\right)\log\left[\frac{H}{h}\left(1+\frac{z}{r}\right)^2\right].$$

same at both stations, whereas it must be affected to a certain extent by the temperature of the air. Let T, T' be the temperature of the mercury at the two stations, which are known by means of a thermometer attached to the barometer. These are not necessarily the same as the temperature of the atmosphere, because it requires some time before the mercury acquires the temperature of the surrounding air. Now, it has been proved that mercury expands $\frac{1}{9742}$th part of its bulk for each degree of Fahrenheit's thermometer; we may consider this, without sensible error, as ·0001. If, then, the temperature of the mercury at the upper station, whose altitude is h, should be increased from T' to T, its altitude in the tube would be increased from h to $h[1 + {\cdot}0001(T - T')]$, which quantity, therefore, must be used instead of h in the preceding formula.

(4). The force of gravity, in the proposition, has been supposed to be constant, but in the note we have investigated a formula, in which it varies inversely as the square of the distance from the centre of the earth. The effect of this change, however, is so trifling, that it may be safely neglected. Again, the force of gravity varies in different latitudes at the surface of the earth; so that, if this force, at the latitude of 45°, be taken as the unit of force, the value of g at the latitude L will be $1 - 0{\cdot}002695 \cos 2L = 1 - \lambda$, putting $0{\cdot}002695 \cos 2L = \lambda$. Hence, if the value of z be calculated for the latitude of 45°, it must be divided by $1 - \lambda$, or, since λ is extremely small, multiplied by $1 + \lambda$, for any other latitude L.

(5). The constant quantity $\dfrac{k}{M}$ can only be obtained from experiment. From several observations of Ramond, Laplace has determined that, in the latitude of 45°, this co-efficient is 18336 metres; but, if the force of gravity be supposed to be constant in the column AZ, Poisson finds that 18393 metres (equal to 60345 English feet) will agree better with these observations. Hence, introducing these several corrections into the value of z, given in the last proposition, we get

$$z = 60345 \left[1 + \frac{t + t' - 64}{900}\right] \log \frac{H}{h[1 + {\cdot}0001\,(T - T')]} \text{ feet};$$

and, by reduction,

$$z = 67{\cdot}05\,(836 + t + t')\,N;$$

where $N = \log H - \log h - \log [1 + {\cdot}0001\,(T - T')]$.

If it be considered necessary to introduce the correction on account of the latitude, this may easily be done by multiplying the preceding value of z by $0{\cdot}002695 \cos 2L$, and adding the product to z for its true value.

Examples.

Ex. I.—To find the height of Mount Etna, when the altitudes of the barometer and thermometers were as follows:—

	Height of Barometer.	Attached Thermometer.	Detached Thermometer.
Level of the sea	30·024	77·5	77·5
Top of Mount Etna	20·146	35·5	32·5

Here $T - T' = 42$; and $836 + t + t' = 946$.

log 1·0042	0·001820	Const. log	1·826399
log h	1·304188	log N	$\bar{1}$·234165
	1·306008	log 946	2·975891
ar. co....	8·693992	10875	4·036455
log H....	1·477469		
N....	0·171461		

Therefore, the height of Mount Etna is 10875 feet.

Ex. 2.—By the observations of Humboldt on the mountain of Quindiu, the altitudes of the barometer and thermometers, reduced to English measure, were as follows: to find the height of Quindiu.

	Height of Barometer.	Attached Thermometer.	Detached Thermometer.	Answer.
	Inch.	°	°	Feet.
Level of the Pacific Ocean	30·036	77·54	77·54	11469·5
Upper station.........	20·0713	68·	65·75	

Ex. 3.—To find the height of a mountain when the altitudes of the barometer and thermometers are as follows :—

	Barometer.	Attached Thermometer.	Detached Thermometer.	Answer.
	Inch.	°	°	Feet.
Lower station	30	68	70	18943
Upper station.........	15	42·4	36	

Ex. 4.—By observations made at Addiscombe, the altitudes of the barometer and thermometers were as follows :—

	Height of Barometer.	Attached Thermometer.	Detached Thermometer.
Addiscombe House	29·812	63	63
Top of Addington Hills...........	29·530	60	58

400. Prop. VII.—*To find the height of a homogeneous atmosphere.*

The mercury in the tube of a barometer is sustained by the pressure of the air; and when two fluids communicate by means of a glass tube, the altitudes at which they stand are inversely as their densities (art. 364). Let z = the altitude of a homogeneous atmosphere; then, since 13·568 is the specific gravity of pure mercury, and ·001225 of air when the barometer stands at 30 inches or 2·5 feet, we have

$$z : 2·5 \text{ ft.} :: 13·568 : 001225,$$

$$z = \frac{13·568 \times 2·5}{·001225} = 27690 \text{ feet} = 5·24 \text{ miles.}$$

401. *Cor.*—Since the weight of a cubic inch of pure water = 252·818 grains at the temperature of 60°, the pressure of a column of mercury, 30 inches in height, on a square inch $= \dfrac{30 \times 13·568 \times 252·818}{7000} =$ 14·70 avoirdupois pounds. This, therefore, is the mean value of the pressure of the atmosphere on the square inch.

402. Prop. VIII.—*To find the altitude of the mercury in the barometer when a portion of the air has been allowed to remain in the tube.* (See fig. at page 281.)

Suppose that as much air is left in the tube as in the natural state of the

atmosphere would occupy the space BP; and when the tube is immersed in the basin, let the mercury stand at Q; then the air, which before occupied the space BP, now fills the space BQ. And since the elasticity of the air is proportional to its density, and the density is inversely as the space occupied by the same quantity, we have

$$\text{elasticity of air in } BP : \text{elasticity in } BQ :: BQ : BP.$$

Let R be the point at which the mercury would stand if there were no air in BP; also, let $BE = a$, $BP = b$, $ER = h$, $EQ = x$. Then the elasticity of the air, when it occupies the space BP, or the elasticity of the air in its natural state, would support a column of mercury h; and the elasticity of the air, when it occupies the space BQ + mercury in QE, are also balanced by the pressure of the atmosphere, and, therefore, would support a column of mercury h; hence the elasticity of ~~the air in~~ BQ would support a column of mercury $= h - QE = h - x$. Hence, therefore,

$$\text{elasticity of air in } BP : \text{do. in } BQ :: h : h - x.$$

We have, therefore, from the preceding proportion,

$$h : h - x :: BQ : BP :: a - x : b,$$

$$\therefore \quad (h - x)(a - x) = bh;$$

and, from the solution of this quadratic, the value of x may be determined.

403. *Cor.*—Both the roots in this equation (Alg. art. 108) are positive; but it is only the less root which will satisfy the conditions of the problem. The other root being greater than h, is evidently inapplicable in this case.

404. *Scholium.*—The altitude in the barometer varies according to the state of the weather from 28 to 31 inches, and, therefore, this instrument is in very general use as a *weather-glass.* Many rules have been given to determine the changes of the weather from the barometer, and even the words *rain, fair, changeable,* &c., have been engraved at different altitudes of the barometer; but they are not entitled to much attention. The changes of weather appear to be indicated not so much by the absolute height of the mercury as by its variations in height. The following rules have been given by Dr. Dalton, in his work on Meteorology:—

(1). The barometer is highest of all during a long frost, and generally rises with a N.E. wind. It is lowest of all during a thaw following a long frost, and is often brought down by a S.W. wind.

(2). When the barometer is near the high extreme for the season of the year, there is very little probability of immediate rain.

(3). When the barometer is low for the season, there is seldom a great weight of rain, though a fair day in such a case is rare. The general tenor of the weather at such times is short, heavy, and sudden showers, with squalls of wind from the S.W., W., or N.W.

(4). In summer, after a long continuance of fair weather, with the barometer high, it generally falls gradually, and for one, two, or more days before there is much appearance of rain.

If the fall be sudden and great for the season, it will probably be followed by thunder.

(5). When the appearances of the sky are very promising for fair, and the barometer at the same time low, it may be depended upon that the

appearances will not continue long. The face of he sky changes very suddenly on such occasions.

(6). Very dark and dense clouds pass over without rain when the barometer is high; whereas, when the barometer is low, it sometimes rains almost without any appearance of clouds.

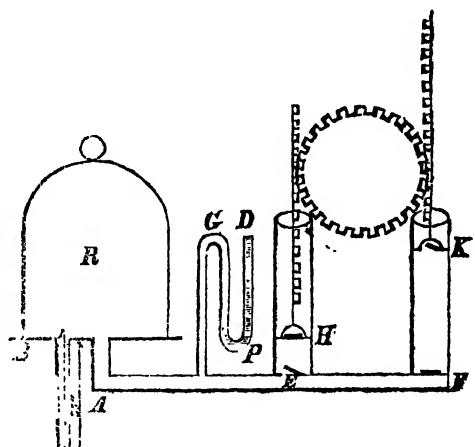
(7). All appearances being the same, the higher the barometer the greater the probability of fair weather.

CHAP. IV.—HYDROSTATIC MACHINES.

THE AIR-PUMP.

405. The *air-pump* is a machine constructed for exhausting the air from a close vessel, which is called a *receiver*. We shall not attempt to give an account of all the various contrivances which are used in constructing this and other machnes, but confine ourselves to explain the general principles on which tihey depend.

A glass receiver R is set on a metal plate, and is made air-tight, by grinding the plate and the edge of the receiver perfectly plain. A pipe communicates with the receiver and with two cylindrical barrels, by means of two valves, E and F, opening upwards. In these barrels are two air-tight pistons, also furnished with valves opening upwards, which are worked up and down by means of a rack-wheel. Suppose the piston H to be at the bottom and K at the top of the barrel; then, as H ascends, a partial vacuum is formed below



the piston, and the elastic force of the air in A and R, pressing upon the valve, opens it and fills the barrel EH. Now, let the wheel be turned back, and the piston H be made to descend: then the valve E is closed by the pressure of the air upon it, and H is opened, and the air in the barrel is forced out. The wheel acts in the same manner upon the piston K, so that a barrel of air is expelled at each turn of the wheel, until the elastic force of the air remaining in the receiver and pipe is insufficient to open the valves E and F, and then all the action of the pump must cease.

406. Prop. I.—*To find the density of the air in a receiver after any number of turns in the wheel.*

Let R be the content of the receiver and pipe, and b the content of

each barrel. Then the air which occupied the space R, when H was at the bottom of the barrel, fills the space $R + b$ when H ascends to the top. If, therefore, ρ was the density before the stroke, and ρ_1 the density afterwards, we shall have

$$R + b : R :: \rho : \rho_1; \quad \therefore \quad \rho_1 = \frac{R\rho}{R + b}.$$

In the same manner it will be found, if ρ_2 be the density after 2 turns, ... ρ_n the density after n turns, that

$$\rho_2 = \frac{R\rho_1}{R + b} = \frac{R^2\rho}{(R + b)^2}; \ldots\ldots\ldots\ldots \rho_n = \frac{R^n\rho}{(R + b)^n}.$$

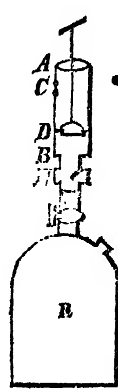
Thus it appears that the density of the air decreases in a geometrical progression; and, therefore, the air in the receiver can never be completely exhausted.

407. *Scholium.*—To ascertain the density of the air left in the receiver, there is a syphon gauge G connected with the pipe. The portion DP is filled with mercury, and it is kept suspended in the tube by the pressure of the air in the receiver. When the elasticity of the air becomes less than the difference of the altitudes of the mercury in this gauge, the mercury will descend in the tube, and if the exhaustion were complete, the mercury would stand at the same height in both tubes. The difference of the altitudes in the two tubes will accurately measure the elastic force of the air which remains in the receiver.

Besides the syphon gauge, there is also a barometer, open at the top B, communicating with the receiver, which is used for the same purpose.

THE CONDENSER.

408. A condenser is an instrument constructed for the purpose of forcing a large quantity of air into a given space.



R is a strong vessel called a receiver. AB is a cylindrical barrel furnished with a valve E opening inwards. D is a solid piston, which is either air-tight or has a valve also opening inwards. C is a small orifice near the top of the cylinder. When the piston D is forced from A to B, the air in the cylinder opens the valve at E, and a barrel of common air is forced into the receiver. Upon raising the piston again, the valve E is closed, and when the piston rises above C the air rushes in and fills the barrel again, which is again forced into the receiver by the descent of the piston; and thus the operation may be continued as far as we please.

409. Prop. II.—*To find the density of the air in the receiver after any number of descents of the piston.*

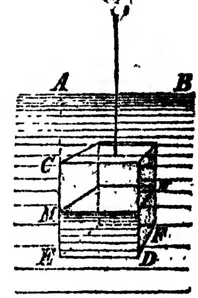
Let R be the content of the receiver, and b the content of the barrel; then, since a barrel of common air is forced into the receiver at every descent, a quantity of air represented by nb will be forced into the receiver after n descents. Let ρ be the density of the atmosphere, ρ_n the density of the air in the receiver after n descents, then

$$\rho_n : \rho :: R + nb : R.$$

There is a *gauge* connected with the receiver similar to that in the air-pump, for the purpose of measuring the density of the air.

THE DIVING BELL.

410. If a vessel be inverted in water, and let down to any depth, the air will occupy the upper part of the vessel, and diminish in bulk as the vessel sinks deeper in the water.



Let AB represent the surface of the water, CEF the diving-bell, EM the height to which the water rises in the bell. Let V be the content of CEF, V' the content of CMN, the space occupied by the air. Let $h = 33$ feet, the weight of a column of water, whose pressure is equal to that of the atmosphere; also, put $AC = a$, and $CM = x$. When the air in the bell occupied the space CEF, it was in its natural state, and its elasticity was measured by the height of a column of water h; but when it occupies the space CN, the pressure of the atmosphere is proportional to h, and the pressure of the water is proportional to the depth $AM = a + x$, therefore the whole pressure upon the air is equal to the weight of a column of water whose height is $h + a + x$. But the elastic force of the air is as its density, or inversely as the space occupied when the quantity remains the same; therefore,

$$h : h + a + x :: \frac{1}{V} : \frac{1}{V'} :: V' : V.$$

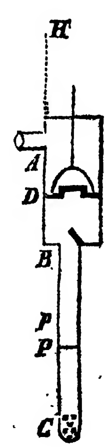
411. *Cor.*—Let the bell be a prismatic chest, whose height is c. Then

$$c : x :: V : V' :: h + a + x : h;$$

$$\therefore \quad x^2 + (a + h)x = ch.$$

And from the solution of this quadratic equation the value of x may be found.

THE COMMON SUCTION PUMP.



412. The common suction pump is thus constructed. AB is a cylindrical barrel, and BC a pipe having its lower end in the water. B is a fixed valve opening upwards. D is an air-tight piston or sucker, moveable by means of a handle fixed to the rod, and having also a valve opening upwards. Suppose the sucker D to descend as low as it can, and each valve to be shut: when D ascends, the air no longer pressing upon the valve B, the valve is opened by the pressure upwards of the air in the pipe, and the air will follow the sucker and fill the empty space DB. Thus the air in the pipe becomes rarefied, and therefore the pressure of the air upon the surface of the water without the pump will be greater than the pressure of the air within it, and, consequently, the water will be forced a little way up the pump until the equilibrium is restored. Again, when D descends, it will condense the air in AB; and when this becomes denser than the air without, the

pressure upwards against the valve D will be greater than the pressure downwards; the valve, therefore, will be opened, and the air in BD will be forced out. When D ascends again, the pressure of the air in BC will again open the valve B, and the air in BC will become further rarefied, and the water again ascends. Thus, at each ascent of D, the water will rise, until at last it reaches the sucker D; it will then force its way through the valve, and at the next rise of the piston will be thrown out at the spout.

413. *Cor.*—In this pump the height of the valve B above the water must be rather less than 32 feet; because, in the rarest state of the atmosphere, the pressure of the air will not raise the water in a vacuum above that altitude.

414. Prop. III.—*To find the height through which the water will rise after any stroke.*

Suppose that, after any number of strokes, the water stands at P, and that after the next stroke it rises to p. Let $CH = 33$ feet $= h$, the height of a column of water equivalent to the pressure of the atmosphere; let $AB = a$, $BP = b$, $HP = c$, and $Pp = x$. Also, let $K =$ the area of a section of the barrel AB, $k =$ area of a section of the suction-pipe BC, and $K = mk$. Suppose the piston to be at B, then the elasticity of the air in BP, together with the weight of a column of water CP, is equal to the pressure of the atmosphere, or is equal to the column of water CH; hence

$$\text{elasticity of the air in } BP = \text{column of water } HP = c,$$

after the next stroke let the water ascend to p, we shall then have

$$\text{elasticity of the air in } Bp = \text{column of water } Hp = c - x.$$

Now, the air which occupied the space BP before the rise of the piston, will, after its ascent, expand itself and occupy the space Ap; therefore,

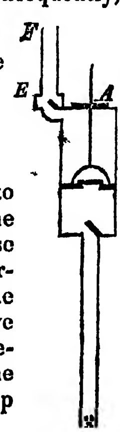
$$\text{density of air in } BP : \text{density of air in } Ap :: \text{space } Ap : \text{space } BP$$
$$:: (b - x)\,k + aK : bk :: b - x + ma : b.$$

And, since the density of air is as its elastic force

$$c : c - x :: b - x + ma : b;$$

we have, therefore, $(c - x)(b - x + ma) = bc$, and, consequently,

$$x^2 - (ma + b + c)\,x + mac = 0.$$



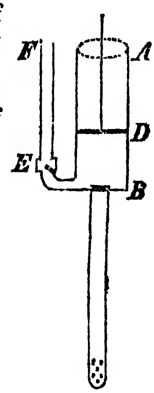
And from the solution of this quadratic equation the value of x may be found.

THE LIFTING PUMP.

415. By the suction pump the water can only be raised to the height of 32 feet, but by means of the lifting pump the water can be raised several hundred feet. In this case the barrel is closed, and the piston-rod works through an air-tight collar at A. And the spout E, through which the water would be discharged in the common pump, has a valve opening upwards into the pipe EF. When the piston descends, the valve at E closes and prevents the return of the water into the barrel; and when the piston ascends, it lifts up

a quantity of water, equal to the content of the barrel, into the pipe *EF* through the valve at *E*.

THE FORCING PUMP.



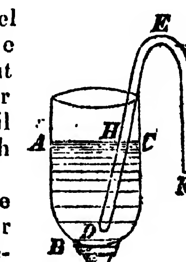
416. *The forcing pump* is another variety, by means of which water may be raised to a greater height than by the common suction pump.

In this case the piston *D* is solid, and the barrel *AB* open to the atmosphere. The water follows the ascent of the piston and fills the barrel, but the piston in its descent forces the water through the valve *E* into the pipe *EF*. The return of the water again, during the ascent of the piston, is prevented by the valve at *E*, which only opens upwards.

The piston of this pump is frequently a solid cylinder, of metal, called a *plunger*, of rather less diameter than the barrel *AB*. This cylinder is of the same dimensions from *D* upwards above *A*; and works through a stuffing box at *A*, which is water-tight. This contrivance has great practical advantages, and dispenses with the necessity of having the barrel and the piston made perfectly true to each other; and even when they are made so, they are continually subject to wear. A view of this force-pump may be seen in Bramah's Hydrostatic Press (art. 420).

417. *Scholium.*—There is an *air-vessel* connected both with lifting and forcing pumps, which is of great advantage, both in keeping a continued supply of water, and also in preventing any injurious strain or jerk, by means of the elasticity of the air condensed in this vessel.

THE SYPHON.



418. If one end of a bent tube be put into a vessel of water, and the other end without be lower than the surface of the water, then, if the air be extracted out of the tube, or the tube be filled with water, the water will flow through the tube in a continued stream, until the surface of the water in the vessel is on a level with the extremity *F*.

For when the air is drawn out of the syphon, the water will rise in it to *E* by the pressure of the air upon the surface of the water *AC*, and then it will descend to *F* by its own gravity. The syphon being full of water, the forces which act upon the water in the tube are the pressure of the atmosphere upon the surface *AC*, and the weight of the column of water *EF*, acting in the direction *DEF*; and the pressure of the air at *F*, and the weight of the column of water *EH*, acting in the opposite direction *FED*. The pressure of the air on *F* and an equal surface of *AC*, may be considered equal to each other, for the difference of the altitudes of *AC* and *F* is too small to produce any appreciable effect on the pressure of the air; these pressures on the tube *DEF* will therefore balance each

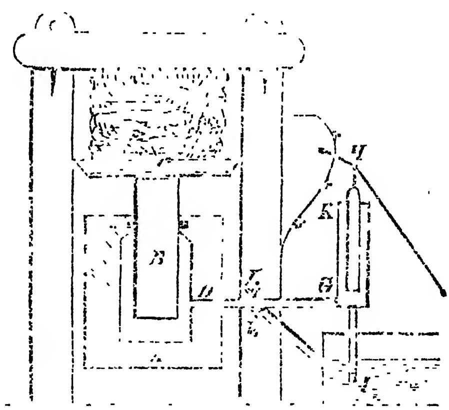
other. But the weight of the column of water EF being greater than the weight of the column of water EH, the sum of the pressures in the direction DEF is greater than the sum of the pressures in the direction FED, the fluid, therefore, will continue to flow in the direction DEF until the surface of the fluid AC is on a level with F.

Cor. 1.—The syphon will not act if the height HE be greater than 33 feet, for then the pressure of the atmosphere on the surface AB will not be sufficient to raise the water to the highest point E.

419. *Scholium.*—The action of intermittent springs, or springs which run and stop alternately, depends on the principle of the syphon. Water is collected from various rills in the cavern AB, and the only outlet by which it can escape is the channel BCD, bent like a syphon. When the cavern is so full that the water stands at the level AC, the water will run out and continue to flow until it is either exhausted, or on a level with D.

THE HYDROSTATIC PRESS.

420. This machine was invented by Bramah, for the purpose of applying enormous forces in various departments of the arts, and is founded upon the principle of the equal transmission of pressure through fluids.



A is a strong cast-iron cylinder, firmly fixed in a vertical position; B a moveable piston, rendered water-tight by a leather collar at a. This piston carries a table C, on which is placed any body that is to be crushed or pressed. DG is a conduit pipe by which the water is conveyed from the injecting pump to A. GK is a plunger pump, made also perfectly water-tight by means of a leather collar at K; and I is the water-cistern below. G is the forcing valve, which opens towards A, L the safety valve, and E the discharge valve.

When the plunger K descends, a quantity of water is forced through the valve at G, which closes after the piston's descent, and prevents the water from returning. The water, therefore, in A continually increases, and causes the piston B to rise. When the required effect has been produced, the discharge valve E is opened, by means of a screw, and the water in A returns to the cistern I.

If P be the force applied at the pump-handle, a the length of the handle, b the distance of the plunger from the fulcrum of the handle,

then the pressure upon the plunger K is $P\dfrac{a}{b}$. Also, since the pressure is transmitted with equal force through the pipe GD, the pressure upon the bottom of the piston B will be proportional to the area of its base. Hence, if R and r be the radii of the bases of these cylinders,

$$\text{pressure upwards on } B \ : \ \text{pressure of plunger} \ :: \ R^2 \ : \ r^2;$$

$$\therefore \quad \text{pressure on the table } C = P\frac{a}{b}\,\frac{R^2}{r^2}.$$

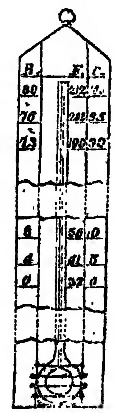
Ex.—If $P = 1$ cwt., $a = 3$ feet, $b = 3$ inches, $R = 5$ inches, $r = \frac{1}{2}$ an inch, then

$$W = 1 \times 12 \times 10^2 = 1200 \text{ cwt.} = 60 \text{ tons}.$$

421. *Scholium.*—There is scarcely any department of the arts where pressure is required, in which this press is not applied. It is used in packing hay and cotton, when large bulks are to be reduced to small compass, to be stowed on shipboard. It is used in extracting oil from seeds, in pressing paper, and in the manufacture of sugar and gunpowder. It is also sometimes applied to extract piles, and to try the strength of iron cables. It is the simplest, and the most easily applicable, of all contrivances for increasing human power; and the only limit to this force, is the want of materials of sufficient strength to resist the enormous strain upon it.

THE THERMOMETER.

422. It does not enter into the plan of this work to explain the properties of *heat* or *caloric*, but the thermometer is an instrument of such general use, and so immediately connected with every department of natural philosophy, that we could not omit explaining the principles on which it is constructed.



423. All bodies (with only one or two exceptions) expand on being heated, and contract again as they become cold. We may, therefore, take the expansion of some known substance as a measure of the variations of heat. Mercury has generally been selected for this purpose, because its expansion is sensible and uniform. It may be exposed to very great heat before it boils, and is not subject to be frozen, unless at a very low temperature.

424. The *thermometer* is a glass tube with a thin bulb at the bottom. The bulb and lower part of the tube are filled with mercury; the air is then driven out from the upper part by boiling the mercury, and the upper extremity is hermetically sealed. To graduate the thermometer, it is immersed in melting ice or snow, and opposite to the point where the mercury stands the number 32 is marked. This is the freezing point. Afterwards the instrument is put in boiling water, and against the point where the mercury stands let the number 212 be marked. Divide the interval into 180 equal parts, or degrees, and also continue equal divisions both above and below these points, and the instrument is properly graduated.

This is called *Fahrenheit's scale*, and is generally used in this country, In the *centigrade* scale, the freezing point is 0, or the *zero* of the scale, and the distance between the freezing and boiling points is divided into 100 degrees. This thermometer is generally used in France. In Reaumur's thermometer the freezing point is also the zero of the scale, but the distance between the freezing and boiling points is divided into 80 degrees. Hence a degree of Fahrenheit's, of the centigrade, and of Reaumur's thermometer, will be inversely as the numbers 180, 100, 80, or inversely as the numbers 9, 5, 4.

425. Mercury freezes at $-40°$ Fahrenheit, and boils at $+640°$. When, therefore, any intense degree of cold is to be measured, a thermometer filled with spirits of wine instead of mercury is used; for pure alcohol has never yet been congealed by the greatest degree of cold to which it has been possible to expose it. When intense heat is to be measured, Daniell's pyrometer is now generally used. It consists of a small rod of platinum $6\frac{1}{2}$ inches long, and $\frac{3}{10}$ of an inch diameter, which is placed in a tube of black lead earthenware, nearly of the same diameter, and an inch deeper. The metallic bar rests at one end against the bottom of this tube, and at its upper extremity there rests upon it a cylindrical piece of porcelain, which is so adjusted as to be pressed forward and *retain its situation* when the bar within has been expanded by heat. The exact amount of this expansion is then measured by means of a scale, when the instrument is cold. A degree of this pyrometer corresponds to seven degrees of Fahrenheit.

PART IV.—HYDRODYNAMICS.

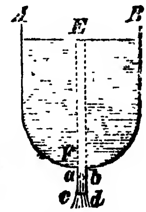
426. Def.—Hydrodynamics is that branch of the science of mechanics which treats of the motion of fluids.

427. Prop. I.—*If a fluid run through any tube kept continually full, and the velocity of the fluid be uniform in every part of the same transverse section; the velocities in the different sections will be inversely as the areas of the section.*

For the same quantity of fluid runs through every section in the same time. But the quantity running through any section is evidently in proportion to the area of the section $\times$ velocity; and as this quantity is constant, the velocity varies inversely as the area of the section.

Cor.—If A be the area of the transverse section, and v the uniform velocity, the quantity of fluid which runs through the section in the time $t = A \times t \times v$.

428. Prop. II.—*The velocity of a fluid issuing from an indefinitely small orifice in the bottom of a vessel kept constantly full, is equal to that which a heavy body would acquire in falling through a space equal to the depth of the orifice below the surface of the fluid.*



Let AB be the surface of the fluid, and F the indefinitely small orifice. Suppose the fluid to be divided into an indefinite number of laminæ, which, during their descent, continue parallel. Then, because the area is indefinitely small compared with the sections of the fluid in the vessel, the velocity of the descending laminæ in the vessel will be indefinitely small. Hence the tendency of the laminæ to descend by the force of gravity will be counteracted, and whatever moving force is lost by the descending fluid will be communicated to the fluid at the orifice. Let Fd be the small column which is discharged in the indefinitely small time t by the pressure of the column EF; and let Fb be the column which would have been discharged by its own weight, or by gravity alone, in the same time t. Also, let V and v be the velocities generated in the columns Fd, Fb, by the moving forces or pressures EF, Fa. Now, the moving forces are proportional to the momenta or quantities of motion generated in a given time (arts. 224, 230); therefore

$$EF : Fa :: \text{column } Fd \times V : \text{column } Fb \times v$$
$$:: Fc \times V : Fa \times v :: \frac{V^2}{2g} : \frac{v^2}{2g};$$

since the spaces described by constant forces in equal times are proportional to the velocities acquired. And because v is the velocity acquired in falling through Fa by the force of gravity, $Fa = \frac{v^2}{2g}$. Hence $EF = \frac{V^2}{2g}$, and, therefore, V is the velocity acquired in falling through EF by the force of gravity (art. 255).

429. *Cor.* 1.—Hence, if z be the depth of the fluid, and $g = 32{\cdot}2$, the velocity with which the water issues $= \sqrt{2gz}$.

430. *Cor.* 2.—The pressure of the air at the orifice F has been supposed equal to its pressure on an equal surface at E. If the fluid is projected into a vacuum, or into a vessel in which the air is partly exhausted, the difference of pressures at the surface and the orifice must be added to the pressure of the water. Thus, if the difference of pressures at the surface and the orifice be equal to that of a column of water whose height is h feet, the velocity of the issuing fluid will be $= \sqrt{2g(h+z)}$.

431. *Cor.* 3.—If the direction of the orifice be turned directly upwards, the velocity with which the fluid issues will carry it to the level of the surface of the fluid.

Examples.

1. To find the velocity with which water issues out of a small orifice at the bottom of a vessel 10 feet high.

2. To find the same when the water issues into a vacuum, its upper surface being exposed to the pressure of the atmosphere.

3. To find the velocity with which water issues into a vessel in which the density of the air is $\frac{1}{2}$ that of the atmosphere; the barometer standing at 30 inches.

4. To find the velocity with which air is impelled by the atmosphere into an exhausted receiver.

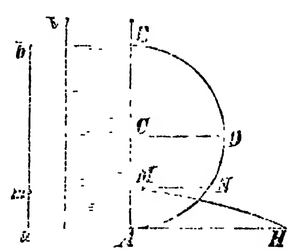
Scholium.

432. The determination of the motion of fluids is a subject involved in the greatest difficulty. M. D'Alembert and D. Bernoulli have attempted, by the aid of analysis, and on particular hypotheses, to find the relation between the velocity of the fluid at the orifice and the altitude of the surface above it; but the general equation which they have obtained can only be integrated in particular cases.

When a vessel, nearly cylindrical, empties itself, it is observed that the particles of the fluid at first descend in vertical lines; but when they have reached within a short distance from the bottom, they incline on every side towards the orifice, forming curves which are convex towards the axis of the vessel. If the thickness of the bottom is inconsiderable, the issuing stream continues to contract externally through a small space in the same converging direction, so that the figure of the fluid without the vessel is a species of conoid, whose altitude is nearly equal to the radius of the orifice. The lower base of this conoid, or the section of the greatest contraction, was called by Sir I. Newton the *vena contracta*, and its area has been found, by careful experiment, to be about $.62k$, or nearly $\frac{5}{8}k$. It is this area which must be substituted for the area of the orifice, either in finding the velocity with which the fluid issues out of the vessel, or the quantity of fluid discharged.

433. Prop. III.—*To find the distance to which fluids will spout, through a very small orifice, from the side of a vessel placed upon a horizontal plane.*

Let VA be a vessel filled with water, and let M be a very small orifice in the side AB, which is perpendicular to the horizontal plane AH. Upon AB describe the semicircle BDA; and draw the ordinate MN. From the last article, the velocity at the vena contracta, which is very near the orifice M, is equal to that which a body would acquire in falling down BM; we must, therefore, consider this as the velocity with which the fluid is projected, and not the velocity at the orifice.



Now, the curve MH, described by the fluid, is a parabola, and MB is a fourth of the parameter belonging to the point M (art. 288); and as the fluid is supposed to spout horizontally, M is evidently the vertex of the parabola. Hence MA is the axis of the parabola, and AH an ordinate; therefore (Par. art. 196), $AH^2 = 4MB \times MA = 4MN^2$, by the property of the circle; and, consequently, $AH = 2MN$.

434. *Cor.*—When the orifice is at C bisecting AB, the distance to which the fluid will spout $= 2CD = AB$, the depth of the fluid. This is manifestly the greatest distance to which the fluid can spout on the horizontal plane AH.

435. Prop. IV.—*To find the time that a cylinder requires to empty itself by a small orifice at the bottom.*

Let the cylinder AV be filled with water to the point B. Let the fluid empty itself by a small orifice at A; and let the surface descend to the point M. Put $AB = a$, $AM = z$; $K =$ the area of the descending surface, and $k =$ the area of a transverse section at the vena contracta. Then the velocity of the fluid at the vena contracta, when the surface of the fluid is at $M = \sqrt{2gz}$. Also, the velocity of the descending surface is to the velocity of the contracta as k to K; therefore the velocity of the surface of the fluid $= \frac{k}{K}\sqrt{2gz}$.

Let $ab = AB$, and suppose a constant accelerating force $f = \frac{k^2}{K^2}g$ to act at the point b, in the direction ab. Then, if a ball be projected from b, in the direction ba, with a velocity $= \sqrt{2f \times ab}$, the body will be continually retarded by the force f, and at the point a the velocity will all be destroyed (art. 253). Also, if $am = z$, the velocity of the ball at the point m will be $\sqrt{2fz}$, which is equal to the velocity of the descending fluid at M. Hence, since the velocity of the ball, and that of the descending fluid, are always equal to one another, and they commence their motion together at b and B, it is evident that the times of describing any equal spaces bm, BM will be equal; and also the whole times of describing the spaces ab, AB. But

$$t \,.\, bm = t \,.\, ab - t \,.\, am = \sqrt{2f \times ab} - \sqrt{2f \times am}$$

$$\therefore \quad t \,.\, BM = \sqrt{2f}\,(\sqrt{a} - \sqrt{z}).$$

436. *Cor.*—Hence a *clepsydra*, or water-clock, may easily be constructed. Suppose the cylinder to be filled to such a height with water that it shall just be discharged in 12 hours; that is, let the height of he water

$$a = \tfrac{1}{2}ft^2 = \tfrac{1}{2}g\,\frac{k^2}{K^2}(12 \times 60 \times 60)^2;$$

then, if the distances measured from A be proportional to

1, 4, 9, 16, ... 144,

these will represent the heights of the water in the cylinder at

1, 2, 3, 4, ... 12

hours from the end of the time.

CONDUIT PIPES AND OPEN CANALS.

437. Prop. V.—*When the water from a reservoir is conveyed in long horizontal pipes of the same aperture, the discharges made in equal times are inversely proportional to the square roots of the lengths.*

This proposition is derived from several experiments made by Bossut

on the actual discharges of water-pipes of different lengths, as far as 2340 toises, or 14930 English feet in length; and he found that this rule was nearly true when the lengths of the pipes were not very unequal.

438. Prop. VI.—*When water runs in rivers or in open canals, it is required to determine the mean velocity with which it runs.*

This proposition also can be determined only from actual experiment. When water runs in an open canal, it is accelerated in consequence of its depth and of the declivity on which it runs, until the resistance increasing with the velocity becomes equal to the acceleration, and then the motion of the stream is uniform. But, as we have no means of determining the amount of the resisting forces from principles already established, it becomes necessary to have recourse to experiment. With this view, M. Du Buat made numerous experiments, and he has given a formula for computing the velocity of running water either in close pipes, open canals, or rivers, which appears to agree remarkably well with experience.

Let V represent the mean velocity of the stream in inches per second, r the quotient obtained from dividing the transverse section of the stream in square inches by its perimeter (omitting the breadth at the surface) in linear inches; this is called by M. Du Buat the mean radius; s the length of the slope of the pipe, or of the surface of the current, whose height is unity; this is manifestly equal to the cosecant of the angle of the slope. Then

$$V = (\sqrt{r} - 0{\cdot}1)\left(\frac{307}{\sqrt{s} - \frac{1}{2}\,\text{Nap.}\log\,(s + 1{\cdot}6)} - 0{\cdot}3\right).$$

439. M. Du Buat has also proved that the greatest velocity is at the surface in the middle of the stream; from which point it diminishes toward the bottom and the sides, where it is least. If the greatest velocity be called v, then the

$$\text{velocity at the bottom} = (\sqrt{v} - 1)^2 = v - 2\sqrt{v} \times 1,$$

$$\text{mean velocity } V = \tfrac{1}{2}[v + (\sqrt{v} - 1)^2] = v - \sqrt{v} + \tfrac{1}{2}.$$

For an account of these experiments, we must refer the student to his *Principes d'Hydraulique*, and to Dr. Robison's Mechanical Philosophy.

Ex.—According to Sir A. Burnes, the velocity at the surface of the river Indus in the middle of the stream, at Tatta, is $2\frac{1}{2}$ miles per hour; the breadth of the river is 670 yards, and its mean depth 15 feet: required the quantity of water discharged per second.

Ans. 67892 cubic feet.

PERCUSSION AND RESISTANCE OF FLUIDS.

440. Prop. VII.—*The force with which a stream impels a plane, when it strikes the plane perpendicularly, is as* $A\rho v^2$; *where* A *is the area of the plane,* ρ *the density of the fluid, and* v *its velocity.*

For the impulsive force of the stream must be proportional to the number of particles which strike against the plane in a given time, multiplied into the force of each. But the number of particles which strike against the plane in a given time is manifestly as $A \times \rho \times v$; also, the

force of each particle is as v; therefore, the force of all the particles against the plane is proportional to $A\varrho v^2$.

We have here supposed that, after the particles strike the plane, the action of these particles immediately ceases; whereas in reality they diverge, and acting upon those which are behind, affect their velocity. Hence, therefore, a difference will arise between theory and experiment.

441. *Cor.* 1.—If f be the impulsive force of the stream against the plane, and k a constant coefficient to be determined by experiment, we have $f = kA\varrho v^2$.

442. *Cor.* 2.—If h be the height due to the velocity v, so that $v^2 = 2gh$, then $f = 2kA\varrho gh$.

443. Prop. VIII.—*If a stream impinges perpendicularly on a plane which is itself in motion, the impulsive force is as* $A\varrho(v - u)^2$; *where* u *is the velocity of the plane.*

For it is evident that both the number of particles which strike the plane, and the force of each particle (art. 242), must be proportional to the *relative* velocity, or to the difference of their absolute velocites; and, therefore, the impulsive force will be as $A\varrho\,(v - u)^2$.

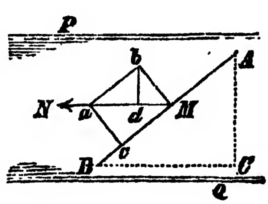
444. *Cor.* 1.—If u be opposed to v, then the force is $= kA\varrho(v+u)^2$. If $v = 0$, the force $= kA\varrho u^2$. The plane, therefore, moving against a fluid at rest, suffers the same impulse as if the fluid were to move with the same velocity, and the plane to remain at rest.

445. *Cor.* 2.—Hence the resistance of a fluid to a body in motion is the same with the percussion of a fluid moving with the same velocity against the body at rest.

446. *Scholium.*—Although this reasoning may be very satisfactory, it appears, from experiment, that the resistance is less than the percussion in the proportion of 5 to 6. This arises, no doubt, from the action of the fluid on the posterior part of the body moving through it, by which the resistance is in some degree counteracted.

We have, in the previous propositions, supposed that the resistance arises only from the inertia of the particles. There is, however, a resistance arising from the cohesion of these particles which must be proportional to the quantity of that cohesion overcome in a given time, that is, to the velocity simply. This resistance is only perceived in slow motions, and is small compared to the other resistance, except in fluids of much viscidity.

447. Prop. IX.—*If a plane m obliquely in a resisting medium, with a uniform velocity, the effective sistance to the plane in the direction of the motion is as* $\sin^3\theta$; θ *being the inclination of the plane to the direction in which it moves.*



Let AB be the plane moving in the medium PQ, in the direction MN; let Ma represent the force with which the plane strikes against any particle of the fluid; resolve this into the two forces Mb, Mc; the one perpendicular and the other parallel to the plane. Then the force Mc, acting in the direction of the plane, will have no effect in

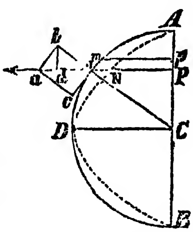
retarding its motion; but the force Mb will be entirely effective, in a direction perpendicular to AB. Again, resolve the effective force Mb into the two forces Md, db; the part Md only is effective in the direction of the plane's motion. The force db will tend to turn the plane about an axis, unless it be counteracted by some other force. Now,

$$Mb = Ma \cos aMb = Ma \sin \theta,$$
$$Md = Mb \cos aMb = Ma \sin^2 \theta;$$

therefore, the effective force of each particle $= Ma \sin^2 \theta$. But the number of particles which oppose the plane's motion is evidently proportional to AC or to $AB \sin \theta$. Hence the whole force of resistance against the plane being as the number of particles multiplied by the force of each, is proportional to $AB \sin \theta \times Ma \sin^2 \theta$, or is proportional to $\sin^3 \theta$, since AB and Ma are supposed to be constant.

448. *Cor.*—If the letters A, ϱ, v, k denote the same quantities as in art. 440, the effective resistance in the direction of the plane's motion $= Ak\varrho v^2 \sin^3 \theta$.

449. Prop. X.—*The resistance on a semicircle* ADB, *moving in its own plane, in a direction perpendicular to the diameter* AB, *is two-thirds of the resistance on* AB.



Let Mm be an indefinitely small arc of the semi-circumference, which may ultimately be considered as an inclined plane. Let Ma represent the force with which Mm strikes against any particle at M. The force Ma may be resolved, as in the last proposition, into the two forces Mb, Mc, of which Mb only is effective in a direction perpendicular to Mm. Again, the force Mb may be resolved into the two forces Md, db; of which db, being perpendicular to the direction of the plane's motion, will have no effect in resisting it; and it will be counteracted by an opposite force in the arc DB. Hence Md is the only effective force in the direction of the plane's motion. Now,

$$Ma : Md :: Ma^2 : Ma \times Md :: Ma^2 : Mb^2$$
$$:: CM^2 : PM^2 :: AC \,.\, CB : AP \,.\, PB;$$

and if $ANDB$ be a parabola, described with the axis DC, vertex D, and double ordinate AB,

$$AC \,.\, CB : AP \,.\, PB :: CD : PN.$$

Hence, whole force of a particle at M : effective force $:: CD : PN$.

Also, since the number of particles which impinge upon Mm is proportional to Pp, we have

resistance on Pp : effective resistance on Mm $:: CD \times Pp : PN \times Pp$;

$\therefore$ whole resistance on AB : effective resistance on ADB

$::$ sum of all $(CD \,.\, Pp)$: sum of all $(PN \,.\, Pp)$.

But the sum of all the rectangles $PN \times Pp$, is equal to the parabolic area $ANDB$ (Con. Sec. art. 220), and the sum of all the rectangles $CD \times Pp$, is evidently the rectangle $CD \times AB$. And since the

area of the parabola $ANDB$ is $\frac{2}{3}$ of the circumscribing parallelogram $CD \times AB$, the effective resistance of the fluid on the semicircle $AMDB$ is equal to $\frac{2}{3}$ of the resistance on the diameter AB.

450. *Cor.*—If the semicircle be supposed to revolve round CD as an axis, it will generate a hemisphere, the rectangle $AB \times CD$ will generate a cylinder, and the parabola $ANDB$ a paraboloid. And as the content of the paraboloid is half that of the circumscribing cylinder, the resistance on the hemisphere is half the resistance on the cylinder, or half that on the great circle AB.

MISCELLANEOUS QUESTIONS IN HYDROSTATICS.

1. Required the pressure of sea-water on the cork in an empty bottle, supposing that its diameter is $\frac{4}{8}$ of an inch, and that it is sunk to the depth of 600 feet. Ans. 134 lbs.

2. A sluice-gate 10 feet square is placed vertically in water, the upper side coinciding with the surface of the fluid; required the pressures on the upper and lower halves of the gate. Ans. 7812·5 and 23437·5 lbs.

3. Required the pressures on the two triangles formed by drawing one of the diagonals. Ans. 10417 and 20833 lbs.

4. A given rectangle is immersed vertically in a fluid, with one side coinciding with the surface; to draw from one of the angles to the base a straight line, so that the pressures on the two parts into which the rectangle is divided may be as $m : n$.

5. The concave surface of a cylinder filled with fluid is divided into 3 annuli, in such a manner that the pressure on each annulus is equal to the pressure on the base. Given the radius of the cylinder to find its height, and also the breadth of each annulus.

6. To find the thickness of an upright wall, necessary to support a body of water; the water being 10 feet deep, and the wall 12 feet high; also, the specific gravity of the wall to that of the water being as 11 to 7. Ans. 4·204 feet.

7. The weight of a body in air is 100 grains, and its weight in water is 800 grains; find its *true* weight, or its weight in a vacuum.

8. A cubical iceberg is 100 feet above the level of the sea, its sides being vertical. Required its dimensions.

9. How deep will a globe of oak sink in water, the diameter being 1 foot? Ans. 10·089 inches.

10. Required the greatest weight which a small cylindrical pontoon, with conical ends, will carry; the length of the cylinder being 12 feet and its diameter 1 ft. 7 in.; also, the height of each cone being 1 ft. 7 in., and the weight of the pontoon 112lbs. Ans. 1494·6 lbs.

11. A piece of lead is dropped into the sea, and, after 5 seconds, it is observed to strike the bottom; required the depth of the water.

12. With what velocity will water issue out of a small orifice 20 feet below the surface of the fluid into another vessel in which $\frac{1}{3}$ of the air has been exhausted? Ans. $44\frac{1}{3}$ feet.

13. In a large vessel 10 feet high, kept constantly full of water, 9 small circular holes, each $\frac{1}{8}$ of an inch diameter, are made at every foot

of the depth; to determine the several distances to which they will spout on the horizontal plane of the base, and the quantity of water discharged by them all in 10 minutes, taking into consideration the vena contracta. Ans. Water discharged = 78·35 imp. gallons.

14. If a spherical balloon of copper of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with hydrogen gas, whose specific gravity is $\frac{1}{10}$ of that of common air; how high will it rise in the atmosphere, the weight of the car, &c., below it, being 500 lbs.? Ans. 14123 feet.

15. What must be the form of a clepsydra, that the water may descend through equal depths in equal portions of time?

16. If a diving-bell of the form of a cone be let down into the sea to the depths of 10 and 20 fathoms; to find the heights to which the water will rise within it, its axis and the diameter of its base being each 10 feet, and the barometer standing at 30 inches. Ans. 2·844 and 3·947 feet.

17. Supposing equal lengths of two fluids, whose specific gravities are as 2 : 1, to be poured into a circular tube; prove that, if x be the length of the tube from the surface of the heavier fluid to the lowest point in the tube, and a the length of each portion of fluid, then

$$\tan x = \frac{2 - \cos a - \cos 2a}{\sin a + \sin 2a}.$$

18. Hiero, king of Syracuse, having employed a goldsmith to make him a crown containing 63 ounces of gold, suspected that silver had been mixed with the gold; and Archimedes being appointed to examine it, he found that the crown raised the water in a vessel 8·2246 cubic inches; and that an inch of gold weighed 10·36 ounces, and an inch of silver 5·85 ounces. It is required to find how much gold had been changed for silver. Ans. 28·8 ounces.

19. How long will it take a cork to rise from the bottom of the sea, 100 feet deep, to the surface? Ans. 1·37 seconds.

20. A hollow cone, whose vertical angle is 2α, is filled with water, and is placed with its base downwards; to determine the distance from the vertex where a small orifice must be made in its side so that the issuing fluid may just strike the bottom of the cone.

Table of Specific Gravities.

Thermometer 60° Fahrenheit. Barometer 30 inches.

A cubic inch of water, at the temperature of 60°, weighs 252·508 grains in air, and 252·818 grains in a vacuum. A cubic inch of air weighs ·3101 grains.

Metals.			
		Mercury	13·568
		Lead, cast	11·352
Platinum, purified	19·500	Pure silver, cast	10·474
——, hammered	20·337	————, hammered	10·511
—— wire	21·042	Bismuth, cast	9·823
Pure gold, cast	19·258	Copper, cast	8·788
————, hammered	19·362	Cobalt, cast	7·812
Gold, 22 carats fine of the standard of London, cast	17·486	Nickel, cast	7·807
		Iron, cast	7·207

Bar iron	7·788
Steel, hard	7·816
——, soft	7·833
Tin, cast	7·291
Zinc, cast	7·191
Antimony, cast	6·702
Arsenic, cast	5·763

Mineral Productions.

Ponderous spar	4·430
Jargon of Ceylon	4·416
Oriental ruby	4·283
Oriental sapphire	3·994
Oriental topaz	4·011
Oriental beryl	3·549
Diamond	3·501 to 3·531
Fluor, red	3·191
White Parian marble	2·838
Green marble	2·742
White marble of Carrara	2·724
Crysolith	2·782
Peruvian emerald	2·775
Red porphyry	2·765
Jasper	2·661 to 2·761
Red Egyptian granite	2·654
Pure rock crystal	2·653
Amorphous quartz	2·647
Agate onyx	2·637
Purbeck stone	2·601
White flint	2·594
Oriental agate	2·590
Portland stone	2·580
Plumbago	1·860
Newcastle coal	1·270
Staffordshire coal	1·240
Pumice stone	·914

Flint glass	3·329
White glass	2·892
Green glass	2·642
Alabaster	2·000
Brick	2 000
Gunpowder, about	·937
Ice, probably	·930

Woods.

Lignum vitæ	1·333
Box, Dutch	1·328
——, French	·912
Heart of oak (60 years old)	1·170
Dry oak	·932
Mahogany	1·063
Beech	·852
Fir, white	·569
Poplar	·383
Cork	·240

Liquids.

Sulphuric acid	1·841
Nitric acid	1·217
Water from the Dead Sea	1·240
Human blood	1·053
Cow's milk	1·032
Malmsey Madeira	1·038
Cider	1·018
Sea water	1·026
Water at 60°	1·000
Bourdeaux wine	·994
Burgundy wine	·991
Olive oil	·915
Pure alcohol	·792
Muriatic ether	·730
Naphtha	·708

Gases.

Atmospheric air	1·000
Compared with water	·001225
Oxygen	1·111
Chlorine	2·500
Hydrogen	0·069
Nitrous oxide	1·524
Carbonic acid	1·527
Coal gas	·450 to ·650

The preceding Table of Specific Gravities has been principally extracted from Dr. Young's Natural Philosophy.

Note.—Since a cubic foot of pure water, at the temperature of 60°, weighs 998·559, or nearly 1000 ounces avoirdupois, in a vacuum; if the decimal point be removed from the numbers in this table, the number opposite to each substance will nearly represent the weight of a cubic foot in ounces.

THE

DIFFERENTIAL AND INTEGRAL CALCULUS.

PART I.—THE DIFFERENTIAL CALCULUS.

CHAP. I.—DEFINITIONS AND PRINCIPLES.

1. In *Algebra* all quantities are supposed to be fixed and determinate in magnitude; but, in the *Differential* and *Integral Calculus*, quantity is conceived to pass through different states of magnitude, and it is the object of this branch of analysis to investigate the corresponding changes which take place in quantities dependent on each other.

2. *Constant* quantities are those which have always the same magnitude, such as the radius of a circle, the parameter of a parabola, &c.; and variable quantities are those which may be supposed to change their value, such as the sine and tangent of a circular arc, the abscissa and ordinate of a curve, &c.

Constant quantities are usually denoted by the first letters of the alphabet, a, b, c, e, f, &c.; and variable quantities by the last letters, x, y, z, v, u, &c.

3. Any expression of calculation containing constant and variable quantities is called a *function* of the variable quantities; thus, if

$$y = ax^2 + b, \quad y = a^x, \quad y = \log(ax + c),$$

y is said to be a function of x. When the variable quantities are not separated from each other, as in the equation $ax^2 + bxy + cy^2 = 0$, since the value of y depends on that of x, y is still said to be a function of x; and the quantity x may also be considered a function of y.

4. If a variable quantity be supposed to change its value, a corresponding change will take place in the value of any function of that quantity. We will examine the nature of this change that takes place in the different functions which have been considered in the previous parts of this work.

5. Let us first suppose that x denoting any variable quantity, u is equal to any power of that quantity, such as x^4. Then x being supposed to be increased by an indefinite quantity h, and to become $x + h$, the function x^4 will become $(x + h)^4$, or

$$x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4.$$

If we compare the new value of this function with its former value, we perceive that the first term x^4 is the original value of the function, and, therefore, the remaining part

$$4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

is the increment which the function has received in consequence of the change in the value of the variable quantity x. It also appears that the first term of the increment is the indefinite quantity h, multiplied by $4x^3$, a function of the variable quantity x; the second term of the increment is h^2, multiplied by the function $6x^2$; and so on.

6. Suppose, now, that u represents any power whatever x^n of the variable quantity x. Then x becoming $x + h$, x^n will become $(x + h)^n$, and (Alg. art. 388) this binomial expanded into a series is

$$x^n + nx^{n-1}h + \frac{n(n-1)}{1 \,.\, 2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}x^{n-3}h^3 + \&c.,$$

where it appears that the first term of the series is the original function x^n, and the following terms are the first, second, and higher powers of h, each multiplied by a new function of x that is independent of h. Let us denote the functions nx^{n-1}, $\frac{n(n-1)}{1 \,.\, 2}x^{n-2}$, &c. by p, q, &c., respectively. Then, according to this notation, when x becomes $x + h$, x^n changes its value and becomes

$$x^n + ph + qh^2 + rh^3 + \&c.$$

7. We will now take a function of the form

$$Ax^\alpha + Bx^\beta + Cx^\gamma + \&c.,$$

which is called a *rational* and *integral* function of x; A, B, C, &c., α, β, γ, &c. being supposed to denote constant quantities. And, to be more distinct, we will suppose the function to consist only of two of these terms $Ax^\alpha + Bx^\beta$. Then, x being supposed to become $x + h$, we have already seen that x^α is changed into the form

$$x^\alpha + ph + qh^2 + rh^3 + \&c.$$

And x^β is also changed into a similar form,

$$x^\beta + p'h + q'h^2 + r'h^3 + \&c.,$$

where p, q, &c., p', q', &c. denote functions of x independent of h. And, therefore, the function $Ax^\alpha + Bx^\beta$ becomes

$$Ax^\alpha + Bx^\beta + (Ap + Bp')h + (Aq + Bq')h^2 + \&c.,$$

and since p, p' are functions of x, $Ap + Bp'$ is a function of x which we may denote by P; and, for the same reason, $Aq + Bq'$, $Ar + Br'$, &c. are functions of x, and may be denoted by Q, R, &c.; thus the expression for the new value of $Ax^\alpha + Bx^\beta$ is

$$Ax^\alpha + Bx^\beta + Ph + Qh^2 + Rh^3 + \&c,$$

in which expression the first part $Ax^\alpha + Bx^\beta$ is the original value of the function, and the other is a series, the terms of which are the successive powers of h, each multiplied by a function of the variable quantity x.

Hence, if u denote any rational and integral function of x, and x be conceived to change its magnitude and become $x + h$; then, if u' de-

note the new value which the function u acquires, in consequence of the change in the value of x, we shall have

$$u' = u + ph + qh^2 + rh^3 + \&c.,$$

where p, q, r, &c. denote certain functions of x independent of the increment h.

8. Let us next suppose the function u to be of the form

$$(Ax^\alpha + Bx^\beta + Cx^\gamma + \&c.)^n,$$

the polynomial within the brackets consisting of any number of terms whatever. Put this polynomial $= v$; we have then $u = v^n$. Now, from the last article, it appears that, when x becomes $x + h$, v becomes

$$v + ph + qh^2 + rh^3 + \&c.,$$

which may be put $=v+Mh$, by substituting M for $p + qh + rh^2 + \&c.$ Hence v^n will become $(v + Mh)^n$; and this expression, when expanded into a series by the binomial theorem, is equal to

$$v^n + nv^{n-1}Mh + \tfrac{1}{2}n(n-1)v^{n-2}M^2h^2 + \&c.$$

If, now, we substitute for M, M^2, &c., their values in the preceding expression, we shall evidently obtain a series of the form

$$v^n + Ph + Qh^2 + Rh^3 + \&c.,$$

where P, Q, &c. denote functions of x independent of h. Hence, if u' be the new value of u, when x becomes $x + h$, we have

$$u' = u + Ph + Qh^2 + Rh^3 + \&c.,$$

a series of the same nature as before.

9. Let us now suppose u equal to the *fractional* function

$$\frac{Ax^\alpha + Bx^\beta + Cx^\gamma + \&c.}{A'x^{\alpha'} + B'x^{\beta'} + C'x^{\gamma'} + \&c.},$$

where A. A', B, B', &c.; also α, α', β, β', &c. denote constant quantities. Put v for the numerator of the fraction and w for its denominator, then this function is $\frac{v}{w}$ or vw^{-1}. Now, when x becomes $x + h$, vw^{-1} becomes (art. 8)

$$(v + ph + qh^2 + \&c.)\ (w^{-1} + p'h + q'h^2 + \&c.),$$

and the product of these two factors, by actual multiplication, is

$$vw^{-1} + (vp' + w^{-1}p)\,h + (vq + pp' + w^{-1}q)\,h^2 + \&c.$$

Now, here, as before, it appears that the coefficients of h, h^2, &c. are functions of x; therefore, denoting these functions by P, Q, R, &c., and observing that vw^{-1} is $\frac{v}{w}$ or u, we have, in putting u' for the value which u acquires when x becomes $x + h$,

$$u' = u + Ph + Qh^2 + Rh^3 + \&c.,$$

a series exactly similar to those which we have found for the other functions of x.

10. We will now consider the *exponential* function $u = a^x$, in which the exponent is the variable quantity x.

When x becomes $x + h$, the function a^x becomes $a^{x+h} = a^x a^h$; and (Alg. art. 395),

$$a^h = 1 + Ah + \frac{A^2}{1 \,.\, 2} h^2 + \frac{A^3}{1 \,.\, 2 \,.\, 3} h^3 + \&c.,$$

where $A = (a - 1) - \frac{1}{2}(a - 1)^2 + \frac{1}{3}(a - 1)^3 - \&c.$ Hence

$$a^{x+h} = a^x + Aa^x h + \frac{A^2 a^x}{1 \,.\, 2} h^2 + \&c.;$$

therefore, putting u' for a^{x+h}, the new value of u, we have

$$u' = u + ph + qh^3 + rh^3 + \&c.,$$

which is a series of the same form as the others.

11. Let us next take the *logarithmic* function $u = \log x$. When x becomes $x + h$, the function $\log x$ will become

$$\log(x + h) = \log\left\{x\left(1 + \frac{h}{x}\right)\right\} = \log x + \log\left(1 + \frac{h}{x}\right)$$

$$= \log x + \frac{1}{A}\left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \&c.\right) \text{ (Alg. art. 400)};$$

so that, putting u' for the value that u acquires when x becomes $x + h$,

$$u' = u + ph + qh^2 + rh^3 + \&c.,$$

a series of the same form as the others.

12. We will now suppose u to be equal to the *circular* function $\sin x$; then x becoming $x + h$, $\sin x$ will become

$$\sin(x + h) = \sin x \cos h + \cos x \sin h; \text{ and (Trig. art. 85)}$$

$$\cos h = 1 - \frac{h^2}{1 \,.\, 2} + \frac{h^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} - \&c., \sin h = h - \frac{h^3}{1 \,.\, 2 \,.\, 3} + \&c.;$$

therefore, substituting these values in the equation above,

$$\sin(x + h) = \sin x + \frac{\cos x}{1} h - \frac{\sin x}{1 \,.\, 2} h^2 - \frac{\cos x}{1 \,.\, 2 \,.\, 3} h^3 + \&c.$$

Hence, if u' be put for $\sin(x + h)$, the new value of u, we obtain

$$u' = u + ph + qh^2 + rh^3 + \&c.,$$

a series, in all respects, analogous to those already found for the other functions of x.

13. Lastly, suppose $u = \cos x$, the

$$u' = \cos(x + h) = \cos x \cos h - \sin x \sin h$$

$$= \cos x - \frac{\sin x}{1} h - \frac{\cos x}{1 \,.\, 2} h^2 + \frac{\sin x}{1 \,.\, 2 \,.\, 3} h^3 + \&c.;$$

$$\therefore\; u' = u + ph + qh^2 + rh^3 + \&c.,$$

a series of the same form as the others.

14. From a due consideration of all that has been said, we may conclude, generally, *That if* u *denote any function of* x, *and* x + h *be substituted for* x *in this function, then* u′, *the new value of* u, *may always be expressed thus,*

$$u' = u + ph + qh^2 + rh^3 + \&c.,$$

where p, q, r, &c. *denote functions of* x *entirely independent of* h.

15. Having examined the general form that any function of a variable quantity x acquires by the substitution of $x + h$ for x, and having found it to be a series, the first term of which is always the function itself, it is evident that the remaining terms, $ph + qh^2 + rh^3 + \&c.$ will express the increment which the function receives in consequence of this substitution. And, since

$$u' - u = ph + qh^2 + rh^3 + \&c.,$$

this increment is called the *difference* of u, being the difference between two succeeding values of u. The first term ph of this difference is called the *differential* of u, and it is denoted by du, where the letter d is not to be considered as a coefficient, but as a symbol of operation. The coefficient p is called the *differential coefficient* of the function u.

16. Since $du = ph$, the first term of the difference of u, we have dx, by the same notation, equal to the first term of the difference of x. But $x' - x = h$, therefore, the difference and the differential of a simple variable x signify the same thing, namely h. Hence we have $dx = h$; substituting, therefore, dx for h in the equation $du = ph$, in order to preserve a uniformity of notation, and also to indicate more distinctly the variable quantity on which the function u depends, we have

$$du = pdx, \quad \text{and} \quad \therefore \quad p = \frac{du}{dx}.$$

17. If we divide the difference of u by the difference of x, we have

$$\frac{u' - u}{h} = p + qh + rh^2 + \&c.;$$

and when h is diminished indefinitely, the ratio of the increments $\frac{u' - u}{h}$ will continually approach towards p as its limit. Many writers, therefore, on this subject, define the differential coefficient to be the *limiting ratio of the increments*, that is, a quantity to which the ratio may approach nearer than by any assignable difference, but to which it can never be considered as becoming absolutely equal.

18. This branch of analysis naturally divides itself into two parts: *Having given* u *any function of* x, *to determine* p *the coefficient of* h *in the second term of the expansion;* and the other is the converse of this: *Having given* p, *the coefficient of* h *in the second term of the expansion, to find the value of* u, *the function from which it was derived.* The first of these is called the *differential calculus*, and the second the *integral calculus.*

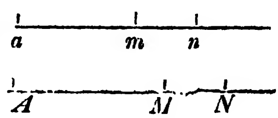
Scholium.

19. In the investigation of many of the properties of the circle and other curve lines, the ancient Geometers were often obliged to have recourse to indirect demonstrations, and to prove their propositions from the consideration of limits. Thus, since a curve may be conceived to be the limit between all inscribed and circumscribed polygons, they inferred

that whatever was proved to be generally true of the po ygons must also be true of the curve. But not satisfied with an inference drawn from these principles, they endeavoured in all cases to show that any supposition to the contrary was necessarily absurd. The demonstration of propositions 87, 89, &c., in the Geometry, will give the student a correct idea of this method of proof.

20. About fifty years before the time of Newton, Cavalieri published his work on *Geometria Indivisibilibus*, &c.; in which he supposed all quantity to be composed of an infinite number of small indivisible elements. Thus, lines were supposed to be made up of an infinite number of points, surfaces of an infinite number of lines, and solids of an infinite number of plane surfaces. We have given an example of this method in the 132nd proposition in the Geometry, and we have shown in the scholium that, however easy and concise this theory may be in practice, it is altogether defective in logical accuracy.

21. Geometry was nearly in this state when Sir I. Newton invented his method of Fluxions.* According to his view, all quantity is considered as generated by motion; a line by the motion of a point; a surface by the motion of a line; and a solid by the motion of a surface. The quantity thus generated he called the *fluent* or flowing quantity; and the velocity with which the flowing quantity increased or decreased at any point of time is called the *fluxion* of the quantity at that instant. Having adopted the theory of motion for the foundation of the doctrine of fluxions, he then proceeds to prove that the ratio of the fluxions is the same as the limiting ratio of the increments generated in the same time. Thus, if we suppose the line am to represent any flowing quantity x; and the line AM to represent u, any function of x; then, if we take $mn = h$, the increment of x, and $MN =$ the corresponding increment of u, it appears, from what we have shown in art. 15, that



$$mn : MN :: h : ph + qh^2 + rh^3 + \&c.$$
$$:: 1 : p + qh + rh^2 + \&c.$$

If, now, we suppose the line am to be described with a uniform velocity v, and the line AM with a variable motion, the line MN will not represent the velocity at M, but will be too great if the motion was accelerated, and too little if the motion was retarded. Suppose the motion to be continually accelerated, and let V be the velocity at M, and $V + w$ the velocity at N, then it is evident that

$$\frac{V}{v} \text{ is } < \frac{MN}{mn}, \quad \text{and} \quad \frac{V+w}{v} \text{ is } > \frac{MN}{mn}.$$

* In speaking of the preceding methods he says, in the scholium to the Lemma in his *Principia*, " Præmisi, vero hæc lemmata, ut effugerem tædium deducend perplexas demonstrationes, more veterum geometrarum, ad absurdum. Contractiore enim redduntur demonstrationes per methodum indivisibilium. Sed quoniam dur est, indivisibilium hypothesis et propterea minus geometrica censetur; malui monstrationes rerum sequentium ad ultimas quantitatum evanescentium summa et rationes, primas que nascentium id est, ad limites summarum et rationum deducere . . . "

Now, if we conceive the corresponding increments *mn* and *MN* to be continually diminished, the augmentation of velocity w from M to N will also be continually diminished, and will at length become less than any assignable quantity, but v and V remain constant. Hence it is evident that the ratio $\frac{MN}{mn}$ being always between the values of $\frac{V}{v}$ and $\frac{V+w}{v}$ must ultimately be equal to $\frac{V}{v}$. But $\frac{MN}{mn} = p + qh + rh^2 + \&c.$, and when h is diminished indefinitely, this is ultimately equal to p. It follows, therefore, that $\frac{V}{v}$ = limit of the ratio $\frac{u'-u}{h} = p$.

This was the method adopted by Sir. I. Newton; and it is clear, from this statement, that the doctrine of fluxions is founded on the theory of limits, and that the theory of motion can only be considered as an illustration of the subject. It is the opinion, therefore, of many mathematicians, that the consideration of motion was introduced into the subject unnecessarily; and several succeeding writers, such as D'Alembert, Euler, &c., have established the theory upon the principles of limiting ratios alone, independently of all ideas of time and velocity, both of which seem foreign to investigations relating to abstract quantity.

22. Nearly about the same time that Newton invented the method of fluxions, Leibnitz published his short tract on the differential calculus, in the first volume of the *Acta Eruditorum*. According to his method, every quantity such as u is composed of an infinite number of infinitely small quantities, which he called *differentials*. The true value of each of these is evidently $ph + qh^2 + rh^3 + \&c.$; but as h is supposed to be infinitely small, qh^2 is infinitely smaller than ph, rh^3 infinitely smaller than qh^2, and so on. Hence $qh^2 + rh^3 + \&c.$, is infinitely smaller than ph, and therefore may be neglected without sensible error. This method of reasoning is now generally considered to be unsatisfactory; and there always remains a latent feeling with the student that it is merely a convenient method of approximation.

23. Since it appears, from what has been stated, that the *ratio of the fluxions*, or the *limiting ratio of the increments*, or the *ratio of the infinitely small differences* of x and u any function of x, is no other than p, the coefficient of h, when $x + h$ is substituted for x in the function u, Lagrange conceived the idea of defining the differential of a quantity to be the first term ph of the entire difference $ph + qh^2 + \&c.$ By this means the whole theory is reduced to the ordinary principles of Algebra, and is independent of every consideration of limits, infinitely small differences, or evanescent quantities.*

24. Before we conclude this scholium, we would say a few words on the algorithm or notation which has been used to denote the process termed differentiation. Sir I. Newton employed different symbols at different times to denote the fluxions of variable quantities; but his

* Il est donc plus naturel et plus simple de considérer immédiatement le développement des fonctions sans employer le circuit métaphysique des infiniment petits ou des limites; et c'est ramener le calcul differentiel à une origine purement algébrique que le faire dépendre uniquement de ce développement.—*Lagrange, Calcul des Fonctions.*

successors generally expressed the fluxion of a quantity by placing a dot over it. Thus, the fluxions of the quantities

$$u, \qquad a^x, \qquad \sin x, \qquad \sqrt{(1+x^2)} \qquad \frac{1}{(1+x^2)^3},$$

are denoted by

$$\dot{u}, \qquad \dot{\overline{a^x}}, \qquad \dot{\overline{\sin x}}, \qquad (\dot{\overline{\sqrt{1+x^2}}}), \qquad \dot{\left(\frac{1}{(1+x^2)^3}\right)},$$

and this was the notation universally adopted in this country, until a very few years ago.

The method adopted by Leibnitz and his followers, was to place the letter d before a function to denote its differential. According to this notation, the differentials of the preceding functions will be denoted thus,

$$du, \qquad d\,.\,a^x, \qquad d\,.\sin x, \qquad d\,.\,\sqrt{(1+x^2)}, \qquad d\,.\frac{1}{(1+x^2)^3}.$$

In simple cases these two methods are perhaps equally clear and distinct, but in complicated calculations the notation of Leibnitz is far superior to the other. Thus, the n^{th} differential $d^n.\,(1+x^2)^m$ is both awkward and indistinct when expressed in the fluxional notation $\Big((1+x^2)^m\Big)^{\overset{(n)}{\cdot}}$. There are also several very elegant and important theorems in the calculus of finite differences and the calculus of variations, which can only be expressed in a notation where the symbol of operation may be removed from the quantity to which it is applied. On these accounts the fluxionary notation is now nearly abandoned in this country, and the notation of Leibnitz adopted in its place.

Besides these two methods of denoting the differential of a function, Lacroix has given six others, which have been adopted by various writers; but as they are very rarely used, it is not necessary to notice them further.

CHAP. II.—DIFFERENTIATION OF FUNCTIONS OF ONE VARIABLE.

25. Def.—The *differential coefficient* of u, any function of x, is the coefficient of h in the second term of the expansion of u, when $x+h$ is substituted for x. And the *differential* of u is the differential coefficient multiplied by the differential of x.

26. Prop. I.—*To find the differential of any power of* x *as* x$^{\text{n}}$, n *being any constant quantity.*

Let us suppose $u = x^n$, and that u becomes u' when $x+h$ is substituted for x, we have then (Alg. art. 387)

$$u' = (x+h)^n = x^n + nx^{n-1}h + qh^2 + rh^3 + \&c.$$

Hence, by the definition of the differential of a quantity (art. 25),

$$\frac{du}{dx} = nx^{n-1} \text{ the coefficient of } h; \quad \therefore \quad du = nx^{n-1}\,dx.$$

To find the differential, therefore, of any power of a variable quantity, we have the following

Rule.—Multiply the differential of the variable quantity itself by the index, and by a power of the quantity whose index is less by unity than the given index, the product will be the differential required.

27. Prop. II.—*To find the differential of* $u = Ax^\alpha + Bx^\beta$.

Substitute $x + h$ for x and the function u becomes

$$\begin{aligned} u' &= A(x+h)^\alpha + B(x+h)^\beta \\ &= (Ax^\alpha + A\alpha x^{\alpha-1}h + qh^2 + \&c.) + (Bx^\beta + B\beta x^{\beta-1}h + q'h^2 + \&c.) \\ &= u + (A\alpha x^{\alpha-1} + B\beta x^{\beta-1})h + (q+q')h^2 + \&c. \end{aligned}$$

Hence, according to the definition (art. 25),

$$\frac{du}{dx} = A\alpha x^{\alpha-1} + B\beta x^{\beta-1}; \quad \therefore \quad du = A\alpha x^{\alpha-1}\,dx + B\beta x^{\beta-1}dx.$$

The first term $A\alpha x^{\alpha-1}\,dx$ is the differential of Ax^α; and the second term $B\beta x^{\beta-1}\,dx$ is the differential of Bx^β.

28. Prop. III.—*To find the differential of* $u = c + v$, *c being a constant quantity and* v *any function of* x *whose differential is known.*

When x becomes $x + h$, suppose v to become v' or $v + ph + qh^2 +$ &c., and u to become u', then

$$u' = c + v' = c + v + ph + qh^2 + \&c.,$$

or,

$$u' = u + ph + qh^2 + \&c.$$

Hence $\frac{du}{dx} = p$. But, according to the definition of a differential, $\frac{dv}{dx} = p$, therefore,

$$\frac{du}{dx} = \frac{dv}{dx}; \quad \text{and} \quad du = d(c+v) = dv,$$

that is, the differential of any variable function is the same as the differential of the same function increased or diminished by any constant quantity.

Cor.—The differential of a constant quantity $= 0$; for, as it undergoes no change, its increment $= 0$, and therefore its differential $= 0$.

29. Prop. IV.—*To find the differential of* u *which is equal to the sum of* v *and* w, *two other functions of* x, *when the differentials of* v *and* w *are given.*

Suppose u', v', w' to be the new values of u, v, w, when x becomes $x + h$; then it appears, from art. 14, that v' and w' are of the form

$$\begin{aligned} v' &= v + ph + qh^2 + rh^3 + \&c. \\ w' &= w + p'h + q'h^2 + r'h^3 + \&c.; \end{aligned}$$

$$\therefore \quad u' = v' + w' = v + w + (p + p')h + (q + q')h^2 + \&c.$$

or, $$u' = u + (p + p')h + (q + q')h^2 + \&c.$$

Hence $\dfrac{du}{dx} = p + p'$. But $p = \dfrac{dv}{dx}$, and $p' = \dfrac{dw}{dx}$ (art. 25);

$$\therefore \quad \frac{du}{dx} = \frac{dv}{dx} + \frac{dw}{dx}; \quad \text{and} \quad du = dv + dw.$$

30. *Cor.*—In the same manner, if $u = av + bw + cy + \&c.$, where a, b, c, &c. are constant quantities, and v, w, y, &c. are functions of x, it may easily be shown, by reasoning as above, that

$$du = adv + bdw + cdy + \&c.$$

31. PROP. V.—*To find the differential of* u = vw, *where* v *and* w *are functions of* x; *supposing the differentials of* v *and* w *to be given.*

Let x change its value and become $x + h$, and let the corresponding values of v and w be $v' = v + ph + qh^2 + \&c.$; $w' = w + p'h + q'h^2 + \&c.$;

$$\begin{aligned} \therefore \quad u' = v'w' &= (v + ph + qh^2 + \&c.)(w + p'h + q'h^2 + \&c.) \\ &= vw + (vp' + wp)h + (vq' + wq + pp')h^2 + \&c. \\ &= u + (vp' + wp)h + Qh^2 + \&c. \end{aligned}$$

Hence $\dfrac{du}{dx} = vp' + wp$. But (art. 25) $p' = \dfrac{dw}{dx}$, and $p = \dfrac{dv}{dx}$,

$$\therefore \quad \frac{du}{dx} = v\frac{dw}{dx} + w\frac{dv}{dx}; \quad \text{and} \quad du = vdw + wdv.$$

Rule.—To find the differential of the product of any two functions, multiply the differential of each function by the other function, and the sum of these products is the differential required.

32. *Cor.*—If we divide the differential equation $du = wdv + vdw$ by the corresponding terms of the equation $u = vw$, we obtain the differential in this form,

$$\frac{du}{u} = \frac{dv}{v} + \frac{dw}{w}.$$

33. PROP. VI.—*To find the differential of the product of three functions of* x, *or of* u = stv.

Put $st = w$, then $u = vw$, and, by the last article, $\dfrac{du}{u} = \dfrac{dv}{v} + \dfrac{dw}{w}$.

But because $w = st$, we have also $\dfrac{dw}{w} = \dfrac{ds}{s} + \dfrac{dt}{t}$. Hence, therefore, we obtain

$$\frac{du}{u} = \frac{ds}{s} + \frac{dt}{t} + \frac{dv}{v}.$$

And, multiplying by the corresponding terms of the equation $u = stv$,

$$du = tvds + svdt + stdv.$$

34. *Cor.*—In like manner, to find the differential of the product of any number of functions s, t, v, w, &c., we obtain, by reasoning in the same manner as in the last article,

$$\frac{du}{u} = \frac{ds}{s} + \frac{dt}{t} + \frac{dv}{v} + \frac{dw}{w} + \ldots.$$

and, if we multiply by the corresponding terms of the equation $u = stvw \ldots$ we shall have this

Rule.—The differential of the product of any number of functions is equal to the sum of the products of the differential of each function multiplied by all the other functions.

35. PROP. VII.—*To find the differential of the fraction* $\frac{\text{v}}{\text{w}}$, *the differentials of* v *and* w *being given.*

Put $u = \frac{v}{w}$, then $v = uw$, and (art. 31) $dv = wdu + udw$; therefore $du = \frac{dv}{w} - \frac{udw}{w^2}$, and substituting $\frac{v}{w}$ for u

$$du = \frac{dv}{w} - \frac{vdw}{w^2} = \frac{wdv - vdw}{w^2}.$$

Hence, to find the differential of a fraction, we have the following

Rule.—Multiply the differential of the numerator by the denominator, and from the product subtract the differential of the denominator multiplied by the numerator; the remainder, divided by the square of the denominator, will be the differential required.

36. PROP. VIII.—*To find the differential of* u *a function of* v, v *itself being a function of the independent variable* x.

Let x change its value and become $x + h$, and let the corresponding value of v be

$$v' = v + ph + qh^2 + \&c.$$

Put $ph + qh^2 + \&c. = k$, and when v becomes $v + k$, suppose u to become

$$u' = u + Pk + Qk^2 + \&c.$$

Substituting for k its value $ph + qh^2 + \&c.$, we obtain

$$u' = u + Pph + (Pq + Qp^2)\,h^2 + \&c.$$

Hence, according to the definition (art. 25), Pp is the differential coefficient of u, considered as a function of x, or $Pp = \frac{du}{dx}$; but P is the differential coefficient of u, considered as a function of v, therefore $P = \frac{du}{dv}$; also $p = \frac{dv}{dx}$: we have, therefore,

$$\frac{du}{dx} = \frac{du}{dv}\frac{dv}{dx}, \quad \text{and} \quad du = \frac{du}{dv}\frac{dv}{dx}dx.$$

Hence, to differentiate u, a function of v, v being a function of x, we have this

Rule.—Multiply the differential coefficient of u, considered as a func- of v, by the differential of v, considered as a function of x, the product will be the differential required.

We will now proceed to apply the preceding rules to some particular examples.

Ex. 1.

Let $u = a + b\sqrt{x} - \frac{c}{x}$. To find the differential of u.

Taking the differential of each term separately (art 29), the differential of $a = 0$, because a is constant (art. 28).

The differential of $b\sqrt{x} = d \cdot bx^{\frac{1}{2}} = \frac{1}{2}bx^{\frac{1}{2}-1}dx = \frac{bdx}{2\sqrt{x}}$ (art. 26).

The differential of $-\frac{c}{x} = d(-cx^{-1}) = cx^{-2}dx = \frac{cdx}{x^2}$.

Hence we have $\quad du = \frac{bdx}{2\sqrt{x}} + \frac{cdx}{x^2}$.

Ex. 2.

Let $\quad u = a + \frac{b}{\sqrt[3]{x^2}} - \frac{c}{x\sqrt[3]{x}} + \frac{e}{x^2}$.

If we put this function into the form

$$u = a + bx^{-\frac{2}{3}} - cx^{-\frac{4}{3}} + ex^{-2},$$

the application of the preceding rules will give us

$$du = -\tfrac{2}{3}bx^{-\frac{5}{3}}dx + \tfrac{4}{3}cx^{-\frac{7}{3}}dx - 2ex^{-3}dx,$$

which becomes, by reduction,

$$du = -\frac{2bdx}{3x\sqrt[3]{x^2}} + \frac{4cdx}{3x^2\sqrt[3]{x}} - \frac{2edx}{x^3}.$$

Ex. 3.

Let $u = (a + bx^m)^n$.

We may bring this form under the preceding rules by putting $a + bx^m = v$, we then have

$$u = v^n \quad \text{and} \quad du = nv^{n-1}dv.$$

But $v = a + bx^m$, therefore $dv = mbx^{m-1}dx$ and $v^{n-1} = (a + bx^m)^{n-1}$; substituting these values in the equation $du = nv^{n-1}dv$, we obtain

$$du = mnb(a + bx^m)^{n-1}x^{m-1}dx.$$

Ex. 4.

Let $\quad u = \sqrt{(a + bx + cx^2)}$.

By proceeding as in the last example, and putting $v = a + bx + cx^2$, we have $u = \sqrt{v} = v^{\frac{1}{2}}$, therefore $du = \frac{1}{2}v^{\frac{1}{2}-1}dv = \frac{dv}{2v^{\frac{1}{2}}} = \frac{dv}{2\sqrt{v}}$.

Again, since $v = a + bx + cx^2$, we find $dv = bdx + 2cxdx$; and substituting for dv and $\sqrt{v}$ in the value of du, we obtain

$$du = \frac{bdx + 2cxdx}{2\sqrt{(a + bx + cx^2)}}.$$

Ex. 5.

Let $u = x(a^2 + x^2)\sqrt{a^2 - x^2}$.

Here u is the product of three factors, x, $a^2 + x^2$, and $\sqrt{a^2 - x^2}$, and therefore its differential will be found by art. 34. Now,

$$d(a^2 + x^2) = d \cdot x^2 = 2xdx;$$

$$\text{also,} \quad d\sqrt{a^2 - x^2} = \frac{d(a^2 - x^2)}{2\sqrt{a^2 - x^2}} = \frac{-xdx}{\sqrt{a^2 - x^2}}.$$

Hence, multiplying the differential of each factor by the product of the other two factors, and taking the sum of all these products, we obtain

$$du = (a^2 + x^2)\sqrt{a^2 - x^2}\,dx + 2x^2\sqrt{a^2 - x^2}\,dx - \frac{x^2(a^2 + x^2)\,dx}{\sqrt{a^2 - x^2}},$$

or, reducing all the terms to a common denominator,

$$du = \frac{(a^4 + a^2x^2 - 4x^4)dx}{\sqrt{a^2 - x^2}}.$$

Ex. 6.

Let $u = \dfrac{\sqrt[3]{(a^2 + x^2)}}{\sqrt{(a - x)}}$.

Put $v = \sqrt[3]{(a^2 + x^2)} = (a^2 + x^2)^{\frac{1}{3}}$,

and $w = \sqrt{(a - x)} = (a - x)^{\frac{1}{2}}$, we have then

$$u = \frac{v}{w} \quad \text{and } du = \frac{wdv - vdw}{w^2}.$$

But $dv = d \cdot (a^2 + x^2)^{\frac{1}{3}} = \frac{1}{3} \times 2xdx \times (a^2 + x^2)^{-\frac{2}{3}} = \dfrac{2xdx}{3(a^2 + x^2)^{\frac{2}{3}}}$.

Also, $dw = d \cdot (a - x)^{\frac{1}{2}} = \dfrac{-dx}{2(a - x)^{\frac{1}{2}}}$;

$$\therefore \quad wdv - vdw = \frac{2xdx\,(a - x)^{\frac{1}{2}}}{3(a^2 + x^2)^{\frac{2}{3}}} + \frac{dx\,(a^2 + x^2)^{\frac{1}{3}}}{2(a - x)^{\frac{1}{2}}}$$

$$= \frac{4xdx\,(a - x) + 3dx\,(a^2 + x^2)}{6\,(a - x)^{\frac{1}{2}}\,(a^2 + x^2)^{\frac{2}{3}}}$$

$$= \frac{(3a^2 + 4ax - x^2)\,dx}{6(a - x)^{\frac{1}{2}}\,(a^2 + x^2)^{\frac{2}{3}}};$$

$$\therefore \quad du = \frac{(3a^2 + 4ax - x^2)\,dx}{6(a - x)^{\frac{3}{2}}\,(a^2 + x^2)^{\frac{2}{3}}}.$$

Examples for Practice.

1. Let $u = a + bx + cx^4$, to find the differential of u.
2. Let $u = a + b\sqrt{x} + c\sqrt[3]{x}$.

3. Let $u = x^3(a + x)^2$.

4. Let $u = x(1 + x)(1 + x^2)$.

5. Let $u = \dfrac{a}{(a - x)^3}$.

6. Let $u = \dfrac{x}{1+x}$. Ans. $du = \dfrac{dx}{(1 + x)^2}$.

7. Let $u = \dfrac{1 + x^2}{1 - x^2}$. Ans. $du = \dfrac{4x dx}{(1 - x^2)^2}$.

8. Let $u = \dfrac{x^n}{(1 + x)^n}$. Ans. $du = \dfrac{nx^{n-1}\,dx}{(1 + x)^{n+1}}$.

9. Let $u = \dfrac{(1 + x)^3}{(1 - x)^4}$. Ans. $du = \dfrac{(1 + x)^2}{(1 - x)^5}(7 + x)\,dx$.

10. Let $u = \dfrac{x^2 - x + 1}{x^2 + x + 1}$. Ans. $du = \dfrac{2dx(x^2 - 1)}{(x^2 + x + 1)^2}$.

11. Let $u = \sqrt{a + x}$. Ans. $du = \dfrac{dx}{2\sqrt{a+x}}$.

12. Let $u = (1 + x)\sqrt{1 - x}$. Ans. $du = \dfrac{(1 - 3x)\,dx}{2\sqrt{1 - x}}$.

13. Let $u = \sqrt{(1 - x)}\,\sqrt{(1 + x^2)}$.

Ans. $du = \dfrac{-dx\,(3x^2 - 2x + 1)}{2\sqrt{(1 - x)}\,\sqrt{(1 + x^2)}}$.

14. Let $u = \dfrac{x}{\sqrt{(1 - x^2)}}$. Ans. $du = \dfrac{dx}{(1 - x^2)^{\frac{3}{2}}}$

15. Let $u = \dfrac{x^3}{(1 - x^2)^{\frac{3}{2}}}$. Ans. $du = \dfrac{3x^2dx}{(1 - x^2)^{\frac{5}{2}}}$.

16. Let $u = \dfrac{\sqrt{(1 + x)}}{\sqrt{(1 - x)}}$. Ans. $du = \dfrac{dx}{(1 - x)\sqrt{(1 - x^2)}}$.

17. Let $u = \dfrac{x^2}{\sqrt{(a^4 + x^4)}}$. Ans. $du = \dfrac{2a^4x dx}{(a^4 + x^4)^{\frac{3}{2}}}$.

18. Let $u = \dfrac{x}{x + \sqrt{(1 + x^2)}}$. Ans. $du = \dfrac{dx(1+2x^2)}{\sqrt{(1+x^2)}} - 2x dx$.

19. Let $u = \dfrac{\sqrt{(1 + x^2)} - x}{\sqrt{(1 + x^2)} + x}$.

Ans. $du = \dfrac{-2dx}{\sqrt{1+x^2}\,(\sqrt{1 - x^2} + x)^2}$.

20. Let $u = \dfrac{\sqrt{(1+x)} + \sqrt{(1-x)}}{\sqrt{(1+x)} - \sqrt{(1-x)}}$. Ans. $du = \dfrac{-dx}{x^2\sqrt{(1-x^2)}} - \dfrac{dx}{x^2}$.

DIFFERENTIATION OF LOGARITHMIC FUNCTIONS.

37. Prop. IX.—*To find the differential of the function* $u = a^x$, *a being considered constant and* x *variable.*

Let x become $x + h$, and let u' be the new value that u acquires by the change in the magnitude of x. We have then

$$u' - u = a^{x+h} - a^x = a^x a^h - a^x = a^x (a^h - 1).$$

Now, if we substitute $1 + b$ for a in a^h, and expand this by the binomial theorem, we obtain

$$a^h = (1 + b)^h = 1 + hb + h\frac{h-1}{2}b^2 + h\frac{h-1}{2}\frac{h-2}{3}b^3 + \&c.$$

$$= 1 + hb + \left(\frac{h^2}{2} - \frac{h}{2}\right)b^2 + \left(\frac{h^3}{6} - \frac{h^2}{2} + \frac{h}{3}\right)b^3 + \&c.$$

And, arranging this series by powers of h, we shall have a series of the form

$$a^h = 1 + Ah + Bh^2 + Ch^3 + \&c.;$$

where $A = b - \frac{1}{2}b^2 + \frac{1}{3}b^3 - \frac{1}{4}b^4 + \&c.$

$$= (a - 1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \&c.,$$

and B, C, &c. are also quantities composed of the powers of b, and consequently are independent of h. Hence

$$u' = a^x a^h = a^x(1 + Ah + Bh^2 + \&c.)$$
$$= a^x + Aa^x h + Ba^x h^2 + \&c.$$
$$= u + Aa^x h + qh^2 + rh^3 + \&c.$$

Hence $\dfrac{du}{dx} = Aa^x$, and $du = Aa^x dx$.

It appears, from Algebra, art. 399, that $A =$ Nap. log a: hence, therefore, we have the following

Rule.—The differential of the exponential quantity a^x is equal to the continued product of the given exponential, the differential of its exponent and the Naperian logarithm of the root.

38. *Cor.*—If $a = e$, the base of the Naperian system of logarithms, then $A = 1$; therefore $d\,.\,e^x = e^x dx$

39. Prop. X.—*To find the differential of* $u = \log x$.

Let a be the base of the system in which u is equal to $\log x$; then, from the nature of logarithms, we have $x = a^u$ (Alg. art. 391). Now, the differential of u is the same, whether we consider it as a function of x, or x as a function of u.* In this latter case we have,

* If u be a function of x, and z a function of u, then $\dfrac{dz}{dx} = \dfrac{dz}{du}\cdot\dfrac{du}{dx}$ (art. 36); and if $z = x$, $\dfrac{dx}{dx} = 1 = \dfrac{dx}{du}\cdot\dfrac{du}{dx}$. Hence it appears that $\dfrac{du}{dx}$, u being a function of x, is the reciprocal of $\dfrac{dx}{du}$, x being considered as a function of u; and, therefore, it is indifferent which of these quantities, x, u, be taken as the independent variable.

$$dx = Aa^u du = Ax du; \qquad \therefore \ du = \frac{1}{A}\frac{dx}{x}.$$

The quantity $\frac{1}{A}$ (Alg. art. 397) is the modulus of the system, and is equal to $\frac{1}{\text{Nap. log } a}$. As Naperian logarithms frequently occur in analytical inquiries, we shall always understand these logarithms to be meant in future, unless the contrary be expressly mentioned; and we shall denote them simply by the abbreviation log x.

Hence, to find the differential of the logarithm of a quantity, we have the following

Rule.—Divide the differential of the quantity by the quantity itself, and multiply the quotient by the modulus of the system.

10. *Cor.*—In the Naperian system of logarithms, the modulus = 1; therefore, the differential of the Naperian logarithm of a quantity is equal to the differential of the quantity divided by the quantity itself.

11. PROP. XI.—*To find the differential of* $u = y^z$, *where* y *and* z *are functions of* x.

Taking the logarithms of each member of the equation, we have $\log u = z \log y$; and then differentiating (arts. 31, 40)

$$\frac{du}{u} = dz \log y + z\frac{dy}{y},$$

$$\therefore \ du = u\left(dz \log y + z\frac{dy}{y}\right) = y^z\left(dz \log y + z\frac{dy}{y}\right).$$

Hence we have the following

Rule.—The differential of an exponential quantity, in which both the root and index are variable, is equal to the sum of the differentials obtained by considering each separately as variable and the other as constant.

Examples for Practice.

1. Let $u = x^n a^x$. Ans. $du = x^{n-1}a^x dx\,(x \log a + n)$.

2. Let $u = e^x(x - 1)$. Ans. $du = e^x x dx$.

3. Let $u = e^x(x^2 - 2x + 2)$. Ans. $du = e^x x^2 dx$.

4. Let $u = \frac{e^x}{1 + x}$. Ans. $du = \frac{e^x x dx}{(1 + x)^2}$.

5. Let $u = \log\left(x + \sqrt{x^2 - 1}\right)$. Ans. $du = \frac{dx}{\sqrt{(x^2 - 1)}}$.

6. Let $u = \log \frac{1 + \sqrt{(1 - x^2)}}{x}$. Ans. $du = \frac{-dx}{x\sqrt{(1 - x^2)}}$.

7. Let $u = \log\left(x\sqrt{-1} - \sqrt{1 - x^2}\right)$. Ans. $du = \frac{dx}{\sqrt{(x^2 - 1)}}$.

8. Let $u = \log\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)$. Ans. $du = \dfrac{dx}{x} - \dfrac{dx}{x\sqrt{(1-x^4)}}$.

9. Let $u = \log \dfrac{1+x}{1-x}$. Ans. $du = \dfrac{2dx}{1-x^2}$.

10. Let $u = \log \dfrac{\sqrt{(x^2+1)}-1}{\sqrt{(x^2+1)}+1}$. Ans. $du = \dfrac{2dx}{x\sqrt{(x^2+1)}}$.

11. Let $u = \log \dfrac{x+\sqrt{(1-x^2)}}{x-\sqrt{(1-x^2)}}$. Ans. $du = \dfrac{-2dx}{(2x^2-1)\sqrt{(1-x^2)}}$.

12. Let $u = x \log x$.

13. Let $u = x^3 \log x - \frac{1}{3}x^3$.

14. Let $u = x^4 (\log x)^2 - \frac{1}{2}x^4 \log x + \frac{1}{8}x^4$.

15. Let $u = e^x \log x$.

16. Let $u = e^{\log x}$.

17. Let $u = x^n e^{\log x}$.

18. Let $u = \dfrac{e^x - 1}{e^x + 1}$. Ans. $du = \dfrac{2e^x dx}{(e^x+1)^2}$.

19. Let $u = \log \dfrac{e^x - 1}{e^x + 1}$. Ans. $du = \dfrac{2e^x dx}{e^{2x} - 1}$.

20. Let $u = y^{xz}$. Ans. $du = y^{xz}\left(\dfrac{xz}{y}\,dy + zdx \log y + xdz \log y\right)$.

DIFFERENTIATION OF CIRCULAR FUNCTIONS.

42. We have already (Trig. art. 85) expanded the functions $\sin x$ and $\cos x$ in a series, by means of Demoivre's formula; but as we wish to find the differentials of circular functions independently of these series, we shall first demonstrate the following lemmas:—

43. Prop. XII.—*If* $\sin h$ *be expanded in a series, it will be of the form* $Ah + Bh^{\beta} + Ch^{\gamma} + \&c.$, *where the exponents* β, γ, &c. *are positive numbers greater than* 1.

Suppose $\sin h$ to be of the form $Ah^{\alpha} + Bh^{\beta} + \&c.$ Now, it is manifest that none of the exponents α, β, &c. can be negative; for if $h = 0$, then $\sin h = 0$; but any term, such as A^{α}, containing a negative index, would in this case be infinite. Neither can there be a term such as A independent of h; for if $h = 0$, $\sin h$ would then $= A$, which is absurd. Hence it follows, that when $\sin h$ is expanded, the series will be of the form

$$\sin h = Ah^{\alpha} + Bh^{\beta} + Ch^{\gamma} + \&c.$$

where α, β, γ, &c. are all positive quantities. Let these indices be taken in order of magnitude, α being the least. We have then, from Trigonometry, art. 77,

$$\sin 3h = 3 \sin h - 4 \sin^3 h.$$

Substitute $3h$ for h in the series above, and we obtain

$$\sin 3h = A(3h)^{\alpha} + B(3h)^{\beta} + \&c.;$$

$$3 \sin h - 4 \sin^3 h = (3Ah^{\alpha} + 3Bh^{\beta} + \&c.) - (4A^3h^{3\alpha} + \&c.)$$

And as these two series are always equal to each other, whatever be the value of h, they must be identical, and the coefficients of like powers of h equal to each other (Alg. art. 385); consequently,

$$A(3h)^{\alpha} = 3Ah^{\alpha}, \quad \therefore \quad 3^{\alpha} = 3 \quad \text{and} \quad \alpha = 1.$$

Hence, $\quad \sin h = Ah + Bh^{\beta} + Ch^{\gamma} + \&c.$

44. *Cor.*—Because $\cos h = \sqrt{(1 - \sin^2 h)} = 1 - \frac{1}{2} \sin^2 h - \frac{1}{8} \sin^4 h - \&c.$ If we substitute for $\sin h$ the expression obtained in the last article, we shall have $\cos h = 1 - \frac{1}{2}A^2h^2 - ABh^{\beta+1} - \&c.$

45. Prop. XIII.—*The coefficient* A *is* $= 1$.

Since it appears from the note, vol. i., page 465, that $\sin h < h$, and $\tan h$ or $\dfrac{\sin h}{\cos h} > h$; we have, from the first of these formulæ $\dfrac{\sin h}{h} < 1$, and from the second $\dfrac{\sin h}{h} > \cos h$. Substitute for $\sin h$ and $\cos h$ the expressions obtained in the two last articles, and we find

$$\frac{\sin h}{h} = A + Bh^{\beta-1} + \&c. < 1;$$

$$\therefore A < 1 - Bh^{\beta-1} - Ch^{\gamma-1} - \&c. < 1 + \delta; \text{ by substitution.}$$

Also, $\quad \dfrac{\sin h}{h} = A + Bh^{\beta-1} + \&c. > \cos h > 1 - \frac{1}{2}A^2h^2 - \&c.;$

$$\therefore \quad A > 1 - \tfrac{1}{2} A^2h^2 - Bh^{\beta-1} - \&c. > 1 - \delta', \text{ also by substitution.}$$

Hence, since A is less than $1 + \delta$ and greater than $1 - \delta'$; and both δ and δ' may evidently become less than any assignable quantity by making h sufficiently small, it is manifest that A must be $= 1$.

46. *Cor.*—Hence it follows that

$$\sin h \quad \text{is of the form} \quad h + Bh^{\beta} + Ch^{\gamma} + \&c.$$

where β, γ, &c., are all positive quantities greater than 1; and

$$\cos h \quad \text{is of the form} \quad 1 - \tfrac{1}{2}h^2 - Bh^{\beta+1} - \&c.$$

47. Prop. XIV.—*To find the differentials of* $\sin x$ *and* $\cos x$.

(1). Suppose x to change its value and become $x + h$, then, if $u = \sin x$,

$$\begin{aligned} u' &= \sin(x + h) = \sin x \cos h + \cos x \sin h \\ &= \sin x \left(1 - \tfrac{1}{2}h^2 - \&c.\right) + \cos x \,(h + \&c.) \\ &= \sin x + \cos x \,.\, h - \tfrac{1}{2} \sin x \,.\, h^2 - \&c. \\ &= u + \cos x \,.\, h + qh^2 + \&c. \end{aligned}$$

Hence, by the definition, art. 25,

$$\frac{du}{dx} = \cos x; \qquad \therefore \quad du = d \,.\, \sin x = dx \cos x.$$

(2). If $u = \cos x$, we have

$$\begin{aligned} u' &= \cos(x + h) = \cos x \cos h - \sin x \sin h \\ &= \cos x\,(1 - \tfrac{1}{2}h^2 - \&c.) - \sin x\,(h + \&c.) \\ &= \cos x - \sin x \,.\, h - \tfrac{1}{2} \cos x \,.\, h^2 - \&c. \\ &= u - \sin x \,.\, h + qh^2 + \&c. \end{aligned}$$

$$\therefore \quad \frac{du}{dx} = -\sin x; \quad \text{and } du = d \,.\, \cos x = -dx \sin x.$$

48. PROP. XV.—*To differentiate* tan x, cot x, sec x, cosec x, *and* vers x.

(1) $$d \,.\, \tan x = d \,.\, \frac{\sin x}{\cos x} = \frac{\cos x \times d \,.\, \sin x - \sin x \times d \,.\, \cos x}{\cos^2 x};$$

$$\therefore \; d \,.\, \tan x = \frac{dx \cos^2 x + dx \sin^2 x}{\cos^2 x} = \frac{dx}{\cos^2 x} = dx \sec^2 x.$$

(2). $$d \,.\, \cot x = d \,.\, \frac{\cos x}{\sin x} = \frac{\sin x \times d \,.\, \cos x - \cos x \times d \,.\, \sin x}{\sin^2 x};$$

$$\therefore \; d \,.\, \cot x = \frac{-dx \sin^2 x - dx \cos^2 x}{\sin^2 x} = \frac{-dx}{\sin^2 x}.$$

(3). $$d \,.\, \sec x = d \,.\, \frac{1}{\cos x} = \frac{dx \sin x}{\cos^2 x} = dx \tan x \sec x.$$

(4). $$d \,.\, \mathrm{cosec}\, x = d \,.\, \frac{1}{\sin x} = \frac{-dx \cos x}{\sin^2 x} = -dx \cot x \,\mathrm{cosec}\, x.$$

(5). $$d \,.\, \mathrm{vers}\, x = d \,.\, (1 - \cos x) = dx \sin x.$$

49. PROP. XVI.—*To differentiate the inverse circular functions,* arc (sin = x), or $\sin^{-1}$x ; arc (cos = x), or $\cos^{-1}$x ; &c.

(1). Let $u = \sin^{-1} x$, that is, let u be the arc whose sine is x, then the differential of u is the same, whether we consider u as a function of x, or x as a function of u. And because $x = \sin u$, $\therefore$ $dx = du \cos u$ (art 47); but $\cos u = \sqrt{(1 - \sin^2 u)} = \sqrt{(1 - x^2)}$; therefore

$$du = \frac{dx}{\cos u} = \frac{dx}{\sqrt{(1 - x^2)}}.$$

In the same manner, if $u = \cos^{-1} x$, we shall find

$$du = \frac{-dx}{\sqrt{(1 - x^2)}}.$$

(2). Let $u = \tan^{-1} x$, then

$$x = \tan u, \quad \text{and} \quad dx = du \sec^2 u \text{ (art. 48)};$$

but $\sec^2 u = 1 + \tan^2 u = 1 + x^2$; therefore

$$du = \frac{dx}{\sec^2 u} = \frac{dx}{1 + x^2}.$$

In like manner, if $u = \cot^{-1}x$, then will

$$du = \frac{-dx}{1 + x^2}.$$

(3). Let $u = \sec^{-1}x$, then

$$x = \sec u, \quad \text{and} \quad dx = du \tan u \sec u \ \text{(art. 18)}.$$

But $\tan u \sec u = \sec u \sqrt{(\sec^2 u - 1)} = x\sqrt{(x^2 - 1)}$;

$$\therefore \quad du = \frac{dx}{\tan u \sec u} = \frac{dx}{x\sqrt{(x^2 - 1)}}.$$

If $u = \operatorname{cosec}^{-1}x$, we shall find, in the same manner,

$$du = \frac{-dx}{x\sqrt{(x^2 - 1)}}.$$

(4). Let $u = \operatorname{vers}^{-1} x$, then

$$x = \operatorname{vers} u, \quad \text{and} \quad dx = du \sin u \ \text{(art. 18)}.$$

But $\sin u = \sqrt{(1 - \cos^2 u)} = \sqrt{1 - (1 - x)^2} = \sqrt{2x - x^2}$;

$$\therefore \quad du = \frac{dx}{\sin u} = \frac{dx}{\sqrt{(2x - x^2)}}.$$

Examples for Practice.

1. Let $u = \sin^n x$. Ans. $du = ndx \sin^{n-1}x \cos x$.
2. Let $u = \cos x \sin 2x$. Ans. $du = (\cos x \cos 2x + \cos 3x)dx$.
3. Let $u = \tan^n x$. Ans. $du = \dfrac{ndx \tan^{n-1}x}{\cos^2 x}$.
4. Let $u = \log \sin x$. Ans. $du = dx \cot x$.
5. Let $u = \log \sqrt{\left(\dfrac{1 + \sin x}{1 - \sin x}\right)}$. Ans. $du = \dfrac{dx}{\cos x}$.
6. Let $u = \log \sqrt{\left(\dfrac{1 + \cos x}{1 - \cos x}\right)}$. Ans. $du = -\dfrac{dx}{\sin x}$.
7. Let $u = x - \sin x \cos x$. Ans. $du = 2dx \sin^2 x$.
8. Let $u = \dfrac{\sin nx}{\sin^n x}$. Ans. $= \dfrac{-n \sin (n - 1)x \cdot dx}{\sin^{n+1} x}$.
9. Let $u = \log(\cos x + \sqrt{-1} \sin x)$. Ans. $du = dx\sqrt{-1}$.
10. Let $u = \log \dfrac{1 + \sqrt{-1} \tan x}{1 - \sqrt{-1} \tan x}$. Ans. $du = 2dx \sqrt{-1}$.
11. Let $u = \sin^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$. Ans. $du = \dfrac{-2dx}{1 + x^2}$.
12. Let $u = \cos^{-1}(4x^3 - 3x)$. Ans. $du = \dfrac{-3dx}{\sqrt{(1 - x^2)}}$.

13. Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$. Ans. $du = \frac{2dx}{1+x^2}$.

14. Let $u = \cot^{-1}\sqrt{\left(\frac{1-x}{x}\right)}$. Ans. $du = \frac{dx}{2\sqrt{(x-x^2)}}$.

15. Let $u = \sec^{-1}\sqrt{(2x+1)}$. Ans. $du = \frac{dx}{(2x+1)\sqrt{2x}}$.

16. Let $u = \text{cosec}^{-1}\left[\frac{1}{2x\sqrt{1-x^2}}\right]$. Ans. $du = \frac{2dx}{\sqrt{(1-x^2)}}$.

17. Let $u = \tan^{-1}\left[\frac{\sqrt{(1+x^2)}-1}{x}\right]$. Ans. $du = \frac{dx}{2(1+x^2)}$.

18. Let $u = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$. Ans. $du = \frac{\sqrt{(a^2-b^2)}dx}{a+b\cos x}$.

19. Let $u = \tan^{-1}\left[\frac{\sqrt{(a-b)}\sqrt{(1-\cos x)}}{\sqrt{(a+b)}\sqrt{(1+\cos x)}}\right]$.

Ans. $du = \frac{\frac{1}{2}\sqrt{(a^2-b^2)}dx}{a+b\cos x}$.

$x\sqrt{3}+x^2$

20. Let $u = \frac{\sqrt{3}}{2}\log\frac{1+\sqrt{3}\,.\,x+x^2}{1-\sqrt{3}\,.\,x+x^2} + \tan^{-1}\left[\frac{3x(1-x^2)}{1-4x^2+x^4}\right]$.

Ans. $du = \frac{6dx}{1+x^6}$.

SUCCESSIVE DIFFERENTIATIONS.

50. It appears, from the reasoning in the preceding articles, that the differential coefficient $\frac{du}{dx}$ or p is a new function of x, whose form depends upon the particular form of the function u. We may then suppose x to vary in the function p, and the differential coefficient $\frac{dp}{dx}$ to be derived from p in the same manner as we previously found the value of $\frac{du}{dx}$. Again, if we put $\frac{dp}{dx} = q$, we may suppose x to vary in q, and the differential coefficient $\frac{dq}{dx}$ to be derived from q in the same manner as we found the values of $\frac{du}{dx}$ and $\frac{dp}{dx}$. And it is evident that we may proceed in this manner as far as we please, unless it happen that, in finding the series of functions p, q, &c. we at last arrive at a result that is constant, and then the operations will terminate. Thus, if the function was ax^4, we should have

$$u = ax^4; \quad \frac{du}{dx} = 4ax^3 = p; \quad \frac{dp}{dx} = 4\,.\,3ax^2 = q;$$

$$\frac{dq}{dx} = 4 . 3 . 2\,ax = r; \quad \frac{dr}{dx} = 4 . 3 . 2 . 1\,a.$$

Here the expression $\frac{dr}{dx}$ is a constant quantity, and, therefore, its differential $= 0$.

51. From the preceding equations $\frac{du}{dx} = p$, $\frac{dp}{dx} = q$, $\frac{dq}{dx} = r$, &c., we obtain $du = pdx$, $dp = qdx$, $dq = rdx$, &c. Now, if we suppose dx to be constant, and find the differential of du by the same operation as we found the differential of u, we have

$$d(du) = ddu = d^2u = d(pdx) = dp . dx,$$

where it must be observed that the index of d indicates the repetition of an operation, and not a power of the letter d, which is never considered as a quantity, but only as a symbol of operation. Substituting for dp its value qdx, we have

$$d^2u = dp . dx = (qdx) \times dx = qdx^2; \quad \therefore \quad q = \frac{d^2u}{dx^2},$$

in which expression it must be remembered that dx^2 denotes the same thing as $(dx)^2$. Differentiating the equation $d^2u = qdx^2$ again, and considering dx^2 constant, we have

$$d(d^2u) = d^3u = d(qdx^2) = dq . dx^2 = (rdx)dx^2 = rdx^3;$$

we find, therefore, that

$$p = \frac{du}{dx}, \quad q = \frac{d^2u}{dx^2}, \quad r = \frac{d^3u}{dx^3}, \quad \&c.$$

The quantities du, d^2u, d^3u, &c. are called the first, second, third, &c. differentials of u; and the quantities denoted by $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c. are called the first, second, &c. differential coefficients of u.

Examples.

1. To find the 5th differential coefficient of $ax^5 + bx + c$.
   Ans. $120a$.
2. To find the 6th differential coefficient of a^x. Ans. $(\log a)^6 a^x$.
3. To find the 4th differential coefficient of $\sin x$. Ans. $\sin x$.
4. To find the 5th differential coefficient of $\frac{a^2}{a^2 + x^2}$.

$$\text{Ans. } \frac{-720a^6x + 2400a^4x^3 - 720a^2x^5}{(a^2 + x^2)^6}.$$

5. To find the n^{th} differential of $\frac{1}{x}$.
6. To find the n^{th} differential of vw, v and w being functions of x.

$$\text{Ans. } d^nv . w + nd^{n-1}v . dw + \frac{n(n-1)}{2} d^{n-2}v\, d^2w \ldots + vd^nw.$$

CHAP. III.—DEVELOPEMENT OF FUNCTIONS AND DIFFERENTIATION OF FUNCTIONS OF TWO OR MORE VARIABLES.

52. PROP. I.—*If there be three series*

$$A + Bh + Ch^2 + Eh^3 + \&c.$$
$$a + bh + ch^2 + eh^3 + \&c.$$
$$\textit{and}\ A + B'h + C'h^2 + E'h^3 + \&c.$$

and the value of the second series be always greater than that of the first series and less than that of the third; then, if the first and third series have the same first term A, it will be equal to the first term of the second series, that is, $A = a$.

Subtract the first series from the second and we obtain

$$(a - A) + (b - B)h + (c - C)h^2 + \&c.;$$

let this be put $= (a - A) + \delta$. Also, subtract the second series from the third and we have

$$(A - a) + (B' - b)h + (C' - c)h^2 + \&c.$$

put this $= (A - a) + \delta'$. It is evident, then, from the hypothesis, that these two expressions will be always positive, whatever be the value of h. Now, if A be not equal to a, it is manifest that we may take h so small that both δ and δ' will be less in magnitude than $(a - A)$. Hence the signs of

$$(a - A) + \delta \quad \text{and} \quad (A - a) + \delta'$$

will depend on the signs of the first terms $(a - A)$ and $(A - a)$. But these terms $(a - A)$ and $(A - a)$ have evidently different signs, and, therefore, $(a - A) + \delta$ and $(A - a) + \delta'$ will have different signs, which is contrary to the supposition. Hence A cannot be unequal to a, that is, $A = a$.

TAYLOR'S THEOREM.

53. It appears, from what we have proved in the preceding articles, that if $x + h$ be substituted for x in the function u, u' is in general capable of being expanded in a series of this form,

$$u + ph + qh^2 + rh^3 + \&c.,$$

where p, q, &c. denote functions of x entirely independent of h.

The second term ph of this expansion is that which we have defined to be the differential of u; and we have denoted p, the coefficient of h, by $\frac{du}{dx}$, which serves to indicate the relation subsisting between the functions u and p. We now proceed to investigate the relation that subsists between q, r, &c., the coefficients of the other terms in the expansion, and the original function u; and we shall pursue the same method as we did in the expansion of the binomial theorem (Alg. art. 388.)

54. Prop. II.—*If* u′ *be the same function of* (x + *h*) *that* u *is of* x, *then* u′ *will be expressed by the series*

$$u' = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

Let u be denoted by $f(x)$, and u' by $f(x+h)$; f in this case being a symbol of operation. We have then (art. 11)

$$f(x+h) = u + ph + qh^2 + rh^3 + \&c. \dots\dots\dots\dots (1),$$

where $p = \frac{du}{dx}$ and q, r, &c. are unknown functions to be determined from the investigation. For h substitute $h + k$, and we obtain

$$f(x+h+k) = u + p(h+k) + q(h+k)^2 + r(h+k)^3 + \&c.,$$

and involving the terms $(h+k)^2$, $(h+k)^3$, &c., and arranging the terms under each other, we have $f(x+h+k)$ equal to the following series,

$$\left.\begin{array}{l} u + ph + qh^2 + rh^3 + sh^4 + \&c. \\ \quad + pk + 2qhk + 3rh^2k + 4sh^3k + \&c. \\ \qquad + qk^2 + \dots\dots\dots\dots\dots\dots \end{array}\right\} \dots\dots (2).$$

Again, if we substitute $x + k$ for x in series (1), we have

$$f(x+h+k) = u_1 + p_1h + q_1h^2 + r_1h^3 + \&c. \dots\dots (3),$$

where u_1, p_1, q_1, &c. denote the functions of u, p, q, &c., when $x + k$ is substituted for x. Now, if we substitute k for h in series (1), we have

$$f(x+k) \text{ or } u_1 = u + pk + qk^2 + \&c. = u + \frac{du}{dx}k + qk^2 + \&c.$$

In like manner, when $x + k$ is substituted for x in the functions p, q, &c., we obtain

$$p_1 = p + \frac{dp}{dx}k + \&c.; \quad q_1 = q + \frac{dq}{dx}k + \&c.; \; r_1 = \&c.$$

Substituting these values of u_1, p_1, q_1, &c., in equation (3), and again arranging the terms under each other, we have $f(x+h+k)$ equal to the following series,

$$\left.\begin{array}{l} u \quad + ph \quad + qh^2 \quad + rh^3 \quad + sh^4 \quad + \&c. \\ + \frac{du}{dx}k + \frac{dp}{dx}hk + \frac{dq}{dx}h^2k + \frac{dr}{dx}h^3k + \frac{ds}{dx}h^4k + \&c. \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \dots (4).$$

Now, the two series (2) and (4) are equal to each other, whatever be the values of h and k, and, therefore (Alg. art. 386), they are identical, and the coefficients of the like powers of h and k are equal to each other. The terms in the first line of the two series are exactly the same; and from the second line we obtain the following equations,

$$\frac{du}{dx} = p; \quad \frac{dp}{dx} = 2q; \quad \frac{dq}{dx} = 3r, \&c.; \text{ therefore,}$$

$$p = \frac{du}{dx}; \quad q = \frac{dp}{2dx} = \frac{d^2u}{2dx^2}; \quad r = \frac{dq}{3dx} = \frac{d^3u}{1.2.3\,dx^3};$$

$s = $ &c. Hence we have

$$f(x+h) = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3u}{dx^3}\frac{h^3}{1\,.\,2\,.\,3} + \&c.\ldots (5).$$

55. The very general theorem which we have just demonstrated is known by the name of *Taylor's theorem*, and is one of the most elegant and important in the whole range of mathematics. It comprehends the binomial theorem, and innumerable others as particular cases; and we shall now apply it to the expansion of some of these functions, which will show its great utility as an instrument of analysis. It will be remembered, that in finding the differential of x^n we assumed nothing except that the second term of the expansion of $(x+h)^n$ is $nx^{n-1}h$, which is proved in the Algebra, art. 387. It was necessary to mention this, lest we should be supposed to be reasoning in a circle.

Examples.

1. Let $u = f(x) = x^n$, and $u' = f(x+h) = (x+h)^n$.

Then $\dfrac{du}{dx} = nx^{n-1}$; $\quad \dfrac{d^2u}{dx^2} = n(n-1)x^{n-2}$;

$$\frac{d^3u}{dx^3} = n(n-1)(n-2)x^{n-3}; \text{ \&c. Hence}$$

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{1\,.\,2}x^{n-2}h^2 + \&c.$$

2. Let $u = \log x$, and $u' = f(x+h) = \log(x+h)$.

Then $\dfrac{du}{dx} = \dfrac{1}{x}$; $\quad \dfrac{d^2u}{dx^2} = -\dfrac{1}{x^2}$; $\quad \dfrac{d^3u}{dx^3} = \dfrac{2}{x^3}$; &c.

$$\therefore\ \log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \&c.$$

3. To expand a^{x+h} by Taylor's theorem.

$$a^{x+h} = a^x\left(1 + Ah + \frac{A^2h^2}{1\,.\,2} + \frac{A^3h^3}{1\,.\,2\,.\,3} + \&c.\right)$$

4. To expand $\sin(x+h)$ by Taylor's theorem.

$$\text{Sin}(x+h) = \sin x + \cos x\,.\,h - \sin x\frac{h^2}{1\,.\,2} - \cos x\frac{h^3}{1\,.\,2\,.\,3} + \&c.$$

5. To expand $\tan(x+h)$ by Taylor's theorem.

$$\text{Tan}(x+h) = \tan x + \sec^2 x\,.\,h + 2\sec^2 x\tan x\frac{h^2}{1\,.\,2}$$

$$+\ 2\sec^2 x(1 + 3\tan^2 x)\frac{h^3}{1\,.\,2\,.\,3} + \&c.$$

6. Let $u = \tan^{-1}x$; then $u' = \tan^{-1}(x+h)$

$$= u + \cos^2 u\,.\,h - \sin 2u\cos^2 u\frac{h^2}{2} - \cos 3u\cos^3 u\frac{h^3}{3} + \&c.$$

MACLAURIN'S THEOREM.

56. PROP. III.—*If* u *be any function of* x, *then*

$$u = U + U'\frac{x}{1} + U''\frac{x^2}{1.2} + U'''\frac{x^3}{1.2.3} + \&c.,$$

where U, U′, U″, &c. *represent the functions* u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c., *when* x = 0.

For by Taylor's theorem (art 54),

$$f(x+h) = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1 \cdot 2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.,$$

and putting $x = 0$, $f(x+h)$ becomes $f(h)$, and u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c. become U, U', U'', &c. Hence

$$f(h) = U + U'\frac{h}{1} + U''\frac{h^2}{1.2} + U'''\frac{h^3}{1.2.3} + \&c.;$$

and as this equation is true, whatever be the value of h, we may substitute x for h, which will not affect the values of the quantities U, U', &c., since they do not contain the letter h. Hence

$$f(x) = U + U'\frac{x}{1} + U''\frac{x^2}{1.2} + U'''\frac{x^3}{1.2.3} + \&c.$$

This is called *Maclaurin's theorem*, and it is manifestly a particular case of Taylor's theorem.

Examples.

1. Let $u = a^x$; then $\frac{du}{dx} = Aa^x$ (art. 37); $\frac{d^2u}{dx^2} = A^2a^x$; $\frac{d^3u}{dx^3} = A^3a^x$; &c. Suppose, now, that $x = 0$, then u or a^x becomes $a^0 = 1$; $\frac{du}{dx}$ or Aa^x becomes $Aa^0 = A$; $\frac{d^2u}{dx^2}$ becomes A^2; and so on. Hence $U = 1$, $U' = A$, $U'' = A^2$, &c. Substituting, therefore, these values in the general formula, it becomes

$$a^x = 1 + A\frac{x}{1} + A^2\frac{x^2}{1.2} + A^3\frac{x^3}{1.2.3} + \&c.$$

2. Let $u = (a+x)^n$.

3. Let $u = \log(1+x)$.

4. Let $u = \sin x$.

5. Let $u = \sin^{-1} x$.

$$\text{Ans. } u = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \&c.$$

6. Let $u = \tan^{-1} x$.

$$\text{Ans. } u = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c.$$

DEVELOPEMENT AND DIFFERENTIATION OF FUNCTIONS CONTAINING TWO OR MORE VARIABLES.

57. Let u be any function of two variable quantities x and y which are entirely independent of each other. We may either suppose x to vary alone and y to remain constant; or y to vary alone and x to remain constant; or we may suppose x and y to vary together. To be more distinct, we will take the particular example $u = x^m y^n$; then, if x varies alone and becomes $x + h$, $x^m y^n$ will become

$$(x + h)^m y^n = x^m y^n + m x^{m-1} y^n h + \tfrac{1}{2} m(m-1) x^{m-2} y^n h^2 + \&c. \ldots (a).$$

If y varies alone, and $y + k$ be substituted for y, $x^m y^n$ will become

$$x^m (y + k)^n = x^m y^n + n x^m y^{n-1} k + \tfrac{1}{2} n(n-1) x^m y^{n-2} k^2 + \&c, \ldots (b).$$

Lastly, if we suppose x and y to vary together, $x^m y^n$ will become $(x + h)^m (y + k)^n$. Now, it is evident that we may obtain the developement of this expression either by multiplying the developement of $(x + h)^m$ by that of $(y + k)^n$, or by substituting $y + k$ for y in series (a); or, lastly, by substituting $x + h$ for x in series (b). If we pursue the second of these methods, or substitute $y + k$ for y in the series (a), we shall obtain $(x + h)^m (y + k)^n$, equal to

$$\begin{array}{llll} x^m y^n & + m x^{m-1} y^n h & + \tfrac{1}{2} m(m-1) x^{m-2} y^n h^2 & + \&c. \\ + n x^m y^{n-1} k & + mn x^{m-1} y^{n-1} hk & + \tfrac{1}{2} m(m-1) n x^{m-2} y^{n-1} h^2 k & + \&c. \\ + \quad \&c. & + \quad \&c. & + \quad \&c., & \end{array}$$

and the same result would have been obtained from either of the other two methods.

58. PROP. IV.—*If* $u = f(x, y)$ *be any function of* x *and* y; *and* $x + h$, $y + k$ *be substituted for* x *and* y; *it is required to expand* $u' = f(x + h, y + k)$ *in a series of the same form as in the last article.*

The same method which we have just adopted may easily be extended to any functions whatever. Let $x + h$ be first substituted for x, then $f(x, y)$ will become $f(x + h, y)$. Considering y, therefore, as constant, and x as variable, and putting $f(x, y) = u$, we have (art. 54)

$$f(x + h, y) = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1 \cdot 2} + \frac{d^3u}{dx^3}\frac{h^3}{1 \cdot 2 \cdot 3} + \&c. \ldots (1).$$

Now, let $y + k$ be substituted for y in each term of this series. Supposing, then, x to be constant and y to be variable, $f(x + h, y)$, will become $f(x + h, y + k)$; also

$$u \text{ will become } u + \frac{du}{dy}\frac{k}{1} + \frac{d^2u}{dy^2}\frac{k^2}{1 \cdot 2} + \&c.$$

And if we put $\dfrac{du}{dx} = p$,

$$p \text{ will become } p + \frac{dp}{dy}\frac{k}{1} + \frac{d^2p}{dy^2}\frac{k^2}{1 \cdot 2} + \&c.$$

But $\frac{dp}{dy} = \frac{d\left(\frac{du}{dx}\right)}{dy}$, which, for the sake of simplicity, may be denoted thus, $\frac{d^2u}{dx\,dy}$; where it is to be understood that the differential of u is taken twice, first considering x as variable, and then y as variable. Also, $\frac{d^2p}{dy^2} = \frac{d^2\left(\frac{du}{dx}\right)}{dy^2}$, which, in like manner, may be written thus, $\frac{d^3u}{dx\,dy^2}$; in which expression the differential of u is first taken, supposing x to be variable, and then twice considering y as variable. In like manner $\frac{d^3p}{dy^3} = \frac{d^3\left(\frac{du}{dx}\right)}{dy^3} = \frac{d^4u}{dx\,dy^3}$; and so on. Hence, by the substitution of $y + k$ for y,

$$\frac{du}{dx} \text{ will become } \frac{du}{dx} + \frac{d^2u}{dx\,dy}\frac{k}{1} + \frac{d^3u}{dx\,dy^2}\frac{k^2}{1\,.\,2} + \&c.$$

In the same way we shall find that

$$\frac{d^2u}{dx^2} \text{ will become } \frac{d^2u}{dx^2} + \frac{d.\frac{d^2u}{dx^2}}{dy}\frac{k}{1} + \frac{d^2.\frac{d^2u}{dx^2}}{dy^2}\frac{k^2}{1\,.\,2} + \&c.,$$

or, by the same notation, $\frac{d^2u}{dx^2} + \frac{d^3u}{dx^2dy}\frac{k}{1} + \frac{d^4u}{dx^2dy^2}\frac{k^2}{1\,.\,2} + \&c.$;

and so on, with the other terms. Hence, by substituting these various expressions in series (1), we obtain

$$f(x + h,\ y + k) =$$

$$\left.\begin{aligned}
&\frac{du}{dy}\frac{k}{1} + \frac{d^2u}{dy^2}\frac{k^2}{1\,.\,2} + \frac{d^3u}{dy^3}\frac{k^3}{1\,.\,2\,.\,3} + \&c.\\
+\ &\frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx\,dy}\frac{k}{1}\,.\,\frac{h}{1} + \frac{d^3u}{dx\,dy^2}\frac{k^2}{1\,.\,2}\frac{h}{1} + \&c.\\
&\qquad + \frac{d^2u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3u}{dx^2\,dy}\frac{k}{1}\,.\,\frac{h^2}{1\,.\,2} + \&c.\\
&\qquad\qquad + \frac{d^3u}{dx^3}\frac{h^3}{1\,.\,2\,.\,3} + \&c.
\end{aligned}\right\} \ldots (2).$$

59. *Cor.*—If we had made these substitutions in an inverse order, we should have found, first, by changing y into $y + k$,

$$f(x, y + k) = u + \frac{du}{dy}\frac{k}{1} + \frac{d^2u}{dy^2}\frac{k^2}{1\,.\,2} + \frac{d^3u}{dy^3}\frac{k^3}{1\,.\,2\,.\,3} + \&c.$$

And putting, then, in each term $x + h$ in place of x, we should have obtained this developement,

$$f(x+h,\,y+k) = \left.\begin{array}{l} u + \dfrac{du}{dx}\dfrac{h}{1} + \dfrac{d^2u}{dx^2}\dfrac{h^2}{1\,.\,2} + \dfrac{d^3u}{dx^3}\dfrac{h^3}{1\,.\,2\,.\,3} + \&c. \\ \quad + \dfrac{du}{dy}\dfrac{k}{1} + \dfrac{d^2u}{dy\,dx}\dfrac{h}{1}\cdot\dfrac{k}{1} + \dfrac{d^3u}{dy\,dx^2}\dfrac{h^2}{1\,.\,2}\dfrac{k}{1} + \&c. \\ \qquad + \dfrac{d^2u}{dy^2}\dfrac{k^2}{1\,.\,2} + \dfrac{d^3u}{dy^2\,dx}\dfrac{h}{1}\dfrac{k^2}{1\,.\,2} + \&c. \\ \qquad\qquad + \dfrac{d^3u}{dy^3}\dfrac{k^3}{1\,.\,2\,.\,3} + \&c. \end{array}\right\}\ldots(3).$$

And since these two series are always equal to each other, whatever be the values of h and k, they are identical, and the coefficients of the like powers of h and k are equal to each other. We have, therefore, $\dfrac{d^2u}{dx\,dy} = \dfrac{d^2u}{dy\,dx}$; $\dfrac{d^3u}{dx^2\,dy} = \dfrac{d^3u}{dy\,dx^2}$; &c. Hence it appears, that if *a function of two variables be differentiated, first with respect to one of the variables, and then with respect to the other, the result will be the same, in whatever order the differentials be taken.* Suppose, for example, that $u = x^m y^n$; if we first differentiate it, considering x only as variable, we have $\dfrac{du}{dx} = mx^{m-1}y^n$; then differentiating this result with respect to y only, we obtain $\dfrac{d^2u}{dx\,dy} = mnx^{m-1}y^{n-1}$. By performing these operations in an inverse order, we find $\dfrac{du}{dy} = nx^m y^{n-1}$; and $\dfrac{d^2u}{dy\,dx} = mnx^{m-1}y^{n-1}$, the same result as before.

60. PROP. V.—*To find the differential of* u *when it is a function of two independent variables* x *and* y.

If $f(x,y)$ be subtracted from both sides of equation (2), we have

$$f(x+h,\,y+k) - f(x,y) = \left.\begin{array}{l} \dfrac{du}{dx}\dfrac{h}{1} + \dfrac{d^2u}{dx^2}\dfrac{h^2}{1\,.\,2} + \&c. \\ + \dfrac{du}{dy}\dfrac{k}{1} + \dfrac{d^2u}{dx\,dy}\dfrac{h}{1}\dfrac{k}{1} + \&c. \\ \quad + \dfrac{d^2u}{dy^2}\dfrac{k^2}{1\,.\,2} + \&c. \end{array}\right\}\ldots(4).$$

And, if we extend the definition (art. 25) of the differential of a function of one variable to those of two variables, we still perceive that the differential of $f(x,y)$ consists of the two terms which form the first column of the preceding developement; therefore, by changing h into dx, and k into dy, we have

$$d\,.f(x,y) = du = \frac{du}{dx}dx + \frac{du}{dy}dy.$$

The quantities $\dfrac{du}{dx}dx$, $\dfrac{du}{du}dy$, are called the ***partial differentials*** of the

function u, and they must not be confounded with du, which is called the *total* or *complete differential* of u. To find the differential, therefore, of a function of two variables, *we must differentiate the given function, first with respect to one of the variables, and then with respect to the other, and the sum of these partial differentials will be the complete differential required.*

61. *Cor.*—In the same manner, if u be a function of any number of variables, t, x, y, z, it may be shown that

$$du = \frac{du}{dt}dt + \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz,$$

denoting by $\frac{du}{dt}$, $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, the differential coefficients of the function u, taken on the supposition that t, or x, or y, or z, alone varies.

Examples.

1. To find the differential of xy.

2. To find the differential of $\frac{x}{y}$.

3. To find the differential of $\frac{ay}{\sqrt{(x^2 + y^2)}}$.

4. To find the differential of $\tan^{-1} \frac{x}{y}$.

DIFFERENTIATION OF EQUATIONS CONTAINING TWO VARIABLES.

62. We have hitherto supposed the equation expressing the relation between x, and u any function of x, to be of such a form that u was found alone in the first member of the equation, as in the examples $u = \log x$, $u = \sin x$, in which case u is said to be an *explicit* function of x. But in the greater number of equations which occur in analytical inquiries, the relation between the variable quantity and its function is expressed by an equation, as in the example $y^2 - 2mxy + x^2 - a^2 = 0$. In this case y is said to be an *implicit* function of the variable quantity x.

If we resolve this quadratic equation, we shall find y in terms of x; and then we may determine the differential of y by the rules already given. But in many cases this method of solution would be impracticable, from the want of a general method of resolving equations. We can, however, in all cases find the differential of y without the necessity of previously resolving the equation.

63. Prop. VI.—*To find the differential of* y *when it is an implicit function of* x.

Let $u = f(x, y) = 0$ be any equation between two variable quantities x and y. If this equation be resolved with respect to y, we shall manifestly have $y = \phi(x)$, a function of x. Let $x + h$ be substituted for x, then the corresponding value of y will be of the form $y + ph + qh^2 + \&c.$ Substitute these values for x and y in the function u, and it will evidently be of the form

$$u' = u + Ph + Qh^2 + Rh^3 + \&c.,$$

where P, Q, R, &c. are functions of x and y. Now, since the equation $u = 0$ is true for all corresponding values of x and y, we have also $u' = 0$, therefore,

$$Ph + Qh^2 + \&c. = 0, \quad \text{and } P + Qh + \&c. = 0.$$

And this equation is always true, whatever be the value of h, which is evidently impossible, unless $P = 0$, $Q = 0$, &c. But P is the complete differential coefficient of u, supposing both x and y to vary, and, therefore, the equation $u = 0$ necessarily leads to the equation $du = 0$; from whence the value of dy may be determined.

Example.

64. Let $u = y^2 - 2mxy + x^2 - a^2 = 0$.
If we differentiate u with respect to both x and y, we obtain

$$du = 2ydy - 2mxdy - 2mydx + 2xdx = 0;$$

$$\therefore \quad (y - mx)\,dy - (my - x)\,dx = 0 \ldots\ldots\ldots\ldots (a).$$

Hence
$$\frac{dy}{dx} = \frac{my - x}{y - mx}.$$

To obtain the value of $\frac{dy}{dx}$ in terms of x alone, we must substitute the value of y deduced from the proposed equation, which is

$$y = mx \pm \sqrt{(a^2 - x^2 + m^2x^2)}, \text{ and this gives}$$

$$\frac{dy}{dx} = \frac{m^2x - x \pm m\sqrt{(a^2 - x^2 + m^2x^2)}}{\pm\sqrt{(a^2 - x^2 + m^2x^2)}};$$

$$\therefore \quad \frac{dy}{dx} = m \pm \frac{m^2x - x}{\sqrt{(a^2 - x^2 + m^2x^2)}},$$

the same result that would have been derived from the original equation if the variables had been first separated.

65. Having, from the equation $y^2 - 2mxy + x^2 - a^2 = 0$, found that $\frac{dy}{dx} = \frac{my - x}{y - mx}$; if it be required to find the second differential of y, we have only to take the differential of this equation, considering dx as constant and y as a function of x; then we have

$$\frac{d^2y}{dx} = \frac{(mdy - dx)\,(y - mx) - (dy - mdx)\,(my - x)}{(y - mx)^2},$$

which may be reduced to
$$\frac{d^2y}{dx} = \frac{(m^2 - 1)\,(ydx - xdy)}{(y - mx)^2}.$$

and from which dy may be eliminated by means of equation (a).

By the same mode of proceeding we may determine the third or any higher differential of the function y.

TO CHANGE THE INDEPENDENT VARIABLE.

66. When we have a differential equation between x and y, in which x has been taken as the independent variable, and y is a function of x,

it is sometimes necessary to change this into an equation where x and y are considered as functions of a third quantity t, taken as the independent variable.

Since x is supposed to be a function of t, when $t+h$ is substituted for t, x will become $x+ph+qh^2+$ &c.; where $p=\frac{dx}{dt}$, $q=\frac{1}{2}\frac{d^2x}{dt^2}$, &c. Again, since x and y are functions of t, it is evident that y may be considered as a function of x; therefore, when $x+k$ is substituted for x, y will become y' or $y+p'k+q'k^2+$ &c., where $p'=\frac{dy}{dx}$, $q'=\frac{1}{2}\frac{d^2y}{dx^2}$, &c. For k substitute $ph+qh^2+$ &c., and we shall obtain

$$y' = y + pp'h + (p'q + q'p^2)\,h^2 + \&c.$$

But, because y is a function of t, if $t+h$ be substituted for t, y will become

$$y' = y + Ph + Qh^2 + \&c.;\ \text{where } P = \frac{dy}{dt},\ Q = \tfrac{1}{2}\frac{d^2y}{dt^2},\ \&c.$$

Hence, equating like coefficients in these two values of y', we have

$$P = pp';\quad Q = p'q + q'p^2,\ \&c.$$

$$\therefore\ p' = \frac{P}{p},\quad q' = \frac{Q - p'q}{p^2} = \frac{pQ - Pq}{p^3};\ \text{\&c., that is,}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt};\quad \frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right) \div \frac{dx^3}{dt^3};\quad \frac{d^3y}{dx^3} = \&c.$$

Substituting these values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c., in the given equation, we obtain an equation where x and y are considered as functions of t.

67. If we suppose $t = y$, or that x is a function of the independent variable y, we have $p = \frac{dx}{dt} = \frac{dx}{dy}$, $q = \frac{1}{2}\frac{d^2x}{dt^2} = \frac{1}{2}\frac{d^2x}{dy^2}$

$$P = \frac{dy}{dy} = 1,\quad Q = \tfrac{1}{2}\frac{d^2y}{dt^2} = \tfrac{1}{2}\frac{d^2t}{dt^2} = 0;$$

$$\therefore\ \tfrac{1}{2}\frac{d^2y}{dx^2} = q' = \frac{pQ - Pq}{p^3} = -\frac{q}{p^3} = -\tfrac{1}{2}\frac{d^2x}{dy^2} \div \frac{dx^3}{dy^3},$$

that is, d^2y, when y is considered a function of x, is equal to $-\frac{d^2x\,dy}{dx}$ when x is considered as a function of y.

CHAP. IV. — THE VALUES OF DIFFERENTIAL COEFFICIENTS IN PARTICULAR CASES.

FAILURE OF TAYLOR'S THEOREM.

68. In the preceding chapter we have given an investigation of Taylor's theorem; and the student will perceive that the form of its developement was derived from an induction of particular cases (art. 14).

It will be seen, also, that the form of these particular functions was ultimately deduced from an observation of the fundamental operations of algebra. (Alg. art. 387).

69. If, for example, we suppose $f(x) = \frac{1}{x}$, and substitute $x+h$ instead of x, we shall have $f(x+h) = \frac{1}{x+h}$, and, by division, we obtain

$$f(x+h) = \frac{1}{x+h} = \frac{1}{x} - \frac{h}{x^2} + \frac{h^2}{x^3} - \frac{h^3}{x^4} + \&c.,$$

and the same result would have been obtained from the expansion of $f(x+h)$ by Taylor's theorem. Now, this developement is true for all values of x, except when $x = 0$. In this particular case

$$\frac{1}{x+h} = \frac{1}{0} - \frac{h}{0} + \frac{h^2}{0} - \&c.,$$

where all the coefficients of the powers of h are infinite. Also, $\frac{1}{x+h}$ becomes $\frac{1}{h} = h^{-1}$, a negative power of h. Here the developement in the ordinary form fails, for we have supposed that it proceeds by positive powers of h, which, when $x = 0$, is not the true form of the expansion.

70. Again, if we take $f(x) = \sqrt{x}$, and substitute $x+h$ for x, we ave $f(x+h) = \sqrt{(x+h)}$, and, by extracting the square root,

$$f(x+h) = \sqrt{(x+h)} = \sqrt{x} + \tfrac{1}{2}\frac{h}{\sqrt{x}} - \tfrac{1}{8}\frac{h^2}{x^{\frac{3}{2}}} + \&c.,$$

which is a general expression for $\sqrt{(x+h)}$. But if $x = 0$, all the coefficients of the powers of h become infinite; and $\sqrt{(x+h)}$ becomes $\sqrt{h} = h^{\frac{1}{2}}$, a fractional power of h, which is not included in the general form of the extraction of the root.

71. In cases of this kind Taylor's theorem is said to be defective. It will, however, be readily seen, from the preceding examples, that this failure does not arise from any defect in the theorem itself, since it only fails in those cases in which it ought to fail, and which are really exceptions to the general form. To obtain the true developement when Taylor's theorem fails, on the supposition of $x = a$, we must substitute $a+h$ for x in the function $f(x)$, and then expand this function by the ordinary operations of algebra. The following example will serve to illustrate these remarks:—

Ex.—Let $f(x) = 2ax - x^2 + a\sqrt{(x^2 - a^2)}$; it is required to obtain the developement when $x = a$.

Here we have $\frac{df(x)}{dx} = 2(a-x) + \frac{ax}{\sqrt{(x^2-a^2)}}$, and this is equal to $\frac{a^2}{0} = \infty$, when $x = a$, and all the other differential coefficients are also infinite. Hence the developement by Taylor's theorem is no longer possible.

Substitute, therefore, $a + h$ for x, according to the preceding rule, and we have

$$f(a + h) = 2a(a + h) - (a + h)^2 + a\sqrt{(a + h)^2 - a^2}$$
$$= a^2 - h^2 + a\sqrt{(2ah + h^2)} = a^2 - h^2 + a\sqrt{2ah} + \&c.,$$

which is the true developement of $f(a + h)$. We cannot dwell longer on these exceptions to the general rule, but must refer the student to the Calcul des Fonctions of Lagrange, Leçon 8, and to Lacroix's Treatise on the Differential Calculus, Chap. 3.

THE VALUE OF A FRACTION WHEN THE NUMERATOR AND DENOMINATOR VANISH AT THE SAME TIME.

72. Since the value of a fraction depends not upon the absolute but the relative magnitude of the numerator and denominator, if in their evanescent state they have a finite ratio, the value of the fraction will be finite. We have an example of this in the fraction $\frac{x^2 - a^2}{x - a}$, which, by supposing $x = a$, becomes $\frac{a^2 - a^2}{a - a} = \frac{0}{0}$, an expression from which nothing can be concluded. But if we consider that its numerator and denominator have a common divisor $x - a$, the fraction $\frac{x^2 - a^2}{x - a}$ becomes $\frac{(x + a)(x - a)}{x - a} = x + a$, and when $x = a$, the value is $2a$. It must here be understood that $2a$ is the value to which the fraction $\frac{x^2 - a^2}{x - a}$ *approaches as its limit;* for when the numerator and denominator are *absolutely nothing*, we cannot in strictness speak of any ratio between them.

73. In general, if we make $x = a$ in an expression of this form $\frac{P(x - a)^m}{Q(x - a)^n}$, it becomes $\frac{0}{0}$; the ratio, however, to which it approaches as its limit, is either nothing, or finite, or infinite, according as $m > n$, or $m = n$, or $m < n$; for, by suppressing the factors common to the numerator and denominator, the fraction becomes $\frac{P(x - a)^{m-n}}{Q}$ in the first case, $\frac{P}{Q}$ in the second, and $\frac{P}{Q(x - a)^{n-m}}$ in the third; it being understood that the quantities P and Q neither become evanescent nor infinite by the supposition of $x = a$.

Whenever, therefore, an expression assumes the form of $\frac{0}{0}$ we must, in order to discover its real value, disengage it from the factors which are common to the numerator and denominator. This may be done by finding their greatest common measure; but the differential calculus will furnish us with another method by which this can more easily be effected.

71. PROP. I.—*To find the value of a vanishing fraction by the differential calculus.*

If we differentiate the expression $P(x-a)^2$, which vanishes when $x=a$, we obtain $(x-a)^2dP + 2(x-a)Pdx$, which also vanishes when $x=a$. But if we differentiate this again, we shall find $(x-a)^2d^2P + 4(x-a)dPdx + 2Pdx^2$. And since P is not supposed to contain the factor $x-a$, this second differential becomes $2Pdx^2$ when $x=a$. By proceeding in this manner, it is easy to see that in differentiating a function of the form $P(x-a)^n$, n times successively (n being a whole number), we shall finally obtain an expression, all the terms of which, except the last, vanish on the supposition of $x=a$; and that the last term will be $1 \,.\, 2 \,.\, 3 \ldots nPdx^n$, an expression free from the factor $(x-a)^n$, and involving only the function P.

It is not necessary that we should know the value of the exponent n, nor that we should exhibit the factor $(x-a)^n$ in order to determine when the expression $P(x-a)^n$ is freed from that factor. We have only to ascertain, after each differentiation, whether the result vanishes or not, when a is substituted instead of x; for in the last case the operation is finished, and the result is the quantity $1 \,.\, 2 \,.\, 3 \ldots nPdx^n$. Suppose, for example, the function to be $x^3 - ax^2 - a^2x + a^3$, which vanishes when $x=a$; its first differential also vanishes when $x=a$, but its second differential is equal to $(6x-2a)dx^2$, which does not vanish. Hence we may conclude that the function has the form $P(x-a)^2$, which is besides obvious, since

$$x^3 - ax^2 - a^2x + a^3 = (x+a)(x-a)^2.$$

In applying these observations to the fraction $\dfrac{P(x-a)^m}{Q(x-a)^n}$, if $m=n$, and we differentiate the numerator and denominator of this fraction m times successively, they will be freed at once from the factor $(x-a)^m$. If a result which does not vanish be obtained first from the numerator, then the factor $x-a$ is raised to a higher power in the denominator than in the numerator, and consequently the fraction is infinite when $x=a$. If, on the contrary, the denominator first affords a result which does not vanish, the numerator contains a higher power of $x-a$ than the denominator; and in this case, when $x=a$, the fraction vanishes. The rule, therefore, for finding the value of a function which becomes $\dfrac{0}{0}$ by giving a particular value to x may be expressed thus—*Differentiate successively the numerator and denominator until a result which does not vanish be obtained from either the numerator or denominator, or from both at the same time. In the first case the function is infinite, in the second it is* $=0$, *and in the last case its value is finite.*

Examples.

Ex. 1.—To find the value of $\dfrac{x^4-1}{x^2-1}$ when $x=1$?

The differential of the numerator is $4x^3dx$; and that of the denomi-

nator is $2xdx$; neither of which quantities vanish when $x = 1$. In this case, therefore, the value of the fraction is $\frac{4x^3}{2x} = 2$.

Ex. 2.—To find the value of $\frac{x^3 - ax^2 - a^2x + a^3}{x^2 - a^2}$ when $x = a$?

By differentiating the numerator and the denominator once we obtain

$$\frac{3x^2dx - 2axdx - a^2dx}{2xdx} = \frac{3x^2 - 2ax - a^2}{2x},$$

and when a is put for x, the numerator alone becomes equal to nothing. Hence the value of the fraction in this case is 0.

The contrary happens in the fraction $\frac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4}$; and, therefore, when $x = a$, this fraction becomes infinite.

Ex. 3.—To find the value of $\frac{a^x - b^x}{x}$ when $x = 0$.

Although this fraction is not apparently of the form given above, yet its value may be found by the preceding rule. By differentiating once we find $\frac{\log a\, a^x dx - \log b\, b^x dx}{dx}$, and, when $x = 0$, we have $\log a - \log b$ for the true value of this fraction.

This result might be immediately obtained by expanding a^x, and b^x; for (Alg. art. 395)

$$a^x = 1 + \log a \frac{x}{1} + (\log a)^2 \frac{x^2}{1 \cdot 2} + \&c.;\ b^x = 1 + \log b \frac{x}{1} + \&c.,$$

$$\therefore\ \frac{a^x - b^x}{x} = \log a - \log b + \frac{(\log a)^2 - (\log b)^2}{1 \cdot 2} x + \&c.,$$

and when $x = 0$, this becomes $\log a - \log b$, the same result as before.

75. *Scholium.*—The rule which we have given in the last article can only be applied where the factors common to the numerator and denominator are integral powers of $x - a$; for since the index of $(x - a)^m$ is diminished by a unit at each successive differentiation; when m is a fraction, we shall at last arrive at a result containing negative powers o $x - a$, which, therefore, when $x = a$, will become infinite. The following mode of proceeding, however, will apply to every case.

76. Prop. II.—*To find the value of a vanishing fraction when Taylor's theorem fails.*

Let $\frac{P}{Q}$ be a fraction of which the numerator and denominator both vanish when $x = a$. By substituting in it $a + h$ instead of x, the functions P and Q may be expanded into two series of the form

$$Ah^\alpha + Bh^\beta + \&c., \qquad A'h^{\alpha'} + B'h^{\beta'} + \&c.,$$

where the exponents α, β, &c., and also α', β', &c., are supposed to be all positive, and arranged in order of magnitude, beginning with the least, since the series must become 0, upon the hypothesis that

$h=0$. Hence, therefore, the function $\frac{P}{Q}$ will become $\frac{Ah^{\alpha} + Bh^{\beta} + \&c.}{A'h^{\alpha'} + B'h^{\beta'} + \&c.}$. Now, it is evident that there will be three different cases, according as $\alpha > \alpha'$, $\alpha = \alpha'$, or $\alpha < \alpha'$; and that this fraction will become, in these three cases,

$$\frac{Ah^{\alpha-\alpha'} + \&c.}{A' + B'h^{\beta'-\alpha'} + \&c.}; \quad \frac{A + Bh^{\beta-\alpha} + \&c.}{A' + B'h^{\beta'-\alpha} + \&c.}; \quad \frac{A + Bh^{\beta-\alpha} + \&c.}{A'h^{\alpha'-\alpha} + \&c.},$$

and when $h = 0$, these fractions become

$$\frac{0}{A'} = 0; \quad \frac{A}{A'}; \quad \frac{A}{0} = \infty.$$

Hence we have the following rule, which is applicable to every function that appears under the form of $\frac{0}{0}$. *Substitute* a + h *for* x *in the given fraction, and expand the numerator and denominator in ascending series of* h; *reduce the resulting fraction to its most simple form, and then make* h = 0, *the result thus obtained will be the value of the proposed fraction when* x = a.

Ex.—To find the value of $\frac{\sqrt{x} - \sqrt{a} + \sqrt{(x-a)}}{\sqrt{(x^2 - a^2)}}$ when $x = a$.

By substituting $a + h$ instead of x, and developing the numerator and denominator into series, this fraction becomes

$$\frac{\sqrt{h} + \frac{h}{2\sqrt{a}} + \&c.}{\sqrt{2ah} + \frac{h^{\frac{3}{2}}}{\sqrt{8a}} + \&c.} = \frac{1 + \frac{\sqrt{h}}{2\sqrt{a}} + \&c.}{\sqrt{2a} + \frac{h}{\sqrt{8a}} + \&c.}$$

and when $h = 0$, it becomes $\frac{1}{\sqrt{2a}}$, the value of the fraction required.

77. *Cor.*—By the same methods the value of the function may be obtained which presents itself under other forms than that given above.

1st. The numerator and denominator of the fraction $\frac{P}{Q}$ may become infinite at the same time; but this fraction being written thus $\frac{1}{Q} \div \frac{1}{P}$ is reduced to the form $\frac{0}{0}$ when P and Q are infinite.

2nd. We may sometimes meet with a product composed of two factors, the one infinite and the other nothing. Let this product be PQ, in which $P = 0$ and $Q = \infty$, when $x = a$; we may write it thus, $PQ = P \div \frac{1}{Q}$, and since Q is infinite, $\frac{1}{Q} = 0$, therefore $PQ = \frac{0}{0}$.

78. Prop. III.—*To find the value of* $\frac{\log x}{x^n}$ *when* x *is infinite.*

To determine the value of this expression we may proceed as follows. We have, from Algebra, art. 395, substituting x for a and n for x,

$$x^n = 1 + \log x \frac{n}{1} + (\log x)^2 \frac{n^2}{1 \cdot 2} + \&c.$$

$$\therefore \quad \frac{\log x}{x^n} = \log x \div \left(1 + \log x \frac{n}{1} + (\log x)^2 \frac{n^2}{1 \cdot 2} + \&c.\right)$$

$$= \quad 1 \quad \div \left(\frac{1}{\log x} + \frac{n}{1} + \log x \frac{n^2}{1 \cdot 2} + \&c.\right),$$

a quantity which becomes evanescent when x is infinite.

Examples for Practice.

1. To find the value of $\dfrac{x - x^{2n+1}}{1 - x^2}$ when $x = 1$? Ans. n.

2. To find the value of $\dfrac{a\sqrt{ax} - x^2}{a - \sqrt{ax}}$ when $x = a$? Ans. $3a$.

3. To find the value of $\dfrac{a\sqrt[3]{(4a^3 + 4x^3)} - ax - a^2}{\sqrt{(2a^2 + 2x^2)} - x - a}$ when $x = a$? Ans. $2a$.

4. To find the value of $\dfrac{\sqrt{(a^2 + ax + x^2)} - \sqrt{(a^2 - ax + x^2)}}{\sqrt{(a + x)} - \sqrt{(a - x)}}$ when $x = 0$? Ans. $\sqrt{a}$.

5. To find the value of $\dfrac{a^n - x^n}{\log a - \log x}$ when $x = a$? Ans. na^n.

6. To find the value of $\dfrac{\log x}{\sqrt{(1 - x)}}$ when $x = 1$? Ans. 0.

7. To find the value of $\dfrac{e^x - 1 + \log(1 + x)}{x^2}$ when $x = 0$? Ans. 1.

8. To find the value of $\dfrac{1 - \frac{2x}{\pi}}{\cot x}$ when $x = \dfrac{\pi}{2}$? Ans. $\dfrac{2}{\pi}$.

9. To find the value of $\dfrac{x}{x - 1} - \dfrac{1}{\log x}$ when $x = 1$? Ans. $\dfrac{1}{2}$.

10. To find the value of $\dfrac{(a^2 - x^2)^{\frac{1}{2}} + (a - x)^{\frac{3}{2}}}{(a - x)^{\frac{1}{2}} + (a^3 - x^3)^{\frac{1}{2}}}$ when $x = a$?

Ans. $\dfrac{(2a)^{\frac{1}{2}}}{1 - (3a^2)^{\frac{1}{2}}}$.

11. To find the value of $\dfrac{\pi}{4x} - \dfrac{\pi}{2x(e^{\pi x} + 1)}$ when $x = 0$? Ans. $\dfrac{\pi^2}{8}$.

12. To find the value of $\dfrac{1}{2x^2} - \dfrac{\pi}{2x \tan \pi x} - \dfrac{1}{1 - x^2}$ when $x = 1$?

Ans. $\frac{3}{4}$.

MAXIMA AND MINIMA OF FUNCTIONS.

79. Def.— If any quantity first increases to a certain limit, and afterwards decreases when it arrives at that limit, at the end of its increase it is said to be a *maximum*. If it first decreases to a certain limit, and afterwards increases, when it arrives at that limit it is said to be a *minimum*.

Let us take, for example, the function $y = b - (x - a)^2$. If we suppose $x = 0$, then $y = b - a^2$. If x increases from 0 to a, $(x - a)^2$ decreases, and therefore y will continually increase from $y = b - a^2$ to $y = b$. When $y = b$, it is a *maximum*; for x being supposed to become greater than a, y will be less then b. By supposing x to increase till $(x - a)^2$ becomes equal to b, y will decrease to 0; and x being still supposed to increase, y will become negative.

Let us next suppose $y = b + (x - a)^2$. In this case, when $x = 0$, $y = b + a^2$; as x increases $(x - a)^2$ decreases, and, consequently, y decreases until $x = a$, and then $y = b$, a *minimum*; for x becoming greater than a, y becomes greater than b.

80. Every function that either increases or decreases continually has neither *maximum* nor *minimum*; for whatever value such a function may acquire, in the one case it may always have a greater, and in the other a less value.

The characteristic property of a maximum consists in its being greater than the values which immediately precede, and also greater than the values which immediately follow it; and that of a minimum consists in its being less than the values immediately preceding, and also less than the values immediately following it.

If a function first increases to a certain limit and then decreases, and afterwards increases again indefinitely, it will at length exceed the maximum which it had before attained. Hence, a function may have values greater than its maximum and less than its minimum; and it is easy to conceive that a function may increase and decrease alternately several times; in such a case it must be considered as having several *maxima* and *minima*.

81. Prop. IV.—*To determine the maximum or minimum of* u *a function of* x.

Let $u = f(x)$ any function of x, and suppose that u has acquired the value which is either a maximum or a minimum. Let $x + h$ be substituted for x; then we have, by Taylor's theorem,

$$\begin{aligned} f(x + h) &= u + ph + qh^2 + rh^3 + \&c. \\ &= u + h(p + qh + rh^2 + \&c.) \end{aligned}$$

where $p = \dfrac{du}{dx}$, $q = \frac{1}{2}\dfrac{d^2u}{dx^2}$, &c. Let $-h$ be substituted for h, and we shall then have

$$\begin{aligned} f(x - h) &= u - ph + qh^2 - rh^3 - \&c. \\ &= u + h(-p + qh - rh^2 + \&c.) \end{aligned} \qquad 2.$$

Now, if p have a finite value, it is evident that h may be taken so

small that the signs of the two series $p + qh + rh^2 + \&c.$, and $-p + qh - rh^2 + \&c.$, shall depend on the signs of their first term. And since the sign of p is $+$ in the one series and $-$ in the other, it follows that one of the functions $f(x + h)$, $f(x - h)$, is greater and the other less than u. But this is contrary to the nature of a maximum or a minimum; and, therefore, p cannot have any finite value.

But if $p = 0$, we have

$$f(x + h) = u + qh^2 + rh^3 + \&c. = u + h^2(q + rh + \&c.)$$
$$f(x - h) = u + qh^2 + rh^2 + \&c. = u + h^2(q - rh + \&c.)$$

and since q has the same sign in the two series $q + rh + \&c.$, and $q - rh + \&c.$, h may evidently be taken so small that these two series shall have the same sign; and, therefore, $f(x + h)$, and $f(x - h)$ will be both greater than u when q is positive, and both less when it is negative. In the first case, therefore, u will be a *minimum*, and in the second case a *maximum*.

Let us now suppose that the value of x, which makes $p = 0$, causes at the same time q to vanish; we have then

$$f(x + h) = u + rh^3 + sh^4 + \&c. = u + h^3(r + sh + \&c.)$$
$$f(x - h) = u - rh^3 + sh^4 - \&c. = u + h^3(-r + sh - \&c.)$$

and since the first terms r, $-r$, of the two series within the brackets, have different signs, h may be taken so small that one of the functions $f(x + h)$, $f(x - h)$ will be greater and the other less than u; and, consequently, u is neither a maximum nor a minimum.

From what has been said it will readily appear, that, *To determine the maximum or minimum of* u *a function of* x, *we must put* $\frac{du}{dx} = 0$; *and if the first of the differential coefficients which is finite be of an* even *order,* u *will be a maximum when this coefficient is negative, and a minimum when it is positive; but if this differential coefficient be of an* odd *order,* u *will be neither a maximum nor a minimum.*

82. Prop. V.—*To determine the maximum or minimum of a function when Taylor's theorem fails.*

In the preceding article we have supposed, when a particular value was substituted for x in the differential coefficients, that they were either evanescent or finite. But when Taylor's theorem fails, the differential coefficients become infinite, on the substitution of particular values for x, and in this case the reasoning in the last proposition will not apply. We may, however, determine the maxima and minima of the function u in the following manner, which is applicable to all cases whatever.

If we suppose u to have its maximum value, it will continue to decrease for some time afterwards: so that if we take the function u not at its maximum state, but immediately afterwards, when Taylor's theorem does not fail, u will be $> f(x + h) > u + ph + qh^2 + \&c.$; consequently $ph + qh^2 + \&c.$ must be negative. But since h may be taken so small that the sign of $ph + qh^2 + \&c.$ depends upon the sign of the first term, ph will be negative; and, therefore, p or $\frac{du}{dx}$ is negative. I

appears, therefore, that if u be a maximum, the value of $\frac{du}{dx}$ *immediately afterwards* will be negative. In the same manner it may be proved that the value of $\frac{du}{dx}$ *immediately before* the maximum state of u will be positive. Hence, when the function $f(x)$ passes through its maximum state, the sign of $\frac{du}{dx}$ passes from $+$ to $-$. Now, a function can only change its sign, either in passing through 0 when the numerator vanishes, or in passing through infinity when the denominator vanishes. Hence, to find the maximum value of u we must make $\frac{du}{dx}$ equal either to 0 or infinity.

In like manner it may be shown, that in the case of a minimum $\frac{du}{dx}$ is negative before this state and positive afterwards; and, consequently, $\frac{du}{dx}$ in this case will also be equal either to 0 or infinity. Hence, therefore, *to find the maximum or minimum of* u *we must make* $\frac{du}{dx}$ *equal either to nothing or infinity. And if the sign of* $\frac{du}{dx}$ *be changed in passing through this state,* u *will be either a maximum or a minimum; but if the sign of* $\frac{du}{dx}$ *continue the same in passing through this state,* u *will be neither a maximum nor a minimum.*

83. In the application of the preceding propositions the following rules will be found extremely useful in making the operations more simple.

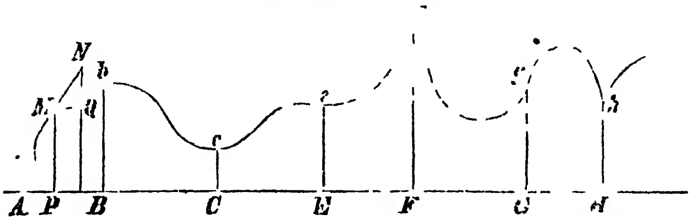
1st. If any function u be a maximum or a minimum, then will Au or $\frac{u}{A}$ be a maximum or a minimum, A being a constant quantity. By this rule we may suppress all constant multipliers and divisors.

2nd. If any function u be a maximum or a minimum, then will u^n be a maximum or a minimum, n being any positive number, integral or fractional. On the contrary, u^{-n} will be a minimum when u is a maximum, and a maximum when u is a minimum. By means of this rule radicals may be removed.

3rd. If any function u be a maximum or a minimum, then will the logarithm of u be also a maximum or a minimum. Hence, when u consists of a number of factors, these may be separated from each other by taking the logarithm of u, and the operation will become less complicated.

Scholium.

84. The preceding propositions may be simply illustrated from the theory of curve lines.



Let AP, measured along the axis AH, be taken $= x$, $PM = u$ any function of x; and let $Mbc \ldots$ be the curve traced by the extremity of the line PM. Also, let MN be a tangent to the curve at any point M, and MQ parallel to the axis AP. Then it will be shown, in art. 93, that $\frac{du}{dx} = \tan \angle NMQ$. Now, if Bb be a maximum, or b the *highest* point in the curve, the tangent at b will be parallel to the axis AH, and, therefore, $\frac{du}{dx} = 0$. In like manner, if Cc be a minimum, or c the lowest point in the curve, the tangent at c will be parallel to the axis, and, in this case, $\frac{du}{dx} = 0$. Lastly, if e be a point of inflexion, and the tangent at e be parallel to AH, $\frac{du}{dx} = 0$ in this case; but Ee is neither a maximum nor a minimum. Thus it appears, that the equation $\frac{du}{dx} = 0$ merely indicates that the tangent at the corresponding point in the curve is *parallel to the axis*; in which case u may either be a maximum or a minimum, or it may be neither.

If $\frac{du}{dx}$ be infinite, u may either be a maximum, as at F; or a minimum, as at H; or it may be neither, as at G, where the point g in the curve is a point of inflexion.

It appears also, from art. 94, that at b, where the curve is concave to the axis, the second differential coefficient $\frac{d^2u}{dx^2}$ is negative; and at c, where the curve is convex to the axis, this coefficient is positive. Also, at the point of inflexion e, the second differential $d^2u = 0$.

Ex. 1.

85. To divide a quantity into two such parts that the cube of the one multiplied by the square of the other shall be the greatest possible.

Let a be the given quantity, and x one of its parts; then will $a - x$ be the other part, and by the question $x^3(a - x)^2$ is to be a maximum. Put $u = x^3(a - x)^2$, we have then

$$\frac{du}{dx} = 3x^2(a - x)^2 - 2x^3(a - x) = x^2(a - x)(3a - 3x - 2x),$$

and in the case of a maximum $\frac{du}{dx} = 0$; putting, therefore, each of these factors $= 0$, we have

$$x = 0, \qquad a - x = 0, \qquad 3a - 5x = 0.$$

To find whether these equations correspond to a maximum value of u, let us take the second differential coefficients,

$$\frac{d^2u}{dx^2} = 2x(a-x)\ (3a-5x) - x^2(3a-5x) - 5x^2(a-x).$$

If $x = 0$, $\dfrac{d^2u}{dx^2} = 0$, therefore nothing can be inferred in this case.

If $x = a$, $\dfrac{d^2u}{dx^2} = -a^2(3a-5a)$, which is $+$, $\therefore$ u is a minimum.

If $x = \dfrac{3a}{5}$, $\dfrac{d^2u}{dx^2} = -\dfrac{45a^2}{25}\left(a - \dfrac{3a}{5}\right)$, which is $-$, $\therefore$ u is a maximum.

If the third differential of u be taken, and 0 be substituted instead of x, $\dfrac{d^3u}{dx^3}$ will be found $= 6a^2$; consequently the corresponding value of is neither a maximum nor a minimum.

Ex. 2.

86. To find when $u = x^4 - 8x^3 + 22x^2 - 24x + 12$ is a maximum or a minimum.

$$\frac{du}{dx} = 4x^3 - 24x^2 + 44x - 24 = 4(x-3)(x-2)(x-1).$$

The roots of the equation $\dfrac{du}{dx} = 0$, therefore, are 3, 2, and 1.

$$\frac{d^2u}{dx^2} = 4(x-2)(x-1) + 4(x-3)\ (x-1) + 4(x-3)(x-2).$$

If $x = 3$, $\dfrac{d^2u}{dx^2} = 8$, therefore u is a minimum.

If $x = 2$, $\dfrac{d^2u}{dx^2} = -4$, therefore u is a maximum.

If $x = 1$, $\dfrac{d^2u}{dx^2} = 8$, therefore u is a minimum.

87. *Scholium*.—If a, b, c, &c., be the roots of the equation $\dfrac{du}{dx} = 0$, then will $\dfrac{du}{dx} = A(x-a)\ (x-b)\ (x-c)\ldots$; therefore,

$$\frac{d^2u}{dx^2} = \begin{cases} A(x-b)\ (x-c)\ldots \\ + A(x-a)\ (x-c)\ldots \\ + A(x-a)\ (x-b)\ldots \end{cases}$$

Now this is evidently the same expression as that which we have given in the Algebra (art. 333); and if A be positive, and a, b, c, &c. be substituted successively for x, the results will be

$A(a-b)\ (a-c)\ldots$ which is positive.
$A(b-a)\ (b-c)\ldots$ - negative.
$A(c-a)\ (c-b)\ldots$ - positive, and so on.

Hence the greatest root a will make u a minimum, b will make u a maximum, c a minimum, and so on. It appears, therefore, that if there be several unequal and real roots, these roots, taken in order, will make u alternately a minimum and a maximum.

If, in the same equation, p roots be equal to e, there will be one maximum or minimum value of u corresponding to e when p is an odd number, but neither a maximum nor minimum if p be an even number.

Ex. 3.

88. To find when $u = b + 6(x - a)^{\frac{2}{3}}$ is a maximum or a minimum.

Here we have $\frac{du}{dx} = \frac{4}{(x-a)^{\frac{1}{3}}}$; put this $= \infty$ (art. 82), then will $(x - a)^{\frac{1}{3}} = 0$, and $x = a$.

To find if u be a maximum or minimum, substitute $a - h$ and $a + h$ for x in $\frac{du}{dx}$, and the results are $\frac{4}{(-h)^{\frac{1}{3}}} = -\frac{4}{h^{\frac{1}{3}}}$ and $\frac{4}{h^{\frac{1}{3}}}$. Hence the corresponding value of u is a minimum.

Ex. 4.

89. Let u be such a function of x that $u^2 - 2mxu + x^2 - a^2 = 0$; it is required to determine when u is a maximum or a minimum.

From this equation we obtain $\frac{du}{dx} = \frac{mu - x}{u - mx} = 0$; therefore,

$$mu - x = 0, \quad \text{and} \quad u = \frac{x}{m}.$$

$mu - x$

Substitute this value in the given equation, and we have

$$\frac{x^2}{m^2} - 2x^2 + x^2 - a^2 = 0;$$

$$\therefore \quad x = \frac{ma}{\sqrt{(1 - m^2)}}, \quad u = \frac{a}{\sqrt{(1 - m^2)}}.$$

Taking the second differential of the proposed equation, we have

$$(u - mx)\frac{d^2u}{dx^2} + \frac{du^2}{dx^2} - 2m\frac{du}{dx} + 1 = 0,$$

$$\text{or,} \quad (u - mx)\frac{d^2u}{dx^2} + 1 = 0, \quad \text{since} \quad \frac{du}{dx} = 0;$$

$$\therefore \quad \frac{d^2u}{dx^2} = \frac{-1}{u - mx} = -\frac{1}{a\sqrt{(1 - m^2)}};$$

and as this result is negative, we conclude that the value which we have found for u is a maximum.

Examples for Practice.

1. To divide a given line a into two such parts that their rectangle may be a maximum.

2. To find when $x^3 - 6x^2 + 9x + 10$ is a maximum, and when it is a minimum.

3. To find when $x^5 - 5x^4 + 5x^3 + 1$ is either a maximum or a minimum.

4. To find the number which bears the least ratio to its logarithm.
Ans. $e = 2{\cdot}718\ldots$

5. To divide the number a into such a number of equal parts that their continued product may be a maximum.

Ans. Number of parts $= \dfrac{a}{e}$.

6. To find when $x \sin x$ is a maximum. Ans. $x = 116^\circ\ 14'\ 21''$.

7. To find when $\sin^m x \sin^n (a - x)$ is a maximum.

Ans. $\sin(a - 2x) = \dfrac{n - m}{n + m} \sin a$.

8. To bisect a triangle by the shortest line.

Ans. The length of this line is $\sqrt{\tfrac{1}{2}(a + b - c)(a + c - b)}$.

9. Given the base and altitude of a triangle, to describe it so that the vertical angle may be a maximum. Ans. The triangle is isosceles.

10. To find the least isosceles triangle which can circumscribe a given circle. Ans The triangle is equilateral.

11. Given the length of a circular arc $= 2a$, to find what portion of a circle it must be, so that the corresponding segment may be a maximum. Ans. The radius of the circle $= \dfrac{2a}{\pi}$.

12. To find the point P in the line CD, so that if A and B be given points in the same plane with CD, $m.AP + n.BP$ may be a minimum.

13. To find the greatest area that can be included by four given lines. Ans. The trapezium is such as may be inscribed in a circle.

14. Of all right cones whose convex surface is given, to find that whose solid content is a maximum.

Ans. Half the vertical angle of the cone is $35^\circ\ 16'$ nearly.

15. To cut the greatest parabola and the greatest ellipse from a given right cone.

16. To find the dimensions of a cylindrical imperial quart, which shall contain the least quantity of silver of a given thickness.

17. Given the altitude of an inclined plane, to find its length, so that a weight P acting freely upon another weight W in a line parallel to the plane may draw it up through the inclined plane in the least time possible.

Ans. $\dfrac{2Wa}{P}$.

18. Given the base BC, to find the altitude AB, such that a body descending from A to B, and thence describing BC with the velocity acquired, the time through AB and BC may be the least possible.

Ans. Altitude $=$ half the base.

19. To find the straight line of quickest descent from a given point to a given circle in the same vertical plane.

20. Given the content of a right cone, to find the base and height

when the time of its vibration shall be a minimum, supposing the point of suspension to be at the vertex.

Ans. Rad. of base = height $\times \sqrt{2}$.

MAXIMA AND MINIMA OF FUNCTIONS CONTAINING TWO OR MORE INDEPENDENT VARIABLES.

90. PROP. VI.—*To determine the maxima and minima of* u *a function of two independent variables.*

Let $u = f(x, y)$ be any function of two independent variables x and y. Substitute $x + h$ and $y + k$ for x and y, then, by art 62, u will be changed into a series of this form,

$$u' = A + Bh + Ck + \tfrac{1}{2}(Dh^2 + 2Ehk + Fk^2) + \&c.,$$

where $A = u$, $B = \dfrac{du}{dx}$, $C = \dfrac{du}{dy}$, $D = \dfrac{d^2u}{dx^2}$, $E = \dfrac{d^2u}{dx\,dy}$, &c.

Now, in order that u may be a maximum or a minimum, u' must be always less or always greater than u, whatever be the values given to the increments h and k, when they are taken very small; but this is only possible when $Bh + Ck$, or (if $k = mh$) when $h(B + Cm)$ is evanescent. For if $B + Cm$ be finite, then, if h be sufficiently small, this term may be rendered greater than the sum of all the following terms; and, therefore, by taking h successively positive and negative, u' will become in one case greater and in the other less than u. Hence, in order that u may be a maximum or a minimum, we must have

$$h(B + Cm) = 0, \quad \text{or} \quad B + Cm = 0.$$

Also, since h and k are entirely independent of each other, m is arbitrary, and, consequently, it follows that*

$$B = \frac{du}{dx} = 0; \qquad C = \frac{du}{dy} = 0.$$

Now, in order to determine whether these conditions make u a maximum or a minimum, or neither, we must take the second differential,

$$\tfrac{1}{2}(Dh^2 + 2Ehk + Fk^2) = \tfrac{1}{2}h^2(D + 2Em + Fm^2),$$

which, if it does not vanish, must always have the same sign, in the case of a maximum or a minimum, whatever value be given to m. Let α, β be the two roots of the quadratic equation $Fm^2 + 2Em + D = 0$, then, by the theory of equations (Alg. art. 270), $Fm^2 + 2Em + D = F(m - \alpha)(m - \beta)$, and if the roots α, β are real, it is evident that, by substituting different values for m, this expression will be sometimes positive and sometimes negative; and, therefore, u will be neither a maxi-

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* This may be shown as follows:—Since m is arbitrary, substitute $2m$ for m; we then have $B + 2Cm = 0$. From this equation subtract $B + Cm = 0$, and we get $Cm = 0$, and, therefore, $C = 0$. Hence also $B = 0$. And in the same manner, if there be any equation,

$$B + Cm + Dn + Ep + \&c. = 0,$$

in which B, C, D, &c. are functions of x, y, &c., and m, n, p, &c. are perfectly arbitrary and independent of these quantities, then will

$$B = 0, \quad C = 0, \quad D = 0, \quad \&c.$$

mum nor a minimum. But if α, β be imaginary, $(m-\alpha)(m-\beta)$ will be always positive (Alg. art. 326) whatever number be substituted for m; therefore, $Fm^2+2Em+D$ will always have the same sign as F, and, consequently, u will be a maximum when F is negative, and a minimum when it is positive. Now, the two roots of the equation $Fm^2+2Em+D=0$ are $\dfrac{-E\pm\sqrt{(E^2-DF)}}{F}$, and these are evidently imaginary when D and F have the same sign, and DF is $>E^2$.

If the coefficients of the second order vanish at the same time with those of the first, it may be shown in the same manner that there can be neither a maximum nor a minimum, unless the first of the differential coefficients which are finite be of an even order, and the factors of this term be all imaginary.

The function u may also admit of a maximum or a minimum when $\dfrac{du}{dx}$, and $\dfrac{du}{dy}$, are infinite; but we must refer the student for further information on this subject to the Differential Calculus of Lacroix.

Ex.

91. To divide the quantity a into three such parts, x, y, $a-x-y$, that the product $x^3y^2(a-x-y)$ may be a maximum.

If we put $u=x^3y^2(a-x-y)$, we have

$$\frac{du}{dx}=3x^2y^2(a-x-y)-x^3y^2=x^2y^2(3a-4x-3y)=0,$$

$$\frac{du}{dy}=2x^3y\,(a-x-y)-x^3y^2=x^3y\,(2a-2x-3y)=0.$$

Hence $\quad 3a-4x-3y=0;\quad 2a-2x-3y=0;$

from which equations we find $\quad x=\frac{1}{2}a,\ \ y=\frac{1}{3}a,\ \ a-x-y=\frac{1}{6}a.$

In order to discover whether these values belong to a maximum or a minimum, we must substitute them in the general expressions of D, E, F. Now,

$$D=\frac{d^2u}{dx^2}=2xy^2(3a-4x-3y)-4x^2y^2=-4x^2y^2;$$

because $3a-4x-3y=0$. Also

$$E=\frac{d^2u}{dx\,dy}=2x^2y(3a-4x-3y)-3x^2y^2=-3x^2y^2;$$

$$F=\frac{d^2u}{dy^2}=\quad x^3(2a-2x-3y)-3x^3y=-3x^3y.$$

And since the quantities D and F are both negative, and

$$DF=12x^5y^3=\frac{12a^8}{32\times27}=\frac{2a^8}{144},$$

$$E^2=\ 9x^4y^4=\frac{9a^8}{16\times81}=\frac{a^8}{144};$$

therefore $DF>E^2$, and u is the required maximum.

Examples for Practice.

1. To divide the quantity a into three such parts that their continued product may be a maximum. Ans. The three parts are equal.

2. To find when $u = x^2 + xy + y^2 - 6x - 9y$ is a maximum or a minimum. Ans. $x = 1$, $y = 4$, make u a minimum.

3. To find when $u = x^3 + y^3 - 15xy$ is a maximum or a minimum.
Ans. If $x = 0$, $y = 0$, u is neither a maximum nor a minimum; if $x = 5$, $y = 5$, u is a [redacted]n.

4. To find when $u = y^4 - 8y^3 + 18y^2 - 8y + x^3 - 3x^2 - 3x$ is a maximum or minimum.

5. Of all triangles having the same perimeter, to determine that whose area is the greatest. Ans. The triangle is equilateral.

6. To determine the dimensions of a rectangular cistern, capable of holding a given quantity of water, so as to be lined with lead at the least expense.

7. To find a point within a triangle from which, if lines be drawn to the angular points, the sum of their squares shall be the least possible. Ans. The point is the centre of gravity of the triangle.

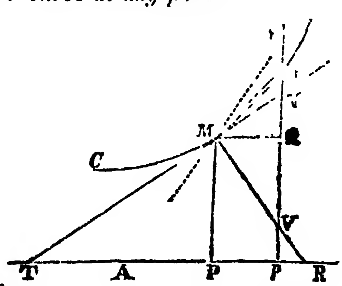
8. Given the weights of two balls A and B, to find two intermediate balls x and y, so that the motion communicated from A to B through x and y may be the greatest possible, the balls being supposed to be perfectly elastic. Ans. A, x, y and B are in continued proportion.

CHAP. V.—THEORY OF CURVES.

METHOD OF DRAWING TANGENTS TO CURVES.

92. Def.—A tangent at any point of a curve is a straight line passing through that point, such that no other line can be drawn between it and the curve.

93. Prop. I.—*To draw a tangent to a curve at any point.*



Let CMm be any curve line defined by an equation between the two co-ordinates AP, PM. Draw any other ordinate pm, and draw MQ parallel to Ap. Put $AP = x$, $PM = y$, and Pp the increment of $x = h$. Then, since y is evidently a function of x, we have, by Taylor's theorem,

$$pm = y + ph + qh^2 + rh^3 + \&c.$$

where $p = \dfrac{dy}{dx}$, $q = \frac{1}{2}\dfrac{d^2y}{dx^2}$, $r = \frac{1}{6}\dfrac{d^3y}{dx^3}$, &c. Let the line MN be

drawn so that p shall be equal to the trigonometrical tangent of the angle QMN; then shall MN be a tangent to the curve at the point M. For let any other line, such as Mn, be drawn through the point M, and let A be the trigonometrical tangent of the angle QMn. Then

$$QN = MQ \times \tan QMN = ph; \; Qn = Ah; \quad \therefore \; Nn = (A-p)h.$$

$$\text{Also, } Qm = ph + qh^2 + rh^3 + \&c.; \quad \therefore \; Nm = qh^2 + rh^3 + \&c.$$

Now, if none of the coefficients $p, q, r,$ &c. become infinite, it is manifest that h may be taken so ~~small that~~ $A-p$ shall be $> qh + rh^2 + \&c.$, or

$$h(A-p) > h(qh + rh^2 + sh^3 + \&c.); \text{ or } Nn > Nm.$$

Hence, therefore, whatever be the value of A, the straight line Mn cannot pass between MN and the curve Mm. In like manner it may be shown that it cannot pass on the other side between MT and the curve MC, and, therefore, by the definition, MT is a tangent to the curve.

94. *Cor.*—Since $pN = y + ph$, and $pm = y + ph + qh^2 + rh^3 +$ &c. therefore,

$$pm - pN = qh^2 + rh^3 + \&c.;$$

and because h may be taken so small that the first term shall be greater than the sum of all the other terms, it follows that the sum of this series shall have the same sign as the first term qh^2. If q, therefore, be positive, pm will be $> pN$, and the curve will be convex to the axis; but if q be negative, pm will $< pN$, and the curve will be *concave*. If the curve be situated on the other side of the axis AP, y will be negative, and, when q is also negative, pm will be greater in magnitude than pN, and the curve will be convex to the axis. Hence it appears that *a curve is convex or concave to the axis according as* y *and* d^2y *have the same or different signs.*

95. Prop. II.—*To find expressions for the tangent, subtangent, normal, and subnormal, at any point of a curve.*

Because the angle $PTM = QMN$, we have

$$\frac{MP}{PT} = \tan PTM = \tan QMN = \frac{dy}{dx}, \text{ and, therefore, the}$$

$$\text{subtangent } PT = PM\frac{dx}{dy} = \frac{ydx}{dy} \ldots\ldots\ldots (1).$$

We have also, in the triangle PMT,

$$\text{tangent } MT = \sqrt{(PM^2 + PT^2)} = y\sqrt{\left(1 + \frac{dx^2}{dy^2}\right)} \ldots . (2).$$

Also, from the similar triangles PMT, PMR, we obtain the proportion $PR : PM :: PM : PT$, therefore the

$$\text{subnormal } PR = PM\frac{PM}{PT} = \frac{ydy}{dx} \ldots\ldots\ldots (3).$$

$$\text{normal } MR = \sqrt{(PM^2 + PR^2)} = y\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} \ldots . (4).$$

Cor.—Hence, if in any curve CM, PT be taken equal to $\frac{ydx}{dy}$, and

TM be joined, TM will be a tangent at M. Also, if PR be taken $= \frac{ydy}{dx}$, and MR be joined, this will be a normal to the curve.

96. Prop. III.—*To find the equations to the tangent and the normal.*

(1). Let x', y' be the co-ordinates of the point M; and x, y the co-ordinates of any point N in the tangent MN. From the theory of curve lines, art. 18, the equation to the straight line MN is

$$y - y' = A(x - x'),$$

where A is the trigonometrical tangent of the angle which MN makes with the axis of x, therefore $A = \frac{dy'}{dx'}$; hence the equation to the tangent is

$$y - y' = \frac{dy'}{dx'}(x - x') \dots\dots\dots\dots\dots (5).$$

(2). Again, by the theory of curve lines (art. 22), the equation to line perpendicular to the tangent is

$$y - y' = -\frac{1}{A}(x - x');$$

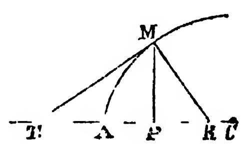
and, therefore, the equation to the normal is

$$y - y' = -\frac{dx'}{dy'}(x - x') \dots\dots\dots\dots (6).$$

Examples.

Ex. 1.—To draw a tangent and normal to the ellipse; that is, to find the subtangent and the subnormal.

The equation to the ellipse, when the co-ordinates are measured from A, the extremity of the transverse axis is (Ellipse, prop. 18)



$$y^2 = \frac{b^2}{a^2}(2ax - x^2), \therefore \frac{dx}{dy} = \frac{a^2}{b^2} \cdot \frac{y}{a-x}.$$

$$\text{Hence } PT = \frac{ydx}{dy} = \frac{a^2}{b^2}\frac{y^2}{a-x} = \frac{2ax - x^2}{a - x};$$

$$\therefore \quad CT = (a - x) + PT = \frac{a^2}{a - x} = \frac{CA^2}{CP}.$$

$$\text{Again,} \quad PR = \frac{ydy}{dx} = \frac{b^2}{a^2}(a - x) = \frac{b^2}{a^2} \times CP.$$

2. To draw a tangent and normal to the parabola; that is, to find the subtangent and the subnormal.

3. To draw a tangent and normal to the circle; that is, to find the subtangent and the subnormal.

4. To find the subtangent and the subnormal to the hyperbola.

5. To draw a tangent to the cissoid of Diocles.

6. To draw a tangent to the hyperbola between the assymptotes.

7. To draw a tangent to a curve of the parabolic kind whose equation is $y = ax^n$.

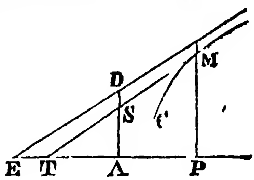
8. To draw a tangent to the trident whose equation is $axy=x^3-a^3$.

9. To draw a tangent to the logarithmic curve.

10. To draw a tangent to the cycloid.

ASYMPTOTES TO CURVES.

97. Prop. IV.—*To find whether a curve admits of an asymptote.*



When any curve CM has an asymptote, as the point M moves to a greater distance from A, the origin of the co-ordinates, the tangent MT continually approaches to the asymptote; and the points S and T continually approach to the points D and E; so that AD and AE are the limits to which the values of AS and AT approach nearer than by any assignable difference, but which they never actually attain. To determine, therefore, whether a curve has an asymptote, we must find the values of AS and AT, and see whether they are susceptible of limits. Now,

$$AT = PT - AP = \frac{y\,dx}{dy} - x.$$

$$AS = AT \tan ATS = AT\frac{dy}{dx} = y - \frac{x\,dy}{dx}.$$

If, therefore, we take the limits of these expressions when x is indefinitely great, we shall determine the values of AD and AE; and the line drawn through the points D, E will be the asymptote required.

Ex. 1.—Let $y^2 = mx + nx^2$, which is the general equation of lines of the second order. We have, from this equation,

$$\frac{dx}{dy} = \frac{2y}{m+2nx}; \quad \therefore \frac{y\,dx}{dy} = \frac{2y^2}{m+2nx} = \frac{2mx+2nx^2}{m+2nx};$$

$$\therefore \quad AT = \frac{2mx+2nx^2}{m+2nx} - x = \frac{mx}{m+2nx}.$$

In like manner $AS = y - \dfrac{mx+2nx^2}{2y} = \dfrac{mx}{2\sqrt{(mx+nx^2)}}$.

Dividing each term in these functions by x, we obtain

$$AT = \frac{m}{\frac{m}{x}+2n}; \quad AS = \frac{m}{2\sqrt{\left(\frac{m}{x}+n\right)}},$$

and, when x becomes infinite, their respective limits are

$$AE = \frac{m}{2n}; \quad AD = \frac{m}{2\sqrt{n}}.$$

If n be negative, the value of AD is imaginary, and, therefore, the curve, which is an ellipse, has no asymptote.

If $n = 0$, the curve is a parabola, and AD and AE are infinite; therefore the parabola has no asymptote.

If n be positive, the curve is a hyperbola, and AD and AE are finite. Hence, therefore, the hyperbola is the only curve of the second order which has an asymptote.

Ex. 2.—To determine the asymptote of the curve whose equation is $y^3 = ax^2 + x^3$.

Ex. 3. To determine the two asymptotes of the curve whose equation is $y = \dfrac{a^2}{x} + 2a + x$.

THE DIFFERENTIALS OF CURVE LINES.

98. PROP. V.—*To find an expression for the differential of the arc of any curve.*

Let CM any arc of a curve $= s$ (see the figure to art. 93), $AP = x$, $PM = y$; also, let the increment of the abscissa $Pp = h$; then, since the arc is evidently a function of the abscissa, when x becomes $x + h$, the arc s will become, by Taylor's theorem,

$$s + \frac{ds}{dx}\frac{h}{1} + \frac{d^2s}{dx^2}\frac{h^2}{1 \, . \, 2} + \frac{d^3s}{dx^3}\frac{h^3}{1 \, . \, 2 \, . \, 3} + \&c.,$$

$$\therefore \quad \text{arc } Mm = \frac{ds}{dx}\frac{h}{1} + \frac{d^2s}{dx^2}\frac{h^2}{1 \, . \, 2} + \frac{d^3s}{dx^3}\frac{h^3}{1 \, . \, 2 \, . \, 3} + \&c.$$

Now (Geom. prop. 88), the arc Mm is $>$ chord Mm, and $< MN + Nm$. And since (art. 93)

$$Qm = ph + qh^2 + rh^3 + \&c. = ph + Ph^2;$$

putting $qh^2 + rh^3 + \&c. = Ph^2$, we have, therefore, chord $Mm =$

$$\sqrt{(QM^2 + Qm^2)} = \sqrt{h^2 + (ph + Ph^2)^2} = h\sqrt{(1 + p^2) + 2Pph + P^2h^2},$$

and this expression for the chord Mm, when expanded, is evidently of the form

$$h\sqrt{(1 + p^2)} + Ah^2 + Bh^3 + \&c.$$

Also, $MN = \sqrt{(QM^2 + QN^2)} = \sqrt{(h^2 + p^2h^2)} = h\sqrt{(1 + p^2)}$,

and $\qquad Nm = Qm - QN = qh^2 + rh^3 + \&c.$

$$\therefore \quad MN + Nm = h\sqrt{(1 + p^2)} + qh^2 + rh^3 + \&c.$$

Hence, then, the series $\dfrac{ds}{dx}\dfrac{h}{1} + \dfrac{d^2s}{dx^2}\dfrac{h^2}{1 \, . \, 2} + \dfrac{d^3s}{dx^3}\dfrac{h^3}{1 \, . \, 2 \, . \, 3} + \&c.$,

is always contained between the values of

$h\sqrt{(1 + p^2)} + Ah^2 + Bh^3 + \&c.$, and $h\sqrt{(1 + p^2)} + qh^2 + rh^3 + \&c.$;

and, therefore, it follows, from art 52, that

$$\frac{ds}{dx} = \sqrt{(1 + p^2)} = \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}; \quad \therefore \quad ds = \sqrt{(dx^2 + dy^2)}.$$

CONTACT OF CURVES AND OSCULATING CURVES.

99. Prop. VI.—*To determine the conditions under which the contact of curves takes place.*

Let $y = f(x)$, and $u = F(x)$, be the equations to two curves which have a common abscissa x. Let x become $x + h$, then y and u will become

$$y' = y + ph + qh^2 + \&c.; \qquad u' = u + Ph + Qh^2 + \&c.$$

where $p = \dfrac{dy}{dx}$, $q = \frac{1}{2}\dfrac{d^2y}{dx^2}$, &c.; $P = \dfrac{du}{dx}$, $Q =$&c. Suppose, now, the curves to have a common point M, then will $y = u$, and if du be also $= dy$, the contact between the curves is such that no other curve, whose equation is $v = \varphi(x)$, drawn through the common point M, can pass between them, unless the differential of its ordinate $= dy$. For when x becomes $x + h$, the ordinate v will become $v' = v + p'h + q'h^2 + \&c.$, p' being $= \dfrac{dv}{dx}$, $q' = \frac{1}{2}\dfrac{d^2v}{dx^2}$, $r' =$ &c.; therefore

$$y' - v' = (p - p')h + (q - q')h^2 + \&c. = ah + bh^2 + \&c.,$$

by substitution. Also, since, by hypothesis, $du = dy$, therefore

$$y' - u' = (q - Q)h^2 + (r - R)h^3 + \&c. = Bh^2 + Ch^3 + \&c.$$

Now, it is evident that h may be taken so small that a shall be greater than $(B - b)h + (C - c)h^2 + \&c.$, or that

$$a + bh + ch^2 + \&c. \text{ shall be } > Bh + Ch^2 + \&c.;$$

$$\text{or,} \qquad ah + bh^2 + ch^3 + \&c. > Bh^2 + Ch^3 + \&c.,$$

that is, $y' - v'$ shall be $> y' - u'$. Hence it follows, that the curve whose equation is $v = \varphi(x)$ cannot pass between the curves whose equations are $y = f(x)$ and $u = F(x)$.

In the same manner it may be proved that if

$$y = u, \ \frac{dy}{dx} = \frac{du}{dx}, \ \text{and} \ \frac{d^2y}{dx^2} = \frac{d^2u}{dx^2},$$

no other curve drawn through the common point M can pass between them, unless $\dfrac{dv}{dx} = \dfrac{dy}{dx}$, and $\dfrac{d^2v}{dx^2} = \dfrac{d^2y}{dx^2}$: and similarly with respect to differentials of higher orders.

100. *Cor.* 1.—When $u = y$, and $\dfrac{du}{dx} = \dfrac{dy}{dx}$, the two curves have a common tangent, and the contact is said to be of the first order. When $u = y$, $\dfrac{du}{dx} = \dfrac{dy}{dx}$, and $\dfrac{d^2u}{dx^2} = \dfrac{d^2y}{dx^2}$, the contact is said to be of the second order; and so on.

101. *Cor.* 2.—The general equation to a straight line is $u = ax + b$, in which there are two constant quantities, a and b. If we consider these quantities as indeterminate, we may assume two equations, but not more, by means of which we may find their values (Alg. art. 84). If, then, we suppose, when they have a common abscissa, that $u = y$, $\dfrac{du}{dx} = \dfrac{dy}{dx}$, are these two equations, the position of the straight line will be com-

pletely determined. Hence, therefore, a straight line will, *in general*, admit only of a contact of the first order.

The parabolic curve, whose equation is $u = at^2 + bt + c$, will admit only of a contact of the second order. For, when $t = x$, if we make

$$u = y; \quad \frac{du}{dt} = \frac{dy}{dx}; \quad \text{and} \quad \frac{d^2u}{dt^2} = \frac{d^2v}{dx^2},$$

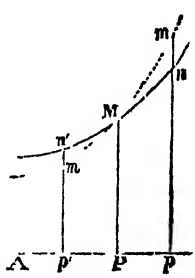
we have three equations, by means of which the three constant quantities a, b, c may be determined.

In like manner, the general equation to a circle being $(u - \beta)^2 + (t - \alpha)^2 = \rho^2$, in which there are three constant quantities, α, β, ρ; the circle, *generally*, will admit only of a contact of the second order. We say *generally*, because it is possible that, besides the equations

$$u = y, \quad \frac{du}{dt} = \frac{dy}{dx}, \quad \frac{d^2u}{dt^2} = \frac{d^2y}{dx^2},$$

from which the values of α, β, ρ, are determined, the equation $\frac{d^3u}{dt^3} = \frac{d^3y}{dx^3}$ may also be satisfied; in which case there will be a contact of the third order. As the curvature of the circle is the same at every point, it is usually taken to measure the degree of curvature at any point of a curve.

102. PROP. VII.—*When any two curves have a contact of an even order, they both touch and cut each other; and when the contact is of an odd order, they only touch each other.*



Let $y = f(x)$, and $u = F(x)$, be the equations to the two curves $m'Mm$ and $n'Mn$, which have a common point M. Let $AP = x$, $PM = y$, $Pp = Pp' = h$; and let us first suppose that the contact is of the second order, or that $u = y$, $du = dy$, $d^2u = d^2y$. We have, then (art 99),

$$pm - pn = (r - R)h^3 + (s - S)h^4 + \&c.;$$

and if the abscissa becomes $x - h$, or if we change h into $-h$, we shall then have

$$p'm' - p'n' = -(r - R)h^3 + (s - S)h^4 - \&c.$$

And since, by taking h sufficiently small, the first term of the series may be made greater than the sum of all the other terms, it follows that $pm - pn$ and $p'm' - p'n'$ will have different signs.

If, therefore, pm be $> pn$, $p'm'$ will be $< p'n'$, consequently the curves $m'Mm$, $n'Mn$, cut each other as well as touch each other at the point M.

But if the contact be of the third order, or $r - R$ also be $= 0$, then

$$pm - pn = (s - S)h^4 + \&c.; \quad p'm' - p'n' = (s - S)h^4 - \&c.$$

Hence, as the first terms in these two series have the same signs, if pm be $> pn$, $p'm'$ will also be $> p'n'$, therefore the curves $m'Mm$, $n'Mn$, will not cut each other; and in the same manner the proposition may be proved in other cases.

103. PROP. VIII.—*To find a general expression for the radius of curvature at any point of a curve.*

Let $y = f(x)$ be the equation to the curve, and let

$$(x-\alpha)^2 + (u-\beta)^2 = \rho^2 \quad \ldots\ldots\ldots (1)$$

be the equation to the circle of curvature, which is supposed to have the same abscissa as the curve. Now, by the definition, we have at the point of contact,

$$u = y, \qquad du = dy, \qquad \text{and} \quad d^2u = d^2y,$$

from which equations, and the equation to the curve, the values of the constant quantities α, β, ρ, are to be determined. By taking the first and second differentials of equation (1), we have

$$(x-\alpha)\,dx + (u-\beta)\,du = 0 \quad \ldots\ldots (2),$$
$$dx^2 + du^2 + (u-\beta)\,d^2u = 0 \quad \ldots\ldots (3)$$

From the last equation we obtain $u-\beta = -\dfrac{dx^2+du^2}{d^2u}$,

and from equation (2), $x-\alpha = -(u-\beta)\dfrac{du}{dx} = \dfrac{du}{dx}\cdot\dfrac{dx^2+du^2}{d^2u}$.

Substitute these values in equation (1), and we find

$$\rho^2 = \left(\frac{dx^2+du^2}{d^2u}\right)^2\left[1+\frac{du^2}{dx^2}\right] = \frac{(dx^2+du^2)^3}{(dx\,d^2u)^2},$$

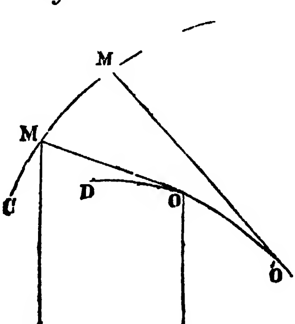
$$\therefore \rho = \pm\frac{(dx^2+du^2)^{\frac{3}{2}}}{dx\,d^2u}.$$

To determine which of the signs $\pm$ is to be used, we must consider that when the ordinate is positive, and the curve is concave to the axis, d^2u is negative (art. 94), and if we suppose ρ to be positive in this case, we must prefix the sign $-$ to d^2u to make it positive. Hence, by this conventional rule, $\rho = \dfrac{(dx^2+du^2)^{\frac{3}{2}}}{-dx\,d^2u}$.

And since $du = dy$, and $d^2u = d^2y$, we have these three equations to determine the values of ρ, α, and β,

$$\rho = \frac{(dx^2+dy^2)^{\frac{3}{2}}}{-dx\,d^2y} = -\frac{ds^3}{dx\,d^2y}. \quad \ldots\ldots (4),$$

$$x-\alpha = \frac{dy\,(dx^2+dy^2)}{dx\,d^2y}, \quad y-\beta = \frac{dx^2+dy^2}{d^2y}. \quad \ldots (5)$$



104. Def.—If the radius of curvature MO be supposed to be drawn at every point of the curve CMM', then DOO', the locus of the point O, will form a curve which is called the evolute of the curve CM, and the curve CM is called the involute of the curve DO.

105. Prop. IX.—*The radius of curvature* MO *is a normal to the involute* CM, *and a tangent to the evolute* DO.

(1). If we substitute y for u, and dy

for du, in equation (2), art. 103, and divide by dy, we shall obtain

$$\beta - y = -\frac{dx}{dy}(\alpha - x),$$

and this equation being the same with the equation to the normal (art. 95,) it is obvious that the point O, whose co-ordinates are α, β, is a point in the normal, and, therefore, the radius of curvature is a normal to the curve.

(2). Since the radius of curvature varies at every point of the curve, if we pass from one point of the evolute to another, not only will x and y vary, but α, β, ρ will also vary at the same time. Hence, therefore, differentiating equation (2), both with respect to x and y, and also α and β, and substituting y, dy, d^2y, for u, du, d^2u, we get

$$(dx - d\alpha)\,dx + (dy - d\beta)\,dy + (y - \beta)\,d^2y = 0.$$

And subtracting equation (3) from this equation, there remains,

$$-\,d\alpha\,dx - d\beta\,dy = 0, \quad \text{or} \quad \frac{d\beta}{d\alpha} = -\frac{dx}{dy}\ldots\ldots(7).$$

Substitute this value in equation (2), and we have

$$y - \beta = \frac{d\beta}{d\alpha}(x - \alpha)\ldots\ldots\ldots\ldots(8),$$

but this is the equation to the tangent of the curve DO at the point O, whose co-ordinates are α, β (art. 96), and, therefore, the radius of curvature MO is a tangent to the evolute.

106. Prop. X.—*The length of any arc OO′ of the evolute is equal to the difference of the radii* MO, M′O′.

If we differentiate equation (1), art. 103, making all the quantities x, y, α, β, ρ, vary, we obtain

$$(x - \alpha)(dx - d\alpha) + (y - \beta)(dy - d\beta) = \rho d\rho,$$

and subtracting equation (2) from this, we get

$$-(x - \alpha)\,d\alpha - (y - \beta)\,d\beta = \rho d\rho.$$

Substituting the value of $y - \beta$, derived from equation (8), both in the last equation, and also in equation (1), we have

$$\rho d\rho = -(x - \alpha)\,d\alpha - (x - \alpha)\frac{d\beta^2}{d\alpha} = -(x - \alpha)\,d\alpha\left(1 + \frac{d\beta^2}{d\alpha^2}\right),$$

$$\rho^2 = (x - \alpha)^2 + (x - \alpha)^2\frac{d\beta^2}{d\alpha^2} = (x - \alpha)^2\left(1 + \frac{d\beta^2}{d\alpha^2}\right).$$

Squaring the first term of these two equations, and dividing it by the corresponding terms of the last, we obtain

$$d\rho^2 = d\alpha^2 + d\beta^2, \quad \text{or} \quad d\rho = \sqrt{(d\alpha^2 + d\beta^2)}.$$

But, if s be the arc of the evolute, we have also $ds = \sqrt{(d\alpha^2 + d\beta^2)}$; $\therefore d\rho = ds$ or $d(\rho - s) = 0$. And since every function whose differential $= 0$ must be constant, we have $\rho - s = C$, a constant quantity. If we suppose $C = 0$, or the radius of curvature and the evolute

to commence together, we shall have $\rho = s$, or the radius of curvature equal to the length of the evolute.

Cor.—Hence, if we suppose a string to be wrapped round the curve DOO', and the line MO taken equal to the radius of curvature at O, if the string be unfolded, its extremity will describe the curve line MM'.

107. Prop. XI.—*To find the equation to the evolute, when that of the involute is known.*

Find the values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$, from the equation of the given curve. Substitute these values in the two equations marked (5), we shall then have two equations between x, α, and β. Eliminate x from these equations, and we shall obtain an equation between α and β, which will be that of the evolute.

Examples.

108. *Ex.* 1.—To find the radius of curvature at any point of the parabola.

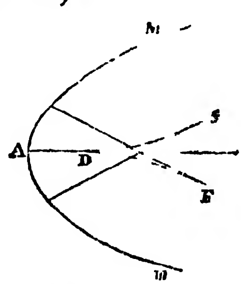
Since $y^2 = 2px$, $\therefore \frac{dy}{dx} = \frac{p}{y}$; and $\frac{d^2y}{dx^2} = -\frac{p}{y^2}\frac{dy}{dx} = -\frac{p^2}{y^3}$.

Also, $\frac{ds}{dx} = \sqrt{(1 + \frac{dy^2}{dx^2})} = \sqrt{(1 + \frac{p^2}{y^2})} = \frac{\sqrt{(p^2 + y^2)}}{y}$.

Hence $\rho = \frac{ds^3}{-dx\,d^2y} = -\frac{ds^3}{dx^3} \div \frac{d^2y}{dx^2} = \frac{(p^2 + y^2)^{\frac{3}{2}}}{p^2}$; that

$$\text{radius of curvature} = \frac{(\text{normal})^3}{(\text{semiparameter})^2}.$$

109. *Ex.* 2.—To find the equation to the evolute of the common parabola.



$$y - \beta = -\frac{dx^2 + dy^2}{d^2y} = \frac{p^2 + y^2}{y^2}\cdot\frac{y^3}{p^2} = y + \frac{y^3}{p^2}.$$

$$x - \alpha = \frac{dy(dx^2 + dy^2)}{dx\,d^2y} = -\frac{p}{y}\,\frac{p^2 + y^2}{y^2}\,\frac{y^3}{p^2}.$$

$$= -\frac{p^2 + y^2}{p};$$

$$\therefore -\beta = \frac{y^3}{p^2} = \frac{(2px)^{\frac{3}{2}}}{p^2}; \quad x - \alpha = -2 - \frac{y^2}{p} = -p - 2x.$$

From the last equation $x = \frac{1}{3}(\alpha - p)$, and from the preceding equation

$$\beta^2 = \frac{8x^3}{p} = \frac{8(\alpha - p)^3}{27p} = \frac{8\alpha'^3}{27p},$$

by putting $\alpha - p = \alpha'$, or transferring the origin of co-ordinats from A to D. Hence the evolute is a curve of the parabolic kinde, and

called a semicubical parabola. It is composed of the two branches DF and Df, the first of which generates by its developement the branch AM, and the other generates the branch Am.

Ex. 3.—To find the radius of curvature in the ellipse.

Ex. 4.—To find the radius of curvature in the hyperbola.

Ex. 5.—To find the radius of curvature to the hyperbola between the asymptotes.

Ex. 6.—To find the radius of curvature and the evolute of the cycloid.

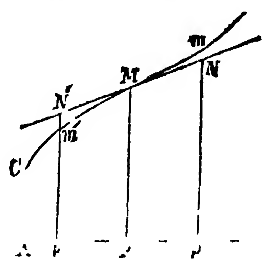
SINGULAR POINTS OF CURVE LINES.

110. Def.—Those points of a curve in which it undergoes any particular changes are called *singular points*. The differential calculus affords the simplest methods of determining their nature and position.

POINTS OF INFLEXION.

111. Def.—When a curve, from being concave to the axis, becomes convex, or from convex becomes concave, the point at which the change takes place is called *a point of inflexion*, or *a point of contrary flexure.*

112.—Prop. XII.—*To find the points of inflexion of a curve.*



(1). Let M be a point of inflexion in the curve CM; and $N'MN$ a tangent at this point. Let $AP = x$, $PM = y$, $Pp = h$; we have then (art. 91)

$$pm - pN = qh^2 + rh^3 + \&c.$$

In like manner, if $Pp' = h$, we shall find, by substituting $-h$ for h,

$$p'm' - p'N' = qh^2 - rh^3 + \&c.$$

Now, if q be finite, it is evident that h may be taken so small that the sign of each of these series shall depend upon the sign of the first term qh^2, and, therefore, $pm - pN$ and $p'm' - p'N'$ will have the same sign, and, consequently, the curve will be convex to the axis both before and after the point M, if q be positive (art. 94); and concave both before and after, if q be negative.

(2). But if $q = 0$, and r be finite, then these two series become $rh^3 + sh^4 + \&c.$ and $-rh^3 + sh^4 - \&c.$; and, since the first terms rh^3, $-rh^3$ have different signs, h may be taken so small that $pm - pN$ and $p'm' - p'N'$ shall have different signs, and, consequently, one of the points m, m' will be above the tangent $N'N$ and the other below it. Hence the curve is concave in one part and convex in the other, therefore M is a point of inflexion.

(3). If $q = 0$, $r = 0$, &c., then, in order that there may be a point of inflexion, the first differential coefficient that is finite must be of an odd order.

(). If the value of x in these series be such that q is infinite, then

each of the coefficients r, s, &c. also will be infinite, and, therefore, no conclusion can be drawn from the preceding demonstration. If, however, immediately before and after the point M, the coefficient q has changed its sign, then the curve is concave in one part and convex in the other. Now, if q be a fraction, it may change its sign either in passing through nothing or infinity; and, therefore, at a point of contrary flexure, q may be either 0 or ∞.

Hence, to determine the point of inflexion in a curve we must put $\frac{d^2y}{dx^2} = 0$, or $\frac{d^2y}{dx^2} = \infty$; and then find the values of this differential coefficient when x is increased or diminished by a very small quantity h: and if, for these new values of x, $\frac{d^2y}{dx^2}$ has different signs, there will be a point of inflexion.

Examples.

Ex. 1.—Let the equation to the curve be $y = ax + bx^2 - cx^3$, to find the point of contrary flexure.

We have $\frac{dy}{dx} = a + 2bx - 3cx^2$; $\frac{d^2y}{dx^2} = 2b - 6cx = 0$.

Hence $x = \frac{b}{3c}$ at the point of contrary flexure.

2. To find the point of contrary flexure in the curve whose equation is $y = x + 36x^2 + 2x^3 - x^4$.

3. To find the same when the equation is $y = 3x^5 - 35x^4 + 140x^3 - 240x^2$.

4. To find the same in a curve of the parabolic kind whose equation is $y = a + \sqrt[3]{(a^3 - 2a^2x + ax^2)}$.

MULTIPLE POINTS.

113. If several branches of a curve meet in one point, it is called a multiple point. If there be two branches which meet in this point, it is said to be a double point; if three branches, a triple point; and so on.

Multiple points are divided into two species,—those in which the branches cut each other, and those in which they touch each other.

114. PROP. XIII.—*To determine the multiple points in which the branches intersect each other.*

(1). When several branches of a curve intersect each other in one point, there will be as many tangents at this point as there are distinct branches of the curve; and, consequently, as many different values for $\frac{dy}{dx}$. Let $u = f(x, y) = 0$ be the equation to the curve; then du will evidently be of the form

$$Mdx + Ndy = 0; \quad \therefore \quad \frac{dy}{dx} = -\frac{M}{N} \ldots\ldots\ldots\ldots (1).$$

If, now, for any given values of x and y, M and N be rational and finite, $\frac{dy}{dx}$ can only have one value, and, therefore, there will be only one tangent at this point. But if M or N be irrational, $\frac{dy}{dx}$ may have several values, consequently there may be several branches passing through this point. Lastly, if $M = 0$, $N = 0$, we have $\frac{dy}{dx} = \frac{0}{0}$. To determine the true value of $\frac{dy}{dx}$ in this case, let us differentiate the equation $M + N\frac{dy}{dx} = 0$; we have then

$$\frac{dM}{dx} + 2\frac{dM}{dy}\frac{dy}{dx} + \frac{dN}{dy}\frac{dy^2}{dx^2} + N\frac{d^2y}{dx^2} = 0 \ldots\ldots\ldots (2)$$

for $\frac{dN}{dx} = \frac{d^2u}{dy\,dx} = \frac{dM}{dy}$: and since $N = 0$, this equation reduces itself to

$$\frac{dM}{dx} + 2\frac{dM}{dy}\frac{dy}{dx} + \frac{dN}{dy}\frac{dy^2}{dx^2} = 0.$$

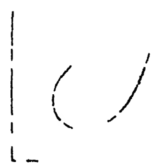
As this is a quadratic equation with respect to the unknown quantity $\frac{dy}{dx}$, if $\frac{dM}{dx}$, $\frac{dM}{dy}$, $\frac{dN}{dy}$, be rational, $\frac{dy}{dx}$ will have two values, and, consequently, there will be two branches of the curve corresponding to these values.

(2). If the terms of equation (2) all vanish, we must differentiate this equation again, and we shall then find an equation of the third degree, from which we may have three different values for the differential coefficient $\frac{dy}{dx}$.

115. *Ex.*—Let $y = x + (x - a)\sqrt{x - b}$;

$$\therefore \quad \frac{dy}{dx} = 1 + \sqrt{(x - b)} + \frac{x - a}{2\sqrt{(x - b)}};$$

and when $x = a$, y also $= a$, and $\frac{dy}{dx} = 1 \pm \sqrt{(a-b)}$.



The curve, therefore, has two branches passing through this point.

If the equation to the curve had been given in this form,

$$(y - x)^2 = (x - a)^2(x - b),$$

we should have had, in differentiating and dividing by 2,

$$\left(\frac{dy}{dx} - 1\right)(y - x) = (x - a)(x - b) + \tfrac{1}{2}(x - a)^2,$$

and when $x = a = y$, each term vanishes. Taking, therefore, the second differential of the given equation, we find

$$\frac{d^2y}{dx^2}(y - x) + \left(\frac{dy}{dx} - 1\right)^2 = 3x - 2a - b,$$

and when $x = a = y$, $\left(\frac{dy}{dx} - 1\right)^2 = a - b$,

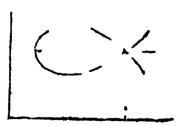
from which we obtain the same value of $\frac{dy}{dx}$ as before.

116. Prop. XIV.—*To find the multiple points when the branches of the curve touch each other.*

If several branches of the curve touch each other, and the contact be of the first order, $\frac{dy}{dx}$ will have only one value, but $\frac{d^2y}{dx^2}$ will have several values; if the contact of the curves be of the second order, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ will each have only one value; but $\frac{d^3y}{dx^3}$ will have several values and similarly with respect to the contacts of superior orders. Let us suppose the contact to be only of the first order, then the value of $\frac{d^2y}{dx^2}$ derived from equation (2) will have several values; and we may reason in the same way as in article 111, that this must arise either from the radicals contained in M and N, or else that $\frac{d^2y}{dx^2}$ is of the form $\frac{0}{0}$. In the latter case we shall have, from equation (2), $N = 0$, and, therefore, from equation (1), M also $= 0$. The following example will be sufficient to illustrate this case.

117. *Ex.*—Let $y = c + (x - a)^2\sqrt{x - b}$.

When $x = a$, $y = c$, $\frac{dy}{dx} = 0$; $\frac{d^2y}{dy^2} = \pm 2\sqrt{(a - b)}$.



Hence there is a double point of the second species at the point where $a = y$, $y = c$.

But if the equation to the curve be given in the form $(y - c)^2 = (x - a)^4 (x - b)$, we have, from the first differential equation, $\frac{dy}{dx} = 0$; from the second, $\frac{2dy^2}{dx^4} = 0$; from the third, $0 = 0$; and from the fourth, $6\left(\frac{d^2y}{dx^2}\right)^2 = 24(a - b)$.

The last equation gives $\frac{d^2y}{dx^2} = \pm 2\sqrt{(a-b)}$, the same result as before.

CUSPS OR POINTS OF REFLEXION.

118. A *cusp* or *point of reflexion* in a curve, is a double point of the second species in which the two touching branches terminate, and do not extend beyond this point in one direction. They are divided into two kinds; the first is when the two branches of the curve are on different sides of the common tangent, and the second is when they are on the same side of it.

It is evident that the curve does not extend beyond this point in one

direction, because the values of the ordinate are imaginary in this direction; and, therefore, some of the differential coefficients are imaginary. The kind of cusp will be determined by considering whether both the branches are concave or convex to the axis, or one is concave and the other convex.

119. *Ex.*—Let $(y - x)^2 = x^9$; or $y = x \pm x^4\sqrt{x}$.

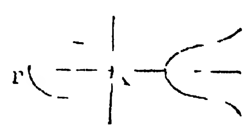
When x is negative y is imaginary, and, therefore, the curve stops at the origin where $x = 0$, $y = 0$. We have also,

$$\frac{dy}{dx} = 1 \pm \frac{9}{2}x^3\sqrt{x}; \quad \frac{d^2y}{dx^2} = \pm \frac{9 \cdot 7}{2 \cdot 2}x^2\sqrt{x}.$$

When, therefore, $x = 0$, $\frac{dy}{dx} = 1$, and the curve has only one tangent at this point. Also, all the other differential coefficients are either 0 or ∞, and, therefore, nothing can be concluded from these values. But if we take a point a little beyond the origin of co-ordinates, or make $x = 0 + h$, a very small quantity, $\frac{d^2y}{dx^2}$ has two values, one of which is positive and the other negative; hence the curve has two branches, the one concave and the other convex to the axis; consequently the origin is a cusp of the first kind.

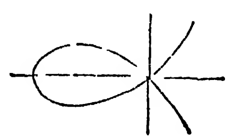
CONJUGATE POINTS.

120. When the co-ordinates for any particular point in a curve are real, but the co-ordinates on either side of this point are imaginary, this point is called an *insulated* or *conjugate point*, and is entirely detached from the rest of the curve.

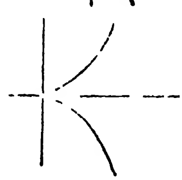


Points of this kind may be supposed to arise from certain portions of the curve disappearing in consequence of the particular values of some of the constant quantities in the equation. Thus, if we had the equation

$$\sqrt{a} \cdot y = \pm\sqrt{x(x - b)(x + c)},$$



and b and c are finite, the curve will resemble figure 1. If $c = 0$, the equation becomes $\sqrt{a} \cdot y = x\sqrt{x - b}$, and the part AF is reduced to a single point A. If $b =$, and c is finite, it will resemble figure 2. If $b = 0$, $c = 0$, it will resemble figure 3.



When the curve has a conjugate point, some of the differential coefficients at this point must be imaginary.

Scholium.

121. From the preceding articles on the singular points of curve lines, it will readily be perceived that they may always be determined from the following simple and general rule.

The determination of any singular point in a curve is obtained by con-

sidering when the differential coefficients of any order become 0 or ∞, or $\frac{0}{0}$: and the species of the point is determined, 1st, by examining what number of branches pass through this point, and whether they extend on both sides of it; 2ndly, by finding the position of their tangent; and 3rdly, the direction in which their concavity or convexity is turned.

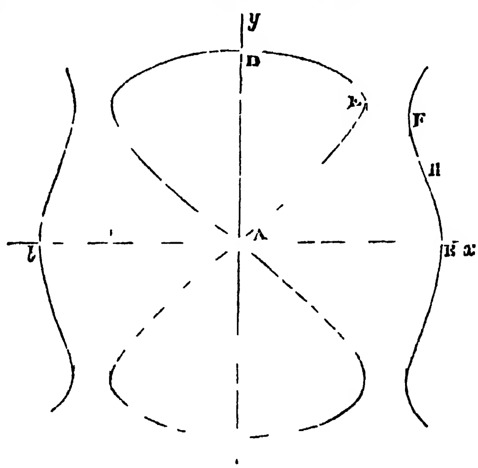
EXAMPLE OF THE ANALYSIS OF A CURVE.

122. If we take, for an example, the equation

$$y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = 0,$$

we shall easily find

$$y = \pm \sqrt{[48a^2 \pm \sqrt{(2304a^4 - 100a^2x^2 + x^4)}]}.$$



By discussing each of these values of y in the same manner as those of the general equation of the second degree (page 10), we may discover the extent and limits of the different branches of the curve. But we propose, at present, to determine them by the application of the differential calculus, which materially abridges the labour, and may be effected, in many cases, where the solution of the equation is impossible.

123. To determine the limits of the curve in the direction of the axis of y, we must find the maximum and minimum values of y; or when the differential coefficient

$$\frac{dy}{dx} = \frac{x^3 - 50a^2x}{y^3 - 48a^2y} = 0.$$

Hence $\qquad x^3 - 50a^2x = 0,$ and, therefore,

$$x = 0; \quad \text{or,} \quad x = \pm 5a\sqrt{2}.$$

The first value of x, substituted in the proposed equation, gives

$$y = 0; \quad \text{or,} \quad y = \pm 4a\sqrt{6}.$$

The corresponding values $x = 0$, $y = \pm 4a\sqrt{6}$, determine the points D and d, while the positive and negative values of y may easily be shown to be maximum values.

The values $x = 0$, $y = 0$, indicate the origin A, and at the same time give $\frac{dy}{dx} = \frac{0}{0}$. To determine the true value of this expression, we will differentiate the equation

$$(y^3 - 48a^2y)\,dy + (50a^2x - x^3)\,dx = 0,$$

and we shall find

$$(y^3 - 48a^2y)\,d^2y + (3y^2 - 48a^2)\,dy^2 + (50a^2 - 3x^2)\,dx^2 = 0.$$

And when $x = 0$, $y = 0$, this reduces itself to

$$-48a^2\,dy^2 + 50a^2\,dx^2 = 0; \quad \text{and} \quad \therefore \quad \frac{dy}{dx} = \pm\frac{5}{12}\sqrt{6}.$$

Hence it follows, that the curve has at the point A two tangents, which make, with the axis of x, two angles, whose trigonometrical tangents are respectively $+\frac{5}{12}\sqrt{6}$, and $-\frac{5}{12}\sqrt{6}$.

The other values $x = \pm 5a\sqrt{2}$, when substituted in the proposed equation, make y imaginary, and, consequently, y is neither a maximum nor a minimum.

124. To obtain the limits of the curve in the direction of the axis of x, we must make $\dfrac{dx}{dy} = 0$, or $y^3 - 48a^2y = 0$, from whence we have

$$y = 0; \qquad y = \pm 4a\sqrt{3}.$$

The first value of y, substituted in the proposed equation, gives

$$x = 0; \qquad x = \pm 10a.$$

The values $y = 0$, $x = 0$ have already been considered. But the two others, $y = 0$, $x = \pm 10a$, correspond to the points B and b, where the curve meets the axis of x.

The other two values, $y = \pm 4a\sqrt{3}$, give us, when substituted for y in the original equation, $x = \pm 6a$, and $x = \pm 8a$. From the equations $x = \pm 6a$, and $y = \pm 4a\sqrt{3}$, we obtain E, and the corresponding points in the other branches; and from the equations $x = \pm 8a$, $y = \pm 4a\sqrt{3}$, we get F and three other corresponding points. At the point E the value of x is a maximum, and at F it is a minimum.

If we substitute tx for y in the proposed equation, it becomes

$$t^4x^2 - 96a^2t^2 + 100a^2 - x^2 = 0.$$

Whence we deduce $\quad x^2 = \dfrac{100a^2 - 96a^2t^2}{1 - t^4}$,

a result which gives $x = \pm\infty$, when $t = 1$.

125. To determine the asymptotes we have

$$x - y\frac{dx}{dy} = \frac{x^4 - 50a^2x^2 - y^4 + 48a^2y^2}{x^3 - 50a^2x} = \frac{50a^2x^2 - 48a^2y^2}{x^3 - 50a^2x},$$

$$y - x\frac{dy}{dx} = \frac{y^4 - 48a^2y^2 - x^4 + 50a^2x^2}{y^3 - 48a^2y} = \frac{48a^2y^2 - 50a^2x^2}{y^3 - 48a^2y}.$$

These expressions diminish continually, whilst x and y increase, and are evanescent when $y = x$. Hence it follows, that the asymptotes are two lines drawn through the origin A, at an angle of 45° with the axis of x.

126. To find the points of inflexion, we have

$$\frac{d^2y}{dx^2} = \frac{(3x^2 - 50a^2) - (3y^2 - 48a^2)\dfrac{dy^2}{dx^2}}{y^3 - 48a^2y},$$

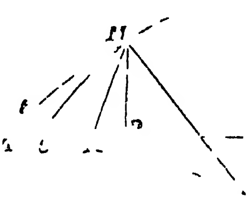
and when x and y are evanescent, $\dfrac{dy^2}{dx^2} = \dfrac{50}{48}$, and, therefore, $\dfrac{d^2y}{dx^2} = \dfrac{0}{0}$, from which equation nothing can be determined. To find the true value of $\dfrac{d^2y}{dx^2}$, we must take the third differential of the original equation, and then making $x = 0$, $y = 0$, we obtain $-144a^2dy\,d^2y = 0$; which gives $\dfrac{d^2y}{dx^2} = 0$, and proves that A is a point of inflexion.

To discover whether the curve has any other points of inflexion, we must make the numerator of the expression for $\dfrac{d^2y}{dx^2}$ equal to 0; and, after proper substitutions and reductions, we shall find the co-ordinates of the point of inflexion I, and the corresponding points in the other branches.

By making $\dfrac{d^2y}{dx^2}$ infinite, or $y^3 - 48a^2y = 0$, we obtain the points A, E, I', and B, which are not points of inflexion, but merely the limits of the curve in the direction of the abscissæ.

CURVES WITH POLAR CO-ORDINATES, AND SPIRALS.

127. PROP. XV.—*To draw a tangent to a polar curve at any point.*



Let A be the pole, and AM the radius vector of any curve line CM. Let MT be a tangent at M, and through A draw At perpendicular to AM, then At is called the subtangent. Put $AM = u$, $\angle MAP = \theta$; $AP = x$, $PM = y$; and then $\angle ATM = \alpha$. We have then

$$At = AM \tan AMt = u\tan(\theta - \alpha)$$

$$= u\frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = u\frac{\dfrac{y}{x} - \dfrac{dy}{dx}}{1 + \dfrac{y}{x}\dfrac{dy}{dx}} = u\frac{ydx - xdy}{xdx + ydy}.$$

But, since $x = u\cos\theta$, $y = u\sin\theta$, we have

$$dx = du\,.\cos\theta - ud\theta\sin\theta, \qquad dy = du\,.\sin\theta + ud\theta\cos\theta.$$

Substituting these values of dx and dy, we obtain

$$\text{the subtangent } At = \frac{u^2d\theta}{du} \ldots\ldots\ldots\ldots (1).$$

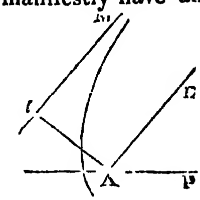
And, in like manner, the subnormal $Ar = \dfrac{du}{d\theta}$ .... (2).

Ex. 1.—To find the subtangent and subnormal at any point of the *logarithmic spiral*, whose equation is $\theta = \log u$.

2. To find the subtangent at any point of the *spiral* of *Archimedes* whose equation is $u = a\theta$.

128. Prop. XVI.—*To determine when a polar curve has an asymptote.*

When the radius vector becomes indefinitely great, if the subtangent has a finite value At for its limit, the curve will manifestly have an asymptote passing through t perpendicular to At. Hence, therefore, if the limit of the value of θ be found when AM is indefinitely great, and AE be drawn making an angle θ with the fixed axis AP, and also tM be drawn parallel to AE at the distance At, tM will be the asymptote required.



Ex. 1. Let the curve be the hyperbolic spiral whose equation is $u\theta = a$.

The subtangent, by the last article, will be found $= a$, and is, therefore, constant. Also $\theta = \dfrac{a}{u}$, therefore, when u is infinite, $\theta = 0$; therefore the asymptote is parallel to the fixed axis, at the distance a from the centre.

Ex. 2. To determine the asymptote to the spiral whose equation is $u = \dfrac{a\theta^2}{\theta^2 - 1}$.

129. Prop. XVII.—*To find an expression for the differential of the arc, and also for the radius of curvature of a polar curve.*

(1). We have, from art. 98, $ds = \sqrt{(dx^2 + dy^2)}$; and, if we substitute the values of dx and dy, art. 127, we shall find

$$ds = \sqrt{(du^2 + u^2d\theta^2)} \ldots\ldots\ldots\ldots\ldots (3).$$

(2). The expression for the radius of curvature referred to rectangular co-ordinates is $\rho = \dfrac{ds^3}{-dx\,d^2y}$; but this formula was deduced on the supposition that dx was constant. If θ be the independent variable, or $d\theta$ be supposed to be constant, dx will be variable, and, therefore, we must substitute for $\dfrac{d^2y}{dx^2}$ (art. 66) the expression $\left(\dfrac{dx^2}{d\theta^2}\dfrac{dy}{d\theta} - \dfrac{dx}{d\theta}\dfrac{d^2y}{d\theta^2}\right) \div \dfrac{dx^3}{d\theta^3}$; or for $-dx\,d^2y$ the expression $d^2x\,dy - dx\,d^2y$.* Hence, on the supposition that x is a function of θ, we have

$$\rho = \frac{ds^3}{d^2x\,dy - dx\,d^2y} \ldots\ldots\ldots\ldots\ldots (a).$$

And if we differentiate the values of dx and dy in art. 127, we shall find

$$d^2x = d^2u\cos\theta - 2du\,d\theta\sin\theta - u\cos\theta\,d\theta^2,$$
$$d^2y = d^2u\sin\theta + 2du\,d\theta\cos\theta - u\sin\theta\,d\theta^2.$$

By substituting the respective values of d^2x, d^2y, and also dx, dy, ds in equation (a), we obtain

$$\rho = \frac{(du^2 + u^2d\theta^2)^{\frac{3}{2}}}{-u\,d^2u\,d\theta + 2du^2d\theta + u^2d\theta^3} \ldots\ldots\ldots (4).$$

Ex.—In the logarithmic curve the radius of curvature is equal to the normal.

PART II.—THE INTEGRAL CALCULUS.

CHAP. I.—INTEGRATION OF DIFFERENTIALS INVOLVING ONE VARIABLE QUANTITY.

130. The *Integral Calculus* is the inverse of the *Differential;* its object being to ascend from the differential coefficients to the functions from which they were derived.

131. PROP. I.—*To find the integral of the monomial* $ax^n dx$.

As when u is such a function of a variable quantity x that $u = Ax^m + C$, where A, m, and C denote constant quantities, the differential of u is $mAx^{m-1}dx$; we infer from thence that the integral of $mAx^{m-1}dx$ is $Ax^m + C$. If, then, we compare $mAx^{m-1}dx$ with the given differential $ax^n dx$, we have $m - 1 = n$, and $mA = a$ or $A = \frac{a}{m} = \frac{a}{n+1}$. Hence, when

$$du = ax^n dx, \quad u = \frac{ax^{n+1}}{n+1} + C.$$

To find the integral, therefore, of a differential of one term, such as $ax^n dx$, we have the following

Rule.—Increase the index by unity, and then divide by the index so increased and by dx.

To indicate this operation, we put before the differential the characteristic $\int$, which signifies *sum* or *integral;* and it is the inverse of the characteristic d, so that $\int du = u$.*

132. The value of the constant quantity C is to be determined from the particular inquiry in which the differential equation $du = ax^n dx$ occurs. If it be known that $u = 0$ when x acquires some known magnitude, such as b, then the general equation $u = \frac{ax^{n+1}}{n+1} + C$ becomes in that case $0 = \frac{ab^{n+1}}{n+1} + C$. Hence, by subtracting each side of this last equation from the corresponding member of the former, we get

* The letter $\int$ was employed by the first writers on the integral calculus, as the initial of the word *summa;* because, according to the method of infinitesimals, they considered every function as the sum of all the infinitesimal increments.

$$u = \frac{a(x^{n+1} - b^{n+1})}{n+1}.$$

133. By giving different values to n in the preceding equation, we may obtain the corresponding integral. There is one case, however, which requires to be particularly noticed, as we cannot get the integral immediately from this rule. If $n = -1$, then $du = ax^{-1}\,dx = \frac{adx}{x}$; and the integral from this formula $= \frac{a(x^0 - b^0)}{0} = \frac{a(1-1)}{0} = \frac{0}{0}$. We, however, find the true value of this function from the example $\frac{a^x - b^x}{x}$, given in page 311, which was found to be $\log a - \log b$, when $x = 0$; consequently, the required integral $= a(\log x - \log b)$.

The expression which we have found for the value of u in this particular case coincides with what we might have found by considering that, when $u = a\log x$, $du = \frac{adx}{x}$; so that, conversely, when $du = \frac{adx}{x}$, we may conclude that $u = a\log x + C$, which agrees with the former result by supposing $C = -a\log b$.

134. It must now be evident, that if

$$du = ax^m dx + bx^n dx + cx^p dx + \&c.,$$

where m, n, p, &c. are constant quantities, then

$$u = \frac{ax^{m+1}}{m+1} + \frac{bx^{n+1}}{n+1} + \frac{cx^{p+1}}{p+1} + \&c. \ldots + C.$$

We only add one arbitrary constant, for it is obvious that, if a constant were added to each integral, their sum would be equivalent to one constant quantity.

135. In general, since we have seen that (art. 32)

$$d(au + bv - cw + \&c. \ldots + C) = adu + bdv - cdw + \&c.$$

where u, v, w, &c. denote any functions of a variable quantity, and C is a constant quantity; we conclude, conversely, that

$$\int(adu + bdv - cdw + \&c.) = au + bv - cw + \&c. + C.$$

Also, since $d(uv + C) = udv + vdu$; therefore, conversely,

$$\int(udv + vdu) = uv + C.$$

And in like manner, since $d\left(\frac{u}{v} + C\right) = \frac{vdu - udv}{v^2}$, we conclude that

$$\int\frac{vdu - udv}{v^2} = \frac{u}{v} + C.$$

136. Prop. II.—*To find the integral of* $(ax + b)^m dx$.

This may be done by expanding $(ax + b)^m$ into a series, multiplying each term by dx, and integrating each term of the result. But the integral may be found more easily thus. Put $ax + b = z$, then

$x = \frac{z-b}{a}$, and $dx = \frac{dz}{a}$. Substituting these values in the expression for du, we find $du = \frac{z^m dz}{a}$, and consequently $u = \frac{z^{m+1}}{(m+1)a} + C$ $= \frac{(ax+b)^{m+1}}{(m+1)a} + C$.

137. PROP. III.—*To find the integral of* $(ax^n + b)^m x^{n-1}dx$.

By putting, as before, $ax^n + b = z$, we have $nax^{n-1}dx = dz$, and $x^{n-1}dx = \frac{dz}{na}$. Hence $du = \frac{z^m dz}{na}$, and, therefore,

$$u = \frac{z^{m+1}}{na(m+1)} + C = \frac{(ax^n + b)^{m+1}}{na(m+1)} + C.$$

138. PROP. IV.—*To find the integral of the fractional function* $\frac{Ax^m dx}{(ax+b)^n}$.

Put $ax + b = z$, then $x = \frac{z-b}{a}$, and $dx = \frac{dz}{a}$. Hence

$$du = \frac{A(z-b)^m dz}{a^{m+1}z^n}.$$

If, therefore, we expand $(z-b)^m$ by the binomial theorem, multiply each of its terms by dz, and divide by z^n, we shall have a series of terms containing only simple powers of z, which may be integrated by the rule in art. 131.

Let us suppose, for example, that $m = 3$ and $n = 2$, then

$$du = \frac{A(z-b)^3 dz}{a^4 z^2} = \frac{A}{a^4}\left(zdz - 3bdz + \frac{3b^2dz}{z} - \frac{b^3dz}{z^2}\right).$$

Hence, taking the integrals of each of the terms, according to the rule in art. 136, we have

$$u = \frac{A}{a^4}\left(\frac{z^2}{2} - 3bz + 3b^2\log z + \frac{b^3}{z}\right) + C.$$

Substituting for z its value, $ax + b$, we find

$$u = \frac{A}{a^4}\left\{\frac{(ax+b)^2}{2} - 3b(ax+b) + 3b^2\log(ax+b) + \frac{b^3}{ax+b}\right\}.$$

Examples for Practice.

1. Integrate the following differentials:—

adx, bx^2dx, $cx^{-3}dx$, ax^7dx, $bx^{\frac{1}{2}}dx$,

$\frac{cdx}{x^2}$, $\frac{bdx}{x^5}$, $adx\sqrt[3]{x}$, $bdx\sqrt[4]{x}$, $\frac{cdx}{\sqrt{x}}$.

2. Integrate the following expressions:—

$adx\,(b + cx)$; $adx\,(b + cx)^3$, $adx\,(b + cx)^4$.

3. Integrate $axdx(b+cx^2)$; $ax^3dx(b+cx^4)$.

4. $\int\frac{xdx}{a+bx}=\frac{x}{b}-\frac{a}{b^2}\log(a+bx)$.

5. $\int\frac{x^3dx}{a+bx}=\frac{x^3}{3b}-\frac{ax^2}{2b^2}+\frac{a^2x}{b^3}-\frac{a^3}{b^4}\log(a+bx)$.

6. $\int\frac{x^2dx}{(a+bx)^2}=\left(\frac{x^2}{b}-\frac{2a^2}{b^3}\right)\frac{1}{a+bx}-\frac{2a}{b^3}\log(a+bx)$.

7. $\int\frac{xdx}{(a+bx)^3}=-\left(\frac{x}{b}+\frac{a}{2b^2}\right)\frac{1}{(a+bx)^2}$.

8. $\int\frac{x^3dx}{(a+bx)^3}=\left(\frac{x^3}{b}-\frac{6a^2x}{b^3}-\frac{9a^3}{2b^4}\right)\frac{1}{(a+bx)^2}-\frac{3a}{b^4}\log(a+bx)$.

9. $\int\frac{x^3dx}{(a+bx)^4}=\left(\frac{3ax^2}{b^2}+\frac{9a^2x}{2b^3}+\frac{11a^3}{6b^4}\right)\frac{1}{(a+bx)}+\frac{\log(a+bx)}{b^4}$.

10. $\int\frac{x^3dx}{(a+bx)^5}=-\left(\frac{x^3}{b}+\frac{3ax^2}{2b^2}+\frac{a^2x}{b^3}+\frac{a^3}{4b^4}\right)\frac{1}{(a+bx)^4}$.

INTEGRATION OF RATIONAL FRACTIONS.

139. All differentials which are rational fractions may be comprehended under this general form,

$$\frac{(ax^m+bx^n+cx^p+\&c.)\,dx}{a'x^{m'}+b'x^{n'}+c'x^{p'}+\&c.},$$

which, for the sake of brevity, we will represent by $\frac{Udx}{V}$. Now, if the highest power of x in the numerator be not less than in the denominator, it may become so by dividing U by V. Calling, then, Q the quotient, and R the remainder, we shall have

$$\int\frac{Udx}{V}=\int Qdx+\int\frac{Rdx}{V}.$$

Now, Q being a rational and integral function of x, $\int Qdx$ may be found, as in art. 136; it only remains, therefore, to find $\int\frac{Rdx}{V}$ an expression in which the highest exponent of x in R is less than in V, so that the fraction $\frac{Rdx}{V}$ may be generally expressed thus,

$$\frac{(ax^{n-1}+bx^{n-2}\ldots\ldots+l)\,dx}{x^n+a'x^{n-1}+b'x^{n-2}+\ldots\ldots l'}.$$

140. Prop. V. — *To integrate a rational fraction when the simple factors of the denominator are all real and unequal.*

The general method of integrating differentials of this form consists in

decomposing them into a series of other fractions whose denominators are more simple. To avoid complicated calculations, we will suppose that the differential which it is required to integrate is

$$\frac{(ax^2 + bx + c)\,dx}{x^3 + a'x^2 + b'x + c'} = \frac{R\,dx}{V}.$$

Let the roots of the equation $x^3 + a'x^2 + b'x + c' = 0$, be $-\alpha$, $-\beta$, $-\gamma$, which are supposed to be all real and unequal. We have then (Alg. art. 270),

$$x^3 + a'x^2 + b'x + c' = (x + \alpha)(x + \beta)(x + \gamma).$$

Assume now $\dfrac{R}{V} = \dfrac{ax^2 + bx + c}{x^3 + a'x^2 + b'x + c'} = \dfrac{A}{x+\alpha} + \dfrac{B}{x+\beta} + \dfrac{C}{x+\gamma}$,

where the numerators A, B, C are constant quantities, but as yet are indeterminate.

By reducing these fractions to a common denominator, we have

$$\frac{R}{V} = \frac{A(x+\beta)(x+\gamma) + B(x+\alpha)(x+\gamma) + C(x+\alpha)(x+\beta)}{(x+\alpha)(x+\beta)(x+\gamma)}.$$

Now, the denominators of the two members of this equation are identical; and as the numerators have the same form, it is manifest that we may make them also identical by equating like coefficients; for we have three equations to determine the three unknown quantities A, B, C. By performing the multiplication, the numerator becomes

$$(A + B + C)\,x^2 + [A(\beta + \gamma) + B(\alpha + \gamma) + C(\alpha + \beta)]\,x + A\beta\gamma + B\alpha\gamma + C\alpha\beta,$$

and comparing this with $ax^2 + bx + c$, the numerator of the given fraction, we obtain these three equations,

$$A + B + C = a,$$
$$A(\beta + \gamma) + B(\alpha + \gamma) + C(\alpha + \beta) = b,$$
$$A\beta\gamma + B\alpha\gamma + C\alpha\beta = c.$$

From these equations, which are all of the first degree, we may determine the values of A, B, C, and thus we have

$$\frac{(ax^2 + bx + c)\,dx}{x^3 + a'x^2 + b'x + c'} = \frac{A\,dx}{x + \alpha} + \frac{B\,dx}{x + \beta} + \frac{C\,dx}{x + \gamma}.$$

If we put $x + \alpha = z$, then $dx = dz$, and the differential $\dfrac{A\,dx}{x+\alpha}$ will be transformed into $\dfrac{A\,dz}{z}$, the integral of which is $A \log z$ or $A \log (x+\alpha)$. In like manner we find

$$\int\frac{B\,dx}{x + \beta} = B \log(x + \beta), \quad \int\frac{C\,dx}{x + \gamma} = C \log(x + \gamma);$$

and, consequently, $\displaystyle\int\frac{(ax^2 + bx + c)\,dx}{x^3 + a'x^2 + b'x + c'}$

$$= A\log(x + \alpha) + B\log(x + \beta) + C\log(x + \gamma).$$

It is easy to extend this mode of proceeding to the general formula

given in art. 139; and it is obvious that, whenever the denominator of a rational fraction can be decomposed into real and unequal simple factors, the integration of this fraction is attended with no other difficulty than this decomposition, which requires the numerical resolution of equations.

141. Prop. VI.—*To integrate a rational fraction when all the simple factors of the denominator are real, but not unequal.*

The method which we have adopted above will not apply when some of the factors are equal. If all the roots are unequal, we may assume

$$\frac{ax^3+bx^2+cx+e}{x^4+a'x^3+b'x^2+c'x+e'}=\frac{A}{x+\alpha}+\frac{B}{x+\beta}+\frac{C}{x+\gamma}+\frac{E}{x+\epsilon};$$

but if $\alpha=\beta=\gamma$, the preceding equation would become

$$\frac{ax^3+bx^2+cx+e}{(x+\alpha)^3(x+\epsilon)}=\frac{A+B+C}{x+\alpha}+\frac{E}{x+\epsilon};$$

and if we put $A+B+C=D$, we may easily perceive that, with the two indeterminate coefficients D and E, we cannot make the second side of the equation identical with the first.

To obviate this difficulty we may assume

$$\frac{ax^3+bx^2+cx+e}{(x+\alpha)^3(x+\epsilon)}=\frac{Ax^2+Bx+C}{(x+\alpha)^3}+\frac{E}{x+\epsilon},$$

for by reducing the second side of the equation to a common denominator, the numerator will be of the same form as the given numerator, ax^3+bx^2+cx+e, and will contain four indeterminate coefficients, A, B, C, E, by means of which the four coefficients of the different powers of x, in the two sides of the equation, may be rendered identical.

The integral of $\frac{(Ax^2+Bx+C)dx}{(x+\alpha)^3}$ may be found from art. 138, by putting $x+\alpha=z$. But we may reduce this to another series of fractions, by assuming

$$\frac{Ax^2+Bx+C}{(x+\alpha)^3}=\frac{G}{(x+\alpha)^3}+\frac{H}{(x+\alpha)^2}+\frac{K}{x+\alpha},$$

for by reducing these fractions to a common denominator, and adding them, the numerators of the two sides of the equation will have the same form, and there will be three indeterminate quantities, G, H, K, to enable us to make the coefficients of the like powers of x equal to each other. To find the integral of $\frac{Gdx}{(x+\alpha)^3}$, put $x+\alpha=z$, then $dx=dz$, and

$$\int\frac{Gdx}{(x+\alpha)^3}=\int\frac{Gdz}{z^3}=\int Gz^{-3}dz=\frac{Gz^{-2}}{-2}=\frac{-G}{2(x-\alpha)^2}.$$

In like manner $\int\frac{Hdx}{(x+\alpha)^2}=\frac{-H}{x+\alpha}$; also $\int\frac{Kdx}{x+\alpha}=K\log(x+\alpha)$.

142. Prop. VII.—*To integrate a rational fraction when all the*

simple factors of the denominator are unequal, but some of them are imaginary.

When some of the factors are imaginary, the same method may be used as was followed in art. 140; but this would lead to tedious calculations, which may easily be avoided by proceeding thus. It appears, from Algebra (art. 267), that every equation is composed of real, simple, and quadratic factors. Let $x^2 + 2\alpha x + \alpha^2 + \beta^2$ be one of the real quadratic factors which contains the two imaginary roots $-\alpha + \beta\sqrt{-1}$ and $-\alpha - \beta\sqrt{-1}$, then, instead of the former substitution of two of the fractions containing the imaginary factors, $x + \alpha + \beta\sqrt{-1}$ and $x + \alpha - \beta\sqrt{-1}$ for the denominators, we may assume $\dfrac{(Ax + B)\,dx}{x^2 + 2\alpha x + \alpha^2 + \beta^2}$, for one of the partial fractions, and find the values of A and B as before.

To integrate this fraction, put $x + \alpha = z$, then $x^2 + 2\alpha x + \alpha^2 + \beta^2 = z^2 + \beta^2$ and $dx = dz$, therefore

$$\frac{(Ax + B)\,dx}{x^2 + 2\alpha x + \alpha^2 + \beta^2} = \frac{(Az - A\alpha + B)\,dz}{z^2 + \beta^2}$$
$$= \frac{Azdz}{z^2 + \beta^2} + \frac{Cdz}{z^2 + \beta^2},$$

by putting $B - A\alpha = C$. The integral of the first of these terms is

$$\int\frac{Azdz}{z^2 + \beta^2} = \frac{A}{2}\int\frac{2zdz}{z^2 + \beta^2} = \frac{A}{2}\log(z^2 + \beta^2);$$

and if we make $z = \beta u$, and $dz = \beta du$, in the second of these integrals, we have

$$\int\frac{Cdz}{z^2 + \beta^2} = \frac{C}{\beta}\int\frac{du}{u^2 + 1} = \frac{C}{\beta}\tan^{-1}u.$$

Adding these two results together, and substituting, we obtain

$$\int\frac{(Ax + B)\,dx}{x^2 + 2\alpha x + \alpha^2 + \beta^2}$$
$$= A\log\sqrt{(x^2 + 2\alpha x + \alpha^2 + \beta^2)} - \frac{A\alpha - B}{\beta}\tan^{-1}\left(\frac{x + \alpha}{\beta}\right).$$

143. PROP. VIII.—*To integrate a rational fraction when some of the simple factors of the denominator are equal and imaginary.*

When the denominator contains several factors equal to $x^2 + 2\alpha x + \alpha^2 + \beta^2$, there will be a factor of the form $(x^2 + 2\alpha x + \alpha^2 + \beta^2)^p$. Corresponding to this factor we may assume a series of fractions,

$$\frac{(Ax + B)\,dx}{(x^2 + 2\alpha x + \alpha^2 + \beta^2)^p} + \frac{(Cx + D)\,dx}{(x^2 + 2\alpha x + \alpha^2 + \beta^2)^{p-1}} \ldots + \frac{(Kx + L)\,dx}{x^2 + 2\alpha x + \alpha^2 + \beta^2};$$

and find the values of the indeterminate coefficients A, B, C, &c. in the same manner as before.

To integrate the first of these fractions $\dfrac{(Ax + B)\,dx}{(x^2 + + 2\alpha x + \beta^2)^p}$, assume $x + \alpha = z$, then, as in the last article,

$$\frac{(Ax+B)\,dx}{(x^2+2ax+a^2+\beta^2)^p} = \frac{Azdz}{(z^2+\beta^2)^p} - \frac{(Aa-B)\,dz}{(z^2+\beta^2)^p},$$

$$\int \frac{Azdz}{(z^2+\beta^2)^p} = \frac{A}{2}\int 2zdz\,(z^2+\beta^2)^{-p} = -\frac{A}{2}\,\frac{(z^2+\beta^2)^{-p+1}}{p-1}$$

$$= -\frac{A}{2p-2}\,\frac{1}{(z^2+\beta^2)^{p-1}}.$$

The integral of $\dfrac{dz}{(z^2+\beta^2)^p}$ may be made to depend upon that of $\dfrac{dz}{(z^2+\beta^2)^{p-1}}$, in which the index of the denominator is less by unity than in the first, by a process that will be more fully explained in art. 160.

Let $u = \dfrac{z}{(z^2+\beta^2)^{p-1}} = z(z^2+\beta^2)^{-p+1}$, then

$$du = dz\,(z^2+\beta^2)^{-p+1} - (2p-2)\,z^2dz\,(z^2+\beta^2)^{-p}$$

$$= \frac{dz}{(z^2+\beta^2)^{p-1}} - \frac{(2p-2)\;(z^2+\beta^2)dz}{(z^2+\beta^2)^p} + \frac{(2p-2)\,\beta^2\,dz}{(z^2+\beta^2)^p}$$

$$= -\frac{(2p-3)\,dz}{(z^2+\beta^2)^{p-1}} + \frac{(2p-2)\,\beta^2\,dz}{(z^2+\beta^2)^p}.$$

Hence, transposing and reducing, we obtain

$$\frac{dz}{(z^2+\beta^2)^p} = \frac{du}{(2p-2)\beta^2} + \frac{2p-3}{(2p-2)\beta^2}\,\frac{dz}{(z^2+\beta^2)^{p-1}};$$

$$\therefore \int\frac{dz}{(z^2+\beta^2)^p} = \frac{1}{(2p-2)\beta^2}\,\frac{z}{(z^2+\beta^2)^{p-1}} + \frac{2p-3}{(2p-2)\beta^2}\int\frac{dz}{(z^2+\beta^2)^{p-1}}.$$

In like manner we can make the integral of $\dfrac{dz}{(z^2+\beta^2)^{p-1}}$ depend on that of $\dfrac{dz}{(z^2+\beta^2)^{p-2}}$, by substituting in this formula $p-1$ for p; and we may proceed in this manner, continually depressing the index of the binomial by unity, until at length we arrive at the term $\dfrac{dz}{z^2+\beta^2}$, the integral of which being known, we shall completely succeed in integrating the differential $\dfrac{dz}{(z^2+\beta^2)^p}$.

144. It appears, from the preceding articles, that if the resolution of equations be granted, every differential which appears under the form of a rational fraction may be integrated either algebraically or by means of logarithms, or circular arcs; and that to prepare it for a solution, we must decompose the fraction into partial fractions, whose denominators are either real binomial, or trinomial quantities. There are several analytical artifices by which the trouble of calculation may to a certain extent be abridged; but as the method of indeterminate coefficients is extremely simple, and not attended with any considerable

labour, we shall confine ourselves, in this work, to that which has already been explained.

Ex. 1.

145. To find the integral of $\dfrac{(x^4 - 2x^3 + x^2 + x - 3)\,dx}{x^2 - 3x + 2}$.

Since the highest exponent of x in the numerator is not less than in the denominator, we divide the numerator by the denominator until it becomes so. We thus find a quotient $x^2 + x + 2$, and a remainder $5x - 7$, so that

$$\frac{(x^4 - 2x^3 + x^2 + x - 3)\,dx}{x^2 - 3x + 2} = (x^2 + x + 2)\,dx + \frac{(5x - 7)\,dx}{x^2 - 3x + 2}.$$

The integral of $(x^2 + x + 2)\,dx$ is found immediately, from art. 134. To find the integral of $\dfrac{(5x - 7)\,dx}{x^2 - 3x + 2}$, put the denominator $x^2 - 3x + 2 = 0$, and we shall find the two roots to be 2 and 1; hence $x^2 - 3x + 2 = (x - 2)(x - 1)$. Assume now,

$$\frac{5x - 7}{x^2 - 3x + 2} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{(A + B)x - (A + 2B)}{(x - 2)(x - 1)},$$

in which A and B are indeterminate coefficients. By equating like coefficients we have

$$A + B = 5, \quad A + 2B = 7; \quad \text{consequently, } B = 2, \quad A = 3;$$

we, therefore, have $\dfrac{(5x-7)\,dx}{x^2 - 3x+2} = \dfrac{3dx}{x-2} + \dfrac{2dx}{x-1}$. Hence

$$\int\frac{(x^4-2x^3+x^2+x-3)dx}{x^2-3x+2} = \int(x^2+x+2)dx + \int\frac{3dx}{x-2} + \int\frac{2dx}{x-1}$$

$$= \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 2x + 3\log(x-2) + 2\log(x-1)$$

$$= \tfrac{1}{6}(2x^3 + 3x^2 + 12x) + \log[(x-2)^3(x-1)^2].$$

Ex. 2.

146. To find the integral of $\dfrac{dx}{x^8 + x^7 - x^4 - x^3}$.

The factors of the denominator are easily found, for it may be put under this form,

$$x^3(x^5 + x^4 - x - 1) = x^3(x + 1)(x^4 - 1)$$

$$= x^3(x + 1)(x^2 - 1)(x^2 + 1) = x^3(x + 1)^2(x - 1)(x^2 + 1),$$

and consequently the proposed differential may be decomposed as follows,

$$\frac{Adx}{x-1} + \frac{Bdx}{(x + 1)^2} + \frac{Cdx}{x+1}$$

$$+ \frac{Ddx}{x^3} + \frac{Edx}{x^2} + \frac{Fdx}{x} + \frac{(Gx + H)\,dx}{x^2 + 1}.$$

By reducing these fractions to a common denominator, and comparing

the numerator of their sum with that of the proposed fraction, we shall find

$$A=\tfrac{1}{8},\ B=\tfrac{1}{4},\ C=\tfrac{9}{8},\ D=-1,\ E=1,\ F=-1,\ G=\tfrac{1}{4},\ H=\tfrac{1}{4}.$$

The integration of each of the preceding fractions has been already explained; and we shall find for the result

$$\begin{cases} \dfrac{1}{8}\log(x-1)-\dfrac{1}{4}\,\dfrac{1}{x+1}+\dfrac{9}{8}\log(x+1)+\dfrac{1}{2x^2}-\dfrac{1}{x} \\ -\log x-\tfrac{1}{8}\log(x^2+1)-\tfrac{1}{4}\tan^{-1}x. \end{cases}$$

Hence, by reduction, we obtain

$$\int\frac{dx}{x^8+x^7-x^4-x^3}=\frac{2-2x-5x^2}{4x^2(1+x)}+\frac{1}{8}\log\frac{x^2-1}{x^2+1}$$
$$+\log\frac{x+1}{x}-\tfrac{1}{4}\tan^{-1}x.$$

Examples for Practice.

1. $\displaystyle\int\frac{2dx}{x^2-4x+3}=\log\frac{x-3}{x-1}.$

2. $\displaystyle\int\frac{4x^3dx-17x^2dx+9xdx+10dx}{x^2-5x+6}=2x^2+3x+8\log\frac{x-2}{x-3}.$

3. $\displaystyle\int\frac{xdx}{x^2+6x+8}=\log\frac{(x+4)^2}{x+2}.$

4. $\displaystyle\int\frac{dx}{x^3+3x^2-4}=\frac{1}{3(x+2)}+\frac{1}{9}\log\frac{x-1}{x+2}.$

5. $\displaystyle\int\frac{x^2dx}{x^3+5x^2+8x+4}=\frac{4}{x+2}+\log(x+1).$

6. $\displaystyle\int\frac{xdx}{x^3+6x^2+11x+6}=\log\frac{(x+2)^2}{\sqrt{\{(x+1)(x+3)^3\}}}.$

7. $\displaystyle\int\frac{dx}{x^3-x^2+x-1}=\frac{1}{2}\log\frac{x-1}{\sqrt{(x^2+1)}}-\frac{1}{2}\tan^{-1}x.$

8. $\displaystyle\int\frac{xdx}{x^3+x^2+x+1}=\tfrac{1}{2}\log\frac{\sqrt{(x^2+1)}}{x+1}+\tfrac{1}{2}\tan^{-1}x.$

9. $\displaystyle\int\frac{x^2dx}{x^4+x^2-2}=\frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}}-\frac{1}{6}\log\frac{1+x}{1-x}.$

10. $\displaystyle\int\frac{dx}{x^4+4x^3+5x^2+4x+4}=\frac{2}{25}\log\frac{(x+2)^2}{x^2+1}+\frac{3}{25}\tan^{-1}x-\frac{1}{5(x+2)}.$

11. $\displaystyle\int\frac{dx}{1-x^2}=\frac{1}{2}\log\frac{1+x}{1-x}.$

12. $\displaystyle\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}.$

$$13.\ \int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}.$$

$$14.\ \int\frac{dx}{1+x^3}=\tfrac{1}{3}\log\frac{1+x}{\sqrt{(1-x+x^2)}}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

$$15.\ \int\frac{dx}{1-x^4}=\tfrac{1}{4}\log\frac{1+x}{1-x}+\tfrac{1}{2}\tan^{-1}x.$$

$$16.\ \int\frac{dx}{1+x^4}=\frac{1}{4\sqrt{2}}\log\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}+\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x\sqrt{2}}{1-x^2}\right).$$

INTEGRATION OF IRRATIONAL FUNCTIONS.

147. When a differential expression involves irrational functions, we must endeavour either to transform it into another which is rational, or to reduce it to a number of irrational quantities, consisting of one term only; and then, in either case, its integral may be found by the preceding rules.

Let us take, for example, the differential $\dfrac{(1+\sqrt{x}-\sqrt[3]{x^2})\,dx}{1+\sqrt[3]{x}}$. It is evident that, by putting $x=z^6$, all the radical signs will disappear, and the differential will be transformed to $\dfrac{6z^5dz\,(1+z^3-z^4)}{1+z^2}$, which, by actual division, becomes

$$-6\left(z^7dz-z^6dz-z^5dz+z^4dz-z^2dz+dz-\frac{dz}{1+z^2}\right),$$

and the integral of this expression is

$$-6\left(\tfrac{1}{8}z^8-\tfrac{1}{7}z^7-\tfrac{1}{6}z^6+\tfrac{1}{5}z^5-\tfrac{1}{3}z^3+z-\tan^{-1}z\right)+\text{const.}$$

148. Prop. IX. — *To find the integral of* $\dfrac{\mathrm{X}dx}{\sqrt{(\mathrm{A}+\mathrm{B}x+\mathrm{C}x^2)}}$, X *being a rational function of* x.

(1). *When* C *is positive.* By taking the coefficient C from under the vinculum, and putting $\dfrac{A}{C}=a$, $\dfrac{B}{C}=b$, this differential becomes $\dfrac{1}{\sqrt{C}}\dfrac{Xdx}{\sqrt{(a+bx+x^2)}}$. Assume, now, $\sqrt{(a+bx+x^2)}=z-x$; then, squaring both sides of the equation, we have $a+bx=z^2-2xz$. Hence we obtain

$$x=\frac{z^2-a}{2z+b},\qquad dx=\frac{2dz(a+bz+z^2)}{(2z+b)^2},$$

$$\sqrt{(a+bx+x^2)}=z-x=\frac{a+bz+z^2}{2z+b}.$$

By means of these values the differential $\dfrac{Xdx}{\sqrt{(A+Bx+Cx^2)}}$ is trans-

formed into another of the form Zdz, where Z is a rational function of z, which is always real when C is positive.

(2). *When* C *is negative.* As the radical quantity $\sqrt{(a + bx - x^2)}$ is supposed to have a real value, the expression $x^2 - bx - a$ may always be decomposed into real factors of the first degree. Let us represent these factors by $x - \alpha$ and $x - \beta$, then it is evident that

$$a + bx - x^2 = -(x^2 - bx - a) = (x - \alpha)(\beta - x).$$

Let us now assume $\sqrt{(x - \alpha)(\beta - x)} = (x - \alpha)z$, then, squaring both sides of the equation, we have $\beta - x = (x - \alpha)z^2$, from which we find

$$x = \frac{\alpha z^2 + \beta}{z^2 + 1}, \quad dx = \frac{2(\alpha - \beta)zdz}{(z^2 + 1)^2},$$

$$\sqrt{(a + bx - x^2)} = (x - \alpha)z = \frac{(\beta - \alpha)z}{z^2 + 1};$$

and these values will render the proposed differential rational.

149. *Cor.*—The differential $Xdx \sqrt{(A + Bx + Cx^2)}$, in which X is a rational function of x, may be immediately reduced to the preceding form by multiplying numerator and denominator by the quantity $\sqrt{(A + Bx + Cx^2)}$, for we then have

$$Xdx \sqrt{(A + Bx + Cx^2)} = \frac{Xdx\,(A + Bx + Cx^2)}{\sqrt{(A + Bx + Cx^2)}},$$

in which the numerator is a rational function of x.

Before we proceed to the general integration of binomial differentials, we shall consider a few simple cases included in the last proposition, whose integrals may be found independently, either by means of circular arcs, or a table of logarithms; and which continually occur in analytical calculations.

CIRCULAR FUNCTIONS.

150. (1). *To find the integral of* $\frac{dx}{\sqrt{(a^2 - x^2)}}$.

Put $x = ay$; then $dx = ady$, and

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \int \frac{ady}{\sqrt{(a^2 - a^2y^2)}} = \int \frac{dy}{\sqrt{(1 - y^2)}}.$$

But this is the differential of an arc whose sine is y or $\frac{x}{a}$ (art 49);

$$\therefore \int \frac{dx}{\sqrt{(a^2 - x^2)}} = \text{arc whose sine is } \frac{x}{a} = \sin^{-1}\frac{x}{a}.$$

In like manner $\int \frac{-dx}{\sqrt{(a^2 - x^2)}} = \cos^{-1}\frac{x}{a}$.

(2). *To find the integral of* $\frac{dx}{\sqrt{(2ax - x^2)}}$.

Put $x = ay$, as before, then $\int \frac{dx}{\sqrt{(2ax - x^2)}} = \int \frac{dy}{\sqrt{(2y - y^2)}}$;
but this is the differential of an arc whose versed sine is y or $\frac{x}{a}$ (art. 49),

$$\therefore \int \frac{dx}{\sqrt{(2ax - x^2)}} = \text{arc whose versed sine is } \frac{x}{a} = \text{vers}^{-1} \frac{x}{a}.$$

(3). *To find the integral of* $\frac{dx}{x\sqrt{(x^2 - a^2)}}$.

Put $x = ay$, then $\int \frac{dx}{x\sqrt{(x^2 - a^2)}} = \frac{1}{a}\int \frac{dy}{y\sqrt{(y^2 - 1)}}$; but this is the differential of an arc whose secant is y or $\frac{x}{a}$,

$$\therefore \int \frac{dx}{x\sqrt{(x^2 - a^2)}} = \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

In like manner, $\int \frac{-dx}{x\sqrt{(x^2 - a^2)}} = \frac{1}{a}\text{cosec}^{-1}\frac{x}{a}$.

Collecting these results together, for the sake of more easy reference; we have

$$(1). \int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a}.$$

$$(2). \int \frac{dx}{\sqrt{(2ax - x^2)}} = \text{vers}_{-1}\frac{x}{a}.$$

$$(3). \int \frac{dx}{x\sqrt{(x^2 - a^2)}} = \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

To which may be added, the rational fraction

$$(4). \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

LOGARITHMIC FUNCTIONS.

151. (1). *To find the integral of* $\frac{dx}{\sqrt{(x^2 + a^2)}}$.

Put $x^2 + a^2 = z^2$; then $xdx = zdz$,
and $xdx + zdx = zdz + zdx$; or, $(x + z)dx = z(dx + dz)$.

Hence $\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{dx}{z} = \int \frac{dx + dz}{x + z} = \log(x + z)$;

$$\therefore \int \frac{dx}{\sqrt{(x^2 + a^2)}} = \log\left(x + \sqrt{x^2 + a^2}\right).$$

In like manner, $\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \log\left(x + \sqrt{x^2 - a^2}\right)$.

(2). *To find the integral of* $\dfrac{dx}{\sqrt{(x^2+2ax)}}$.

Let $x+a=z$; then $dx=dz$, and $\sqrt{(x^2+2ax)}=\sqrt{(z^2-a^2)}$;

$$\therefore \int\frac{dx}{\sqrt{(x^2+2ax)}}=\int\frac{dz}{\sqrt{(z^2-a^2)}}=\log\left(z+\sqrt{z^2-a^2}\right);$$

$$\therefore \int\frac{dx}{\sqrt{(x^2+2ax)}}=\log\left(x+a+\sqrt{x^2+2ax}\right).$$

In like manner $\displaystyle\int\frac{dx}{\sqrt{(x^2-2ax)}}=\log\left(x-a+\sqrt{x^2-2ax}\right)$.

(3). *To find the integral of* $\dfrac{dx}{x\sqrt{(a^2+x^2)}}$.

Let $a^2+x^2=z^2$; then $xdx=zdz$, therefore,

$$\int\frac{dx}{x\sqrt{(a^2+x^2)}}=\int\frac{1}{x}\frac{dx}{z}=\int\frac{1}{x}\frac{dz}{x}=\int\frac{dz}{z^2-a^2}$$

$$=\frac{1}{2a}\int\left[\frac{dz}{z-a}-\frac{dz}{z+a}\right]=\frac{1}{2a}\left[\log(z-a)-\log(z+a)\right];$$

$$\therefore \int\frac{dx}{x\sqrt{(a^2+x^2)}}=\frac{1}{2a}\log\frac{\sqrt{(a^2+x^2)}-a}{\sqrt{(a^2+x^2)}+a}.$$

In like manner $\displaystyle\int\frac{dx}{x\sqrt{(a^2-x^2)}}=\frac{1}{2a}\log\frac{a-\sqrt{(a^2-x^2)}}{a+\sqrt{(a^2-x^2)}}$.

Hence, collecting them together, we have

$$(1).\ \int\frac{dx}{\sqrt{(x^2\pm a^2)}}=\log\left(x+\sqrt{x^2\pm a^2}\right).$$

$$(2).\ \int\frac{dx}{\sqrt{(x^2\pm 2ax)}}=\log\left(x\pm a+\sqrt{x^2\pm 2ax}\right).$$

$$(3).\ \int\frac{dx}{x\sqrt{(a^2+x^2)}}=\frac{1}{2a}\log\frac{\sqrt{(a^2+x^2)}-a}{\sqrt{(a^2+x^2)}+a}.$$

$$(4).\ \int\frac{dx}{x\sqrt{(a^2-x^2)}}=\frac{1}{2a}\log\frac{a-\sqrt{(a^2-x^2)}}{a+\sqrt{(a^2-x^2)}}.$$

To which may be added the rational fractions,

$$(5).\ \int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\frac{a+x}{a-x}.$$

$$(6).\ \int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}.$$

Examples for Practice.

1. $\displaystyle\int\frac{dx}{\sqrt{(x^2+x)}}=\log\left(2x+1+2\sqrt{x^2+x}\right)$.

2. $\displaystyle\int\frac{dx}{\sqrt{(x-x^2)}}=\text{vers}^{-1}2x=2\sin^{-1}\sqrt{x}$.

3. $\int \frac{dx}{\sqrt{(a + bx + x^2)}} = \log (2x + b + 2\sqrt{a + bx + x^2})$.

4. $\int \frac{dx}{\sqrt{(a + bx - x^2)}} = -2 \tan^{-1} \sqrt{\left(\frac{\beta - x}{x - \alpha}\right)}$.

5. $\int \frac{dx}{x\sqrt{(1 + x + x^2)}} = \log \frac{2 + x - 2\sqrt{(1 + x + x^2)}}{x}$.

6. $\int \frac{dx}{x\sqrt{(x^2 + x - 1)}} = \sin^{-1} \left(\frac{x-2}{x\sqrt{5}}\right)$.

7. $\int \frac{dx}{(1 + x)\sqrt{(1 - x)}} = \frac{1}{\sqrt{2}} \log \frac{-3 + x + \sqrt{8 - 8x}}{1 + x}$.

8. $\int \frac{dx}{(1 + x)\sqrt{(1 + x + x^2)}} = \log \frac{1 - x - 2\sqrt{(1 + x + x^2)}}{1 + x}$.

INTEGRATION OF BINOMIAL DIFFERENTIALS.

152. These differentials are represented by the formula

$$x^{m-1}dx\,(a + bx^n)^{\frac{p}{q}}$$

in which we may suppose m and n to be whole numbers without affecting the generality of the expression. For if we had $x^{\frac{1}{3}}\,dx\,(a + bx^{\frac{1}{2}})^{\frac{p}{q}}$, we may assume $x = z^6$, and the differential becomes $6z^7dz\,(a + bz^3)^{\frac{p}{q}}$. We may also suppose n to be positive, for if it were negative, and the differential were $x^{m-1}dx\,(a + bx^{-n})^{\frac{p}{q}}$, we may put $x = \frac{1}{z}$, and the differential will be transformed to $-z^{-m-1}dz\,(a + bz^n)^{\frac{p}{q}}$, where n is positive.

153. Prop. X.—*To determine in what cases the differential* $x^{m-1}dx(a + bx^n)^{\frac{p}{q}}$ *may be made rational.*

(1). Assume $a + bx^n = z^q$, then $(a + bx^n)^{\frac{p}{q}} = z^p$; therefore,

$$x^n = \frac{z^q - a}{b},\ x^m = \left(\frac{z^q - a}{b}\right)^{\frac{m}{n}},\ x^{m-1}dx = \frac{q}{nb}\,z^{q-1}dz\left(\frac{z^q - a}{b}\right)^{\frac{m}{n}-1}.$$

Hence the proposed differential is transformed to

$$\frac{q}{nb}\,z^{p+q-1}\,dz\left(\frac{z^q - a}{b}\right)^{\frac{m}{n}-1},$$

an expression which is evidently rational whenever $\frac{m}{n}$ is a whole number.

(2). The differential $x^{m-1}dx\,(a + bx^n)^{\frac{p}{q}}$ may also be put under the form

$$x^{m-1}\,dx\,(x^n)^{\frac{p}{q}}\,(ax^{-n}+b)^{\frac{p}{q}} = x^{m+\frac{np}{q}-1}\,dx\,(ax^{-n}+b)^{\frac{p}{q}},$$

and, from the reasoning in the first case, the last expression may be made rational, whenever $\left(m+\frac{np}{q}\right)\div n$, or $\frac{m}{n}+\frac{p}{q}$ is a whole number.

The differential $x^8dx\,(a+bx^3)^{\frac{p}{q}}$ satisfies the first condition, since $\frac{m}{n}=\frac{9}{3}=$ a whole number; and the differential $x^4dx\,(a+bx^3)^{\frac{1}{3}}$ satisfies the second condition, since $\frac{m}{n}+\frac{p}{q}=\frac{5}{3}+\frac{1}{3}=2$.

154. As it is not possible, in every case, to find the integral of $x^{m-1}dx\,(a+bx^n)^{\frac{p}{q}}$ in a finite number of terms, we must endeavour to reduce it to its most simple case. This may be done by continually diminishing in magnitude the index $\pm m$ by n, or by continually diminishing in magnitude the exponent $\pm\frac{p}{q}$ by unity. The way in which we shall effect this is by means of the formula $d\,.\,uv=udv+vdu$; from whence we have, by integration, $uv=\int udv+\int vdu$ and $\int udv=uv-\int vdu$. This is called the method of *Integration by parts*, and it is of very great utility in the Integral Calculus. For the sake of conciseness, we shall in future write p in the place of the fraction $\frac{p}{q}$.

155. Prop. XI. *To reduce the integral of* $x^{m-1}dx\,(a+bx^n)^p$ *to another in which the exponent* m *shall be diminished by* n.

If we write the proposed differential in this manner,

$$x^{m-n}\times x^{n-1}dx\,(a+bx^n)^p,$$

the factor $x^{n-1}dx\,(a+bx^n)^p$ is always integrable, whatever be the value of p. Let us denote this factor by dv, and the factor x^{m-n} by u, then

$$v=\frac{(a+bx^n)^{p+1}}{(p+1)nb},\ \text{and}\ du=(m-n)x^{m-n-1}dx;$$

$$\therefore\ \int x^{m-1}dx\,(a+bx^n)^p=\int udv=uv-\int vdu$$

$$=\frac{x^{m-n}(a+bx^n)^{p+1}}{(p+1)nb}-\frac{m-n}{(p+1)n}\int x^{m-n-1}dx(a+bx^n)^{p+1}\ldots(1).$$

But $x^{m-n-1}dx\,(a+bx^n)^{p+1}=x^{m-n-1}dx\,(a+bx^n)^p\,(a+bx^n)$

$$=ax^{m-n-1}dx\,(a+bx^n)^p+bx^{m-1}dx\,(a+bx^n)^p.$$

Substituting now this last value in equation (1), and collecting together the terms involving the integral $\int x^{m-1}dx\,(a+bx^n)^p$, we find

$$\left(1+\frac{m-n}{(p+1)n}\right)\int x^{m-1}dx\,(a+bx^n)^p$$

$$=\frac{x^{m-n}(a+bx^n)^{p+1}}{(p+1)nb}-\frac{a(m-n)}{(p+1)nb}\int x^{m-n-1}dx\,(a+bx^n)^p,$$

from which we get $\int x^{m-1}dx(a + bx^n)^p$

$$= \frac{x^{m-n}(a + bx^n)^{p+1}}{b(m + pn)} - \frac{a(m - n)}{b(m + pn)}\int x^{m-n-1}dx\,(a + bx^n)^p \ldots (\mathrm{A}).$$

156. It is easy to see that as we have, by this formula, reduced the determination of $\int x^{m-1}dx(a+bx^n)^p$ to that of $\int x^{m-n-1}dx(ab+x^n)^p$, we may also reduce this last to that of $\int x^{m-2n-1}dx\,(a + bx^n)^p$, by writing $m-n$ for m in equation (A): then, by changing m into $m-2n$, we may reduce $\int x^{m-2n-1}dx(a+bx^n)^p$ to that of $\int x^{m-3n-1}dx(a+bx^n)^p$; and so on.

In general, if r denote the number of reductions, we shall at last come to $\int x^{m-rn-1}dx(a + bx^n)^p$, and if m be a multiple of n, the integral of $x^{m-1}dx\,(a + bx^n)^p$ will be a finite algebraical quantity. Thus, if $m = 3n$, the index of x without the vinculum will be reduced from $3n - 1$ to $2n - 1$, $n - 1$ successively, and the integral of $x^{n-1}dx\,(a + bx^n)^p$ being known, the proposed differential can be integrated in a finite number of terms.

157. Prop. XII.—*To reduce the integral of* $x^{m-1}dx\,(a + bx^n)^p$ *to another in which the index* m *shall be changed to* $m + n$.

When m is a negative quantity, the index of x without the vinculum will be increased in magnitude instead of being diminished, by formula (A). If, however, we reverse this formula, we shall find that it is applicable to the case when m is negative.

From formula (A) we deduce $\int x^{m-n-1}dx\,(a + bx^n)^p$

$$= \frac{x^{m-n}(a + bx^n)^{p+1}}{a(m - n)} - \frac{b(m + np)}{a(m - n)}\int x^{m-1}dx\,(a + bx^n)^p.$$

Substitute now $m + n$ for m, and we obtain $\int x^{m-1}dx\,(a + bx^n)^p$

$$= \frac{x^m(a + bx^n)^{p+1}}{am} - \frac{b(m + n + np)}{am}\int x^{m+n-1}dx(a+bx^n)^p \ldots (\mathrm{B}),$$

an expression which diminishes in magnitude the exponent m when it is negative.

158. Prop. XIII.—*To reduce the integral of* $x^{m-1}dx\,(a + bx^n)^p$ *to another in which the index* p *shall be diminished by unity.*

Because $\int x^{m-1}dx\,(a + bx^n)^p = \int x^{m-1}dx\,(a + bx^n)^{p-1}(a + bx^n)$

$$= a\int x^{m-1}dx\,(a + bx^n)^{p-1} + b\int x^{m+n-1}dx\,(a + bx^n)^{p-1}.$$

And from the formula $\int udv = uv - \int vdu$, we have

$$\int x^{m+n-1}dx\,(a + bx^n)^{p-1} = \int x^m \times x^{n-1}dx\,(a + bx^n)^{p-1}$$

$$= \frac{x^m \times (a + bx^n)^p}{bnp} - \int\frac{mx^{m-1}dx\,(a + bx^n)^p}{bnp};$$

therefore, substituting this value in the preceding equation, and collect-

ing together the terms involving the integral $\int x^{m-1}dx\,(a+bx^n)^p$, we obtain $\left(1+\dfrac{m}{np}\right)\int x^{m-1}dx\,(a+bx^n)^p$

$$= \frac{x^m(a+bx^n)^p}{np} + a\int x^{m-1}dx\,(a+bx^n)^{p-1}.$$

Hence, therefore, $\quad \int x^{m-1}dx\,(a+bx^n)^p$

$$= \frac{x^m(a+bx^n)^p}{m+np} + \frac{anp}{m+np}\int x^{m-1}dx\,(a+bx^n)^{p-1} \ldots\ldots (\text{C}).$$

159. By means of this general formula we may take away successively from p as many units as it contains; and by the application of this formula, and formula (A), we may make the integral $\int x^{m-1}dx(a+bx^n)^p$ depend on $\int x^{m-rn-1}dx\,(a+bx^n)^{p-s}$, rn being the greatest multiple of n contained in $m-1$, and s the greatest whole number contained in p.

For example, the integral $\int x^7dx\,(a+bx^3)^{\frac{5}{2}}$ may, by formula (A), be reduced successively to

$$\int x^4dx\,(a+bx^3)^{\frac{5}{2}}, \qquad \int xdx\,(a+bx^3)^{\frac{5}{2}},$$

and, by formula (C), $\int xdx\,(a+bx^3)^{\frac{5}{2}}$ is reduced successively to

$$\int xdx\,(a+bx^3)^{\frac{3}{2}}, \qquad \int xdx\,(a+bx^3)^{\frac{1}{2}}.$$

160. PROP. XIV.—*To reduce the integral of* $x^{m-1}dx\,(a+bx^n)^p$ *to another in which the exponent* p *shall be changed to* $p+1$.

By reversing formula (C), we have $\int x^{m-1}dx\,(a+bx^n)^{p-1}$

$$= -\frac{x^m(a+bx^n)^p}{anp} - \frac{m+np}{anp}\int x^{m-1}dx\,(a+bx^n)^p,$$

and, writing $p+1$ in the place of p, we obtain $\int x^{m-1}dx\,(a+bx^n)^p$

$$= -\frac{x^m(a+bx^n)^{p+1}}{an(p+1)} - \frac{m+n+np}{a(p+1)}\int x^{m-1}dx(a+bx^n)^{p+1} \ldots (\text{D}).$$

This formula will evidently diminish in magnitude the exponent p when it is negative.

161. These four formulæ cannot be applied when their denominators vanish. This is the case, for example, with the formula (B), when $m=0$; but in all such cases the proposed differential may be integrated either algebraically or by logarithms.

162. In the application of the four preceding formulæ, the student will often find it desirable to work out each example separately, instead of substituting in the general forms. In this case, his best plan will be to invert the process which we have given above, and, making a proper assumption for P, to differentiate P, and divide it into two parts which may agree with the proposed differentials.

Thus, in art. 157, put $P = x^{m-n}(a+bx^n)^{p+1}$, then

$$dP = (m-n)\,x^{m-n-1}dx(a+bx^n)^{p+1} + nb(p+1)x^{m-1}dx(a+bx^n)^p$$

$=(m-n)x^{m-n-1}dx(a+bx^n)^p(a+bx^n)+nb(p+1)x^{m-1}dx(a+bx^n)^p$
$=(m-n)^{m-n-1}dx(a+bx^n)^p+(m+bnp)x^{m-1}dx(a+bx^n)^p$,
from whence we shall obtain formula (A). The same method manifestly is applicable for obtaining formula (B).

To obtain formula (C), assume $P = x^m(a+bx^n)^p$, then

$$dP = nx^{m-1}dx\,(a+bx^n)^p + bnp\,x^{m+n-1}\,dx\,(a+bx^n)^{p-1}.$$

But $np\,x^{m-1}dx\,(a+bx^n)^p = npx^{m-1}dx\,(a+bx^n)^{p-1}(a+bx^n)$

$$= anpx^{m-1}dx\,(a+bx^n)^{p-1} + bnpx^{m+n-1}dx(a+bx^n)^{p-1}.$$

Subtracting this equation from the last, and transposing,

$$dP = (m+np)\,x^{m-1}dx\,(a+bx^n)^p - anpx^{m-1}dx\,(a+bx^n)^{p-1},$$

from which equation formula (C) is obtained, and thence formula (D). The only difficulty in these cases is that of making the proper assumption, which may easily be remembered from the following

Rule.—In formulæ (B) and (D) omit dx, and add unity to each of the indices $m-1$ and p; in formulæ (A) and (C), add unity to the indices to which the integrals are to be reduced.

163. PROP. XV.—*To find the integral of* $\frac{dx}{x}(a+bx^n)^p$.

Let $y = a + bx^n$, then $\log(y-a) = \log(bx^n) = \log b + n\log x$;

$$\therefore\ \frac{dy}{y-a} = \frac{ndx}{x} \text{ and } \int\frac{dx}{x}(a+bx^n)^p = \frac{1}{n}\int\frac{y^p\,dy}{y-a},$$

the integral of which may easily be found.

Ex. 1.

164. To find the integral of $\frac{x^4dx}{\sqrt{(1-x^2)}}$.

By comparing this with the general formula $x^{m-1}dx\,(a+bx^n)^p$, we have

$$a=1,\quad b=-1,\quad n=2,\quad p=-\tfrac{1}{2},\quad \text{and } m=5.$$

Hence we find, from formula (A),

$$\int\frac{x^4dx}{\sqrt{(1-x^2)}} = -\frac{x^3\sqrt{(1-x^2)}}{4} + \frac{3}{4}\int\frac{x^2dx}{\sqrt{(1-x^2)}}.$$

Again, making $m = 3$, we have,

$$\int\frac{x^2dx}{\sqrt{(1-x^2)}} = -\frac{x\sqrt{(1-x^2)}}{2} + \frac{1}{2}\int\frac{dx}{\sqrt{(1-x^2)}}$$

$$= -\frac{x\sqrt{(1-x^2)}}{2} + \frac{1}{2}\sin^{-1}x + \text{const.}$$

Substituting this value in the preceding equation, we obtain

$$\int\frac{x^4dx}{\sqrt{(1-x^2)}} = -\left(\frac{x^3}{4}+\frac{1\cdot3}{2\cdot4}x\right)\sqrt{(1-x^2)} + \frac{1\cdot3}{2\cdot4}\sin^{-1}x+\text{const.}$$

Ex. 2.

165. To find the integral of $\dfrac{x^2dx}{\sqrt{(hx-x^2)}} = \dfrac{x^{\frac{3}{2}}\,dx}{\sqrt{(h-x)}}$.

By comparing this differential with the general formula, we find

$$a = h,\quad b = -1,\quad n = 1,\quad p = -\tfrac{1}{2},\quad \text{and } m = \tfrac{5}{2}.$$

Hence we have, from formula (A),

$$\int\frac{x^{\frac{3}{2}}dx}{\sqrt{(h-x)}} = -\frac{x^{\frac{3}{2}}\sqrt{(h-x)}}{2} + \frac{3h}{4}\int\frac{x^{\frac{1}{2}}dx}{\sqrt{(h-x)}}.$$

Again, making $m = \frac{3}{2}$, we have

$$\int\frac{x^{\frac{1}{2}}dx}{\sqrt{(h-x)}} = -x^{\frac{1}{2}}\sqrt{(h-x)} + \frac{h}{2}\int\frac{x^{-\frac{1}{2}}dx}{\sqrt{(h-x)}}.$$

And since $\displaystyle\int\frac{x^{-\frac{1}{2}}dx}{\sqrt{(h-x)}} = \int\frac{dx}{\sqrt{hx-x^2)}} = \text{vers}^{-1}\frac{x}{\frac{1}{2}h} + \text{const.}$;

also, $x^{\frac{3}{2}}\sqrt{(h-x)} = x\sqrt{(hx-x^2)}$; $x^{\frac{1}{2}}\sqrt{(h-x)} = \sqrt{(hx-x^2)}$;

we have, therefore, by substitution,

$$\int\frac{x^2dx}{\sqrt{(hx-x^2)}} = -\left(\frac{x}{2}+\frac{3h}{4}\right)\sqrt{hx-x^2} + \frac{1\,.\,3}{2\,.\,4}\,h^2\ \text{vers}^{-1}\frac{x}{\frac{1}{2}h} + \text{const.}$$

We have to determine this integral in finding the time of an oscillation in a circular arc, page 212; and, since the integral is supposed to be $= 0$ when $x = h$, we have, by substituting h for x in the last equation,

$$0 = 0 + \frac{1\,.\,3}{2\,.\,4}\,h^2\,\text{vers}^{-1}2 + \text{const.};$$

$$\therefore\ \text{const.} = -\frac{1\,.\,3}{2\,.\,4}\,h^2\,\text{vers}^{-1}2 = -\frac{1\,.\,3}{2\,.\,4}\,h^2\pi.\quad \text{Hence}$$

$$\int\frac{-x^2dx}{\sqrt{(hx-x^2)}} = \frac{1\,.\,3}{2\,.\,4}h^2\pi + \left(\frac{x}{2}+\frac{3h}{4}\right)\sqrt{hx-x^2} - \frac{1\,.\,3}{2\,.\,4}\,h^2\,\text{vers}^{-1}\frac{x}{\frac{1}{2}h}.$$

and when $x = 0$, $\displaystyle\int\frac{-x^2dx}{\sqrt{(hx-x^2)}} = \frac{1\,.\,3}{2\,.\,4}\,h^2\pi$, which is the expression we have given in page 242. In the same manner we may find all the other integrals in that proposition.

Examples for Practice.

1. $\displaystyle\int\frac{dx}{x^2\sqrt{(1+x)}} = -\frac{\sqrt{(1+x)}}{x} - \frac{1}{2}\log\frac{\sqrt{(1+x)}-1}{\sqrt{(1+x)}+1}.$

2. $\displaystyle\int\frac{dx}{x^3\sqrt{(x-1)}} = \left(\frac{1}{2x^2}+\frac{3}{4x}\right)\sqrt{x-1} + \frac{3}{4}\tan^{-1}\sqrt{(x-1)}.$

3. $\displaystyle\int\frac{dx}{x^2(1+x)^{\frac{3}{2}}} = -\left(\frac{1}{x}+3\right)\frac{1}{\sqrt{(1+x)}} - \frac{3}{2}\log\frac{\sqrt{(1+x)}-1}{\sqrt{(1+x)}+1}.$

4. $$\int \frac{dx}{x\sqrt{(a+bx)}} = \frac{1}{\sqrt{a}} \log \frac{\sqrt{(a+bx)}-\sqrt{a}}{\sqrt{(a+bx)}+\sqrt{a}}.$$

5. $$\int \frac{dx}{(a+bx^2)^{\frac{3}{2}}} = \frac{x}{a\sqrt{(a+bx^2)}}.$$

6. $$\int \frac{x^5 dx}{\sqrt{(x^2-1)}} = \left(\frac{x^4}{5} + \frac{4x^2}{5.3} + \frac{4.2}{5.3.1}\right)\sqrt{x^2-1}.$$

7. $$\int \frac{x^6 dx}{\sqrt{(x^2+1)}} = \left(\frac{x^5}{6} - \frac{5x^3}{6.4} + \frac{5.3x}{6.4.2}\right)\sqrt{x^2+1} - \frac{5}{16}\log(x+\sqrt{x^2+1}).$$

8. $$\int \frac{dx}{x^4\sqrt{(1-x^2)}} = -\left(\frac{1}{3x^3} + \frac{2}{3x}\right)\sqrt{1-x^2}.$$

9. $$\int \frac{dx}{x^5\sqrt{(1-x^2)}} = -\left(\frac{1}{4x^4} + \frac{3}{4.2x^2}\right)\sqrt{1-x^2}$$
$$+ \frac{3.1}{4.2}\log\frac{1-\sqrt{1-x^2}}{x}.$$

10. $$\int \frac{x^4 dx}{(1-x^2)^{\frac{3}{2}}} = -\left(\frac{x^3}{2} - \frac{3x}{2}\right)\frac{1}{\sqrt{1-x^2}} - \frac{3}{2}\sin^{-1}x.$$

INTEGRATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

166. PROP. XVI.—*It is required to integrate the differential* $x^m dx (\log x)^n$ *when* n *is a positive integer, and* m *any number whatever.*

If we make $x^m dx = dv$, and $(\log x)^n = u$, then $v = \dfrac{x^{m+1}}{m+1}$, $du = \dfrac{ndx}{x}(\log x)^{n-1}$. Hence, integrating by parts, we have

$$\int x^m dx (\log x)^n = \int u dv = uv - \int v du$$
$$= \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}\int x^m dx (\log x)^{n-1}.$$

In like manner, if we change n into $n-1$, we shall find

$$\int x^m dx (\log x)^{n-1} = \frac{x^{m+1}(\log x)^{n-1}}{m+1} - \frac{n-1}{m+1}\int x^m dx (\log x)^{n-2}.$$

Proceeding continually in this manner until the index n is reduced to 1, we shall have, lastly,

$$\int x^m dx \log x = \frac{x^{m+1}\log x}{m+1} - \frac{x^{m+1}}{(m-1)^2}.$$

Hence we have, in ascending from the last of these integrals to the first,

$$\int x^m dx (\log x)^n =$$
$$\frac{x^{m+1}}{m+1}\left\{(\log x)^n - \frac{n}{m+1}(\log x)^{n-1} .. \pm \frac{n(n-1)\ldots 2.1}{(m+1)^n}\right\}.$$

When $m = -1$, we have $\int \frac{dx}{x} (\log x)^n = \frac{(\log x)^{n+1}}{n+1} + \text{const.}$

167. *Cor.*—In like manner may we find the integral of $Pdx(\log x)^n$, when we know the integrals $\int Pdx = Q$, $\int Qdx = R$, $\int Rdx = S$, &c.

168. PROP. XVII.—*To find the integral of* $\frac{x^m dx}{(\log x)^n}$ *when* n *is a positive integer, and* m *any number whatever.*

Since $\int \frac{x^m dx}{(\log x)^n} = \int x^{m+1} \times \frac{dx}{x} (\log x)^{-n}$, if we put $u = x^{m+1}$, $dv = \frac{dx}{x} (\log x)^{-n}$, then $du = (m+1)x^m dx$, $v = \frac{(\log x)^{-n+1}}{-n+1}$ $= \frac{-1}{(n-1)(\log x)^{n-1}}$. Hence, integrating by parts, we have

$$\int \frac{x^m dx}{(\log x)^n} = \int udv = uv - \int vdu$$

$$= \frac{-x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m dx}{(\log x)^{n-1}}.$$

In like manner, if we change n into $n-1$, we shall find

$$\int \frac{x^m dx}{(\log x)^{n-1}} = \frac{-x^{m+1}}{(n-2)(\log x)^{n-2}} + \frac{m+1}{n-2} \int \frac{x^m dx}{(\log x)^{n-2}}.$$

By continuing these reductions we shall find the last integral to be $\int \frac{x^m dx}{\log x}$, which cannot be obtained in a finite number of terms, unless $m = -1$. It may, however, be reduced to a more simple form by making $x^{m+1} = z$; for then we have $(m+1) \log x = \log z$,

$$(m+1) x^m dx = dz, \text{ and, consequently, } \int \frac{x^m dx}{\log x} = \int \frac{dz}{\log z}.$$

When $m = -1$, the preceding integral becomes

$$\int \frac{x^{-1} dx}{\log x} = \int \frac{dx}{x} \frac{1}{\log x} = \int \frac{d \cdot \log x}{\log x} = \log (\log x).$$

169. *Cor.*—In the same manner we may reduce the integral $\int \frac{Pdx}{(\log x)^n}$, in which P is a function of x, to another of the form $\int \frac{Vdx}{\log x}$, by making successively

$$d(Px) = Qdx, \quad d(Qx) = Rdx \ldots\ldots\ldots d(Tx) = Vdx.$$

170. PROP. XVIII.—*To integrate the differential* $a^x x^n dx$, *when* n *is a positive integer.*

Since $d \cdot a^x = \log a \cdot a^x dx$, therefore, $a^x dx = \frac{d \cdot a^x}{\log a}$, and

$\int a^x\,dx = \frac{a^x}{\log a} + \text{const}$. Hence, from the formula $\int u dv = uv - \int v du$, we have

$$\int x^n \times a^x dx = x^n \frac{a^x}{\log a} - \int \frac{n \cdot x^{n-1} dx\, a^x}{\log a}.$$

Similarly, $\int x^{n-1} \times a^x dx = x^{n-1} \frac{a^x}{\log a} - \int \frac{(n-1)x^{n-2} dx\, a^x}{\log a}$;

and so on. Hence we have, by substitution,

$$\int a^x x^n dx = \frac{a^x}{\log a}\left\{x^n - \frac{n}{\log a} x^{n-1} \ldots \pm \frac{n(n-1)\ldots 2 \cdot 1}{(\log a)^n}\right\} + \text{const.}$$

171. PROP. XIX.—*To integrate the differential* $\frac{a^x dx}{x^n}$, *when* n *is a positive integer.*

Integrating by parts, as before, we have

$$\int \frac{a^x dx}{x^n} = \int a^x \times x^{-n} dx = a^x \frac{x^{-n+1}}{-n+1} - \int \log a \cdot a^x dx \frac{x^{-n+1}}{-n+1}$$

$$= \frac{-a^x}{(n-1)x^{n-1}} + \frac{\log a}{n-1}\int \frac{a^x dx}{x^{n-1}}.$$

Again, substituting $n-1$ for n, we obtain

$$\int \frac{a^x dx}{x^{n-1}} = \frac{-a^x}{(n-2)x^{n-2}} + \frac{\log a}{n-2}\int \frac{a^x dx}{x^{n-2}}.$$

By proceeding in this manner, we shall at last arrive at the integral $\int \frac{a^x dx}{x}$, which cannot be found in a finite number of terms. If we make $a^x = z$, we have $x \log a = \log z$, $dx \log a = \frac{dz}{z}$, therefore, $\frac{dx}{x} = \frac{dz}{z \log z}$, and $\frac{a^x dx}{x} = \frac{dz}{\log z}$, which is the same expression as that to which we reduced the integral in art. 168.

172. PROP. XX.—*To find the integrals of* $\frac{a^x dx}{x}$ *and* $\frac{dz}{\log z}$ *in a series.*

We have, from Algebra, art. 395,

$$\int \frac{a^x dx}{x} = \int \frac{dx}{x}\left(1 + x \log a + \frac{x^2(\log a)^2}{1 \cdot 2} + \frac{x^3(\log a)^3}{1 \cdot 2 \cdot 3} + \&c.\right)$$

$$= \log x + \log a \frac{x}{1} + \frac{(\log a)^2}{1 \cdot 2}\frac{x^2}{2} + \frac{(\log a)^3}{1 \cdot 2 \cdot 3}\frac{x^3}{3} + \&c.$$

If, likewise, we make $a^x = z$, we have $x \log a = \log z$, therefore,

$$a^x = 1 + \log z + \frac{(\log z)^2}{1 \cdot 2} + \frac{(\log z)^3}{1 \cdot 2 \cdot 3} + \&c.;$$

Also, $\dfrac{dx}{x} = \dfrac{dz}{z\log z}$, $\dfrac{a^x\,dx}{x} = \dfrac{dz}{\log z}$. Hence

$$\int\frac{dz}{\log z} = \int\frac{a^x\,dx}{x} = \int\frac{dz}{z}\left(\frac{1}{\log z} + 1 + \frac{\log z}{1\,.\,2} + \frac{(\log z)^2}{1\,.\,2\,.\,3} + \&c.\right)$$
$$= \log(\log z) + \log z + \frac{1}{2}\,\frac{(\log z)^2}{1\,.\,2} + \&c.$$

Examples for Practice.

1. $\int dx \log x = x\log x - x$.
2. $\int x^2 dx \log x = \frac{1}{3}x^3\log x - \frac{1}{9}x^3$.
3. $\int x^m dx (\log x)^2 = \dfrac{x^{n+1}}{n+1}\left\{(\log x)^2 - \dfrac{2}{m+1}\log x + \dfrac{2\,.\,1}{(m+1)^2}\right\}$.
4. $\int\dfrac{dx}{(1-x)^2}\log x = \dfrac{x\log x}{1-x} + \log(1-x)$.
5. $\int\dfrac{dx}{x(\log x)^3} = -\dfrac{1}{2(\log x)^2}$.
6. $\int a^x x dx = \dfrac{a^x x}{\log a} - \dfrac{a^x}{(\log a)^2}$.
7. $\int\dfrac{a^x dx}{x^3} = -\dfrac{a^x}{2x^2} - \dfrac{a^x\log a}{2x} + \dfrac{(\log a)^2}{2}\int\dfrac{a^x dx}{x}$.
8. $\int\dfrac{a^x dx}{1-x} = \dfrac{a^x}{(1-x)\log a}\left\{1 - \dfrac{1}{(1-x)\log a} + \dfrac{1\,.\,2}{(1-x)^2(\log a)^2} - \&c.\right\}$.

INTEGRATION OF CIRCULAR FUNCTIONS.

173. From articles 47 and 48 it appears that

$$d\,.\sin nx = ndx\cos nx, \quad \therefore \int dx\cos nx = \frac{\sin nx}{n}.$$
$$d\,.\cos nx = -ndx\sin nx, \quad \therefore \int dx\sin nx = -\frac{\cos nx}{n}.$$
$$d\,.\tan nx = \frac{ndx}{\cos^2 nx}, \quad \therefore \int\frac{dx}{\cos^2 nx} = \frac{\tan nx}{n}.$$
$$d\,.\cot nx = \frac{-ndx}{\sin^2 nx}, \quad \therefore \int\frac{dx}{\sin^2 nx} = \frac{-\cot nx}{n}.$$
$$d\,.\sec nx = \frac{ndx\sin nx}{\cos^2 nx}, \quad \therefore \int\frac{dx\sin nx}{\cos^2 nx} = \frac{\sec nx}{n}.$$
$$d\,.\operatorname{cosec} nx = \frac{-ndx\cos nx}{\sin^2 nx} \quad \therefore \int\frac{dx\cos nx}{\sin^2 nx} = \frac{-\operatorname{cosec} nx}{n}.$$

174. PROP. XXI.—*To find the integral of* $dx \cos^n x$, *when* n *is an integer.*

For the sake of simplicity, we will suppose $n = 5$. We have then, from Trigonometry, art. 89.

$$2^4 \cos^5 x = \cos 5x + 5 \cos 3x + 10 \cos x.$$

Hence $\int 16\, dx \cos^5 x = \int dx \cos 5x + \int 5 dx \cos 3x + \int 10 dx \cos x$;

$$\therefore \int dx \cos^5 x = \frac{1}{16}\left(\frac{\sin 5x}{5} + \frac{5 \sin 3x}{3} + 10 \sin x\right).$$

There are several other methods of integrating this and other circular functions. 1st. We may integrate them by parts, as in art. 176. 2d. We may put $\cos x = y$, then $x = \cos^{-1} y$, and $dx = \dfrac{-dy}{\sqrt{(1-y^2)}}$ (art. 49), therefore, $\int dx \cos^5 x = \int \dfrac{-y^5\, dy}{\sqrt{(1-y^2)}}$. The integral of this expression may be found from formula (A), art. 155. 3d. We may substitute the exponential expression for $\cos x$, given in Trigonometry, art. 86, and then find the integral by art. 170.

175. PROP. XXII.—*To find the integral of* $dx \sin^n x$, *when* n *is an integer.*

If n be an even number, 4, for example, we have, from Trigonometry, art. 90,

$$2^3 \sin^4 x = \cos 4x - 4 \cos 2x + 3\,;$$

$$\therefore \int dx \sin^4 x = \tfrac{1}{8} \int dx \cos 4x - \tfrac{1}{2} \int dx \cos 2x + \int \tfrac{3}{8} dx$$
$$= \tfrac{1}{32} \sin 4x - \tfrac{1}{4} \sin 2x + \tfrac{3}{8} x.$$

If n be an odd number, 5, for example, then

$$2^4 \sin^5 x = \sin 5x - 5 \sin 3x + 10 \sin x.$$

Hence, multiplying by dx, and integrating and reducing

$$\int dx \sin^5 x = -\frac{\cos 5x}{80} + \frac{\cos 3x}{48} - \frac{5 \cos x}{8}.$$

176. PROP. XXIII.—*To find the integral of* $dx \sin^m x \cos^n x$.

The method which we shall adopt in this case is that of integration by parts.

Since $dx \sin^m x \cos^n x = dx \sin x \cos^n x \times \sin^{m-1} x$, if we make $dv = dx \sin x \cos^n x$, $u = \sin^{m-1} x$, then $v = \int dx \sin x \cos^n x = -\dfrac{\cos^{n+1} x}{n+1}$,

$du = (m-1) \sin^{m-2} x \cos x\, dx$;

$$\therefore \int dx \sin x \cos^n x \times \sin^{m-1} x = \int u dv = uv - \int v du$$
$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int dx \sin^{m-2} x \cos^{n+2} x.$$

But $\int dx \sin^{m-2} x \cos^{n+2} x = \int dx \sin^{m-2} x \cos^n x\, (1 - \sin^2 x)$

$$= \int dx \sin^{m-2} x \cos^n x - \int dx \sin^m x \cos^n x.$$

Substituting this in the preceding equation, and reducing, we find

$$\int dx \sin^m x \cos^n x$$

$$= -\frac{\sin^{m-1}x \cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int dx \sin^{m-2}x \cos^n x \ldots\ldots\ldots\ldots \text{(E)}$$

If we apply the same method of reduction to the cosine as we have done to the sine, we shall find

$$\int dx \sin^m x \cos^n x = \frac{\sin^{m+1}x \cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int dx \sin^m x \cos^{n-2}x \ldots \text{(F)}$$

By proceeding in this manner we shall diminish the exponents m and n successively by 2, until at length, if they are both integers, we shall arrive at one of the four differentials,

$$dx, \quad dx \sin x, \quad dx \cos x, \quad dx \sin x \cos x,$$

the integrals of which are

$$x, \quad -\cos x, \quad \sin x, \quad \tfrac{1}{2}\sin^2 x.$$

177. Prop. XXIV.—*To find the integral of* $\frac{dx \sin^m x}{\cos^n x}$.

The index m may be continually diminished by 2, by means of formula (E), by changing n into $-n$. But if we change n into $-n$, in formula (F), we shall find that the index of the denominator will be increased instead of being diminished. We may, however, easily reduce this exponent by a change similar to that which was introduced in art. 159.

We have then, from formula (F),

$$\int\frac{dx \sin^m x}{\cos^n x} = \frac{1}{m-n}\frac{\sin^{m+1}x}{\cos^{n+1}x} - \frac{n+1}{m-n}\int\frac{dx \sin^m x}{\cos^{n+2}x};$$

$$\therefore \int\frac{dx \sin^m x}{\cos^{n+2} x} = \frac{1}{n+1}\frac{\sin^{m+1}x}{\cos^{n+1}x} - \frac{m-n}{n+1}\int\frac{dx \sin^m x}{\cos^n x}.$$

and changing n into $n-2$,

$$\int\frac{dx \sin^m x}{\cos^n x} = \frac{1}{n-1}\frac{\sin^{m+1}x}{\cos^{n-1}x} - \frac{m-n+2}{n-1}\int\frac{dx \sin^m x}{\cos^{n-2}x}.$$

178. *Cor.*—In the same manner we may continually reduce the indices m and n in the expression $\int\frac{dx \cos^n x}{\sin^m x}$; and it is easy to see, that if m and n be whole numbers, these two integrals will be finally reduced to one of the forms dx, $dx \sin x$, $dx \cos x$, the integrals of which are immediately given; or to one of the four forms,

$$\frac{dx}{\sin x}, \quad \frac{dx}{\cos x}, \quad \frac{dx \sin x}{\cos x}, \quad \frac{dx \cos x}{\sin x},$$

the integrals of which expressions will be found in art. 180.

179. Prop. XXV.—*To find the integral of* $\frac{dx}{\sin^m x \cos^n x}$.

By changing m into $-m$ and n into $-n$, in formulæ (E) and (F), art. 176, we find

$$\int \frac{dx}{\sin^m x \cos^n x} = \frac{1}{m+n} \frac{1}{\sin^{m+1}x \cos^{n-1}x} + \frac{m+1}{m+n} \int \frac{dx}{\sin^{m+2} \cos^n x}.$$

$$\int \frac{dx}{\sin^m x \cos^n x} = \frac{-1}{m+n} \frac{1}{\sin^{m-1}x \cos^{n+1}x} + \frac{n+1}{m+n} \int \frac{dx}{\sin^m x \cos^{n+2}x}.$$

Also, changing m into $m - 2$ in the first of these two equations, and n into $n - 2$ in the second, and reducing, as in art. 179, we obtain two new formulæ, (G) and (H),

$$\int \frac{dx}{\sin^m x \cos^n x} = \frac{-1}{m-1} \frac{1}{\sin^{m-1}x \cos^{n-1}x} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^{m-2}x \cos^n x},$$

$$\int \frac{dx}{\sin^m x \cos^n x} = \frac{1}{n-1} \frac{1}{\sin^{m-1}x \cos^{n-1}x} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m x \cos^{n-2}x}.$$

By the continued application of these two formulæ, we shall at last arrive at one of the four differentials

$$dx, \quad \frac{dx}{\sin x}, \quad \frac{dx}{\cos x}, \quad \frac{dx}{\sin x \cos x}.$$

180. PROP. XXVI.—*To find the integrals of the five differentials*

$$\frac{dx}{\sin x}, \quad \frac{dx}{\cos x}, \quad \frac{dx \cos x}{\sin x}, \quad \frac{dx \sin x}{\cos x}, \quad \frac{dx}{\sin x \cos x}.$$

(1). We have, first,

$$\frac{dx}{\sin x} = \frac{dx \sin x}{\sin^2 x} = \frac{dx \sin x}{1 - \cos^2 x} = \frac{-dy}{1-y^2},$$

in making $\cos x = y$. Hence we have the integral

$$\int \frac{dx}{\sin x} = \frac{1}{2} \log \frac{1-y}{1+y} = \log \sqrt{\frac{1-\cos x}{1+\cos x}} = \log \tan \frac{x}{2}.$$

(2). In the same way we have

$$\frac{dx}{\cos x} = \frac{dx \cos x}{\cos^2 x} = \frac{dx \cos x}{1 - \sin^2 x} = \frac{dy}{1-y^2},$$

in making $\sin x = y$. Hence, therefore,

$$\int \frac{dx}{\cos x} = \frac{1}{2} \log \frac{1+y}{1-y} = \log \sqrt{\frac{1+\sin x}{1-\sin x}} = \log \sqrt{\frac{1+\cos(90^\circ - x)}{1-\cos(90^\circ - x)}}$$
$$= \log \cot \tfrac{1}{2}(90^\circ - x) = \log \tan (45^\circ + \tfrac{1}{2}x) + \text{const.}$$

(3.) $\displaystyle\int \frac{dx \cos x}{\sin x} = \int \frac{d \,.\, \sin x}{\sin x} = \log \sin x.$

(4.) $\displaystyle\int \frac{dx \sin x}{\cos x} = \int \frac{-d \,.\, \cos x}{\cos x} = -\log \cos x.$

(5.) $\displaystyle\int \frac{dx}{\sin x \cos x} = \int \frac{dx \cos x}{\sin x} + \int \frac{dx \sin x}{\cos x}$

$$= \log \sin x - \log \cos x = \log \tan x.$$

Hence it appears that the integral of $dx \sin^m x \cos^n x$ can be obtained in all cases in which m and n are positive or negative whole numbers. If these exponents are fractional, we must in general have recourse to series.

181. PROP. XXVII.—*To find the integral of* $zx^n\,dx$, *where* $z = \sin^{-1}x$.

We have, from the formula $\int u dv = uv - \int v du$,

$$\int z.x^n\,dx = z\frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}dx = \frac{zx^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}\frac{dx}{\sqrt{(1-x^2)}},$$

and the integral of $\dfrac{x^{n+1}dx}{\sqrt{(1-x^2)}}$ may be obtained from art. 159.

In the same manner we may find the integral of $zx^n\,dx$, when z is put for $\cos^{-1} x$, $\tan^{-1} x$, &c.

Examples.

1. $\int dx \sin^2 x = -\frac{1}{2}\sin x \cos x + \frac{1}{2}x$.
2. $\int dx \sin^3 x = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x$.
3. $\int dx \sin^4 x = -\frac{1}{4}\sin^3 x \cos x - \frac{3}{8}\sin x \cos x + \frac{3}{8}x$.
4. $\int dx \cos^4 x = \frac{1}{4}\sin x \cos^3 x + \frac{3}{8}\sin x \cos x + \frac{3}{8}x$.
5. $\int dx \sin x \cos^n x = -\dfrac{1}{n+1}\cos^{n+1}x$.
6. $\int dx \sin^2 x \cos^2 x = \frac{1}{4}\sin^3 x \cos x - \frac{1}{8}\sin x \cos x + \frac{1}{8}x$.
7. $\int dx \sin^3 x \cos^2 x = (\frac{1}{5}\sin^4 x - \frac{1}{15}\sin^2 x - \frac{2}{15})\cos x$.
8. $\int \dfrac{dx}{\sin^4 x} = -\cot x - \frac{1}{3}\cot^3 x$.
9. $\int \dfrac{dx}{\cos^4 x} = \tan x + \frac{1}{3}\tan^3 x$.
10. $\int \dfrac{dx}{\sin^3 x \cos^2 x} = \dfrac{1}{\sin^2 x \cos x} - \dfrac{3\cos x}{2\sin^2 x} + \dfrac{3}{2}\log\tan\dfrac{x}{2}$.

ARBITRARY CONSTANTS AND INTEGRATION BY SERIES.

182. In finding the integral of any proposed differential, we always add an arbitrary constant to it, to give it all the generality that is requisite. In ordinary cases, we determine this constant by making the integral vanish for a given value of x. Thus, since $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$, C denoting the arbitrary constant, then, if this integral ought to vanish for a value of $x = a$, we shall have the equation $0 = \dfrac{a^{n+1}}{n+1} + C$, and

subtracting this equation from the preceding one

$$\int x^n\, dx = \frac{x^{n+1} - a^{n+1}}{n+1}.$$

183. The value of $x = a$, which makes the integral vanish, is called its *origin*, and the integral is then said to *commence when* x = a. If we stop at the value of $x = b$, we then say that *the integral is complete when* x = b. In this case

$$\int x^n\, dx = \frac{x^{n+1} - a^{n+1}}{n+1} = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

The two values $x = a$, and $x = b$, are called *the limits of the integral.* This is now conveniently written thus: $\int_a^b x^n\, dx$, the first limit being placed at the bottom of the $\int$, and the other at the top.

184. Every integral which we express without indicating its limits, is called an *indefinite integral*, and ought, in order to be *complete*, to include an *arbitrary constant.* When we assign these limits, the integral is *definite.* If they are $x = a$, and $x = b$, for instance, we then say that the integral $\int x^n\, dx$ ought to be taken from $x = a$ to $x = b$. As these expressions are frequently used, the student should make himself familiar with them.

185. When we cannot determine the exact integral of a proposed differential, we must then have recourse to the method of approximation by series. Thus, if it be required to integrate the expression Xdx, where X denotes any function of x, we must endeavour to expand the function X into a series, according to the ascending or descending powers of x; then multiplying by dx, we can find the integral of each term separately.

186. To find the integral of $\dfrac{dx}{a+x}$.

Since $\dfrac{1}{a+x} = \dfrac{1}{a} - \dfrac{x}{a^2} + \dfrac{x^2}{a^3} - \dfrac{x^3}{a^4} + \text{\&c.};$

$$\therefore \frac{dx}{a+x} = \frac{dx}{a} - \frac{xdx}{a^2} + \frac{x^2dx}{a^3} - \frac{x^3dx}{a^4} + \text{\&c.};$$

$$\therefore \log(a+x) = C + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{\&c.}$$

To find the value of the constant quantity, put $x = 0$, and then this equation becomes $\log a = C$, Hence

$$\log(a+x) = \log a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{\&c.}$$

187. In the same manner we may find

$$\int \frac{dx}{1+x^2} = \int dx\,(1 - x^2 + x^4 - x^6 + \text{\&c.})$$

$$\therefore \tan^{-1} x = C + x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \text{\&c.}$$

Also, $\int \frac{dx}{\sqrt{(1-x^2)}} = \int dx\left(1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \&c.\right)$;

$$\sin^{-1}x = C + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \&c.$$

If we suppose this to be the least arc whose sine or tangent is x, the arbitrary constant disappears, for when $x = 0$, the arc is also $= 0$.

188. Since the object of expanding the differential Xdx is to transform it into a series of terms, each of which may be integrated separately, it is not necessary that the form of this expansion should be always according to the powers of x. The following example gives the length of the arc of an ellipse whose semitransverse axis $= 1$, excentricity $= e$, and abscissa measured from the centre $= x$.

189. Prop. XXVIII.—*To find the integral of* $\frac{dx\sqrt{(1-e^2x^2)}}{\sqrt{(1-x^2)}}$ *in a series.*

When e is small, we may expand $\sqrt{(1 - e^2x^2)}$ into a converging series, since x^2 is always < 1. Now

$$\sqrt{(1 - e^2x^2)} = 1 - \frac{1}{2}e^2x^2 - \frac{1}{2.4}e^4x^4 - \frac{1.3}{2.4.6}e^6x^6 - \&c.;$$

and, therefore, the integral becomes

$$\int \frac{dx}{\sqrt{(1-x^2)}}\left(1 - \frac{1}{2}e^2x^2 - \frac{1}{2.4}e^4x^4 - \frac{1.3}{2.4.6}e^6x^6 - \&c.\right)$$

Let $\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x = A$, we have then (art. 161.)

$$\int \frac{dx\sqrt{(1-e^2x^2)}}{\sqrt{(1-x^2)}} = A + \frac{e^2}{2}\left(\frac{x\sqrt{1-x^2}}{2} - \frac{A}{2}\right.$$
$$\left. + \frac{e^4}{2.4}\left\{\left(\frac{x^3}{4} + \frac{3x}{2.4}\right)\sqrt{1-x^2} - \frac{3A}{2.4}\right\} + \&c.\right.$$

190. Prop. XXIX.—*To find an approximate series for the definite integral* $\int_a^b$ Xdx, *taken from* x = a *to* x = b.

Let $u = \int Xdx$; then, if u' be the value of u when x becomes $x + h$, we have, by Taylor's theorem,

$$u' = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

And because $u = \int Xdx$, therefore $\frac{du}{dx} = X$, $\frac{d^2u}{dx^2} = \frac{dX}{dx}$, &c. Hence

$$u' - u = Xh + \frac{dX}{dx}\frac{h^2}{1.2} + \frac{d^2X}{dx^2}\frac{h^3}{1.2.3} + \&c.$$

To deduce from this formula the value of $\int Xdx$ from $x = a$ to $x = b$, we must assume $u =$ value of this integral when $x = a$, and $u' =$ its value when $x = b$. Hence a must be substituted for x in the function

X, and its differential coefficients, which we will represent by A, A', A'', &c.; and $b-a$ for h. We shall, therefore, have

$$\int_a^b Xdx = u' - u = A\,(b-a) + A'\frac{(b-a)^2}{1\,.\,2} + A''\frac{(b-a)^3}{1\,.\,2\,.\,3} + \text{\&c.}$$

191. This series is so much the more convergent the less is the interval $b-a$. When this interval is too great to be convergent, we must divide it into a number of equal parts, each of which shall be sufficiently small to make the series converge rapidly, and then we can calculate the value of the integral for each part separately. Let n be the number of equal parts into which $b-a$ is divided, and let α be the value of each of these parts, so that $n\alpha = b-a$; also, let A, B, C, &c. be the value of X when a, $a+\alpha$, $a+2\alpha$, &c., are respectively substituted for x in the function X. Then the value of this integral between

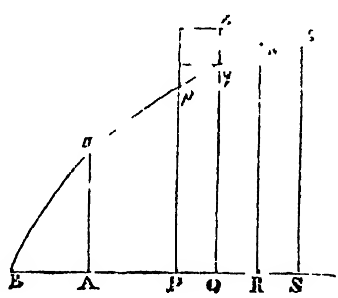
$$\begin{array}{lllll} a & \text{and} & a+\alpha & \text{is} & A\alpha + \frac{1}{2}A'\alpha^2 + \frac{1}{6}A''\alpha^2 + \text{\&c.,} \\ a+\alpha & \text{and} & a+2\alpha & - & B\alpha + \frac{1}{2}B'\alpha^2 + \frac{1}{6}B''\alpha^2 + \text{\&c.,} \\ a+2\alpha & \text{and} & a+3\alpha & - & C\alpha + \frac{1}{2}C'\alpha^2 + \frac{1}{6}C''\alpha^2 + \text{\&c.} \end{array}$$

If α be taken so small that its second and higher powers may be neglected, we shall have the approximate value of $\int Xdx$ between the limits of $x=a$ and $x=b$ equal to

$$A\alpha + B\alpha + C\alpha + D\alpha + \text{\&c.}$$

Scholium.

192. The consideration of curve lines will serve to illustrate the preceding remarks. Since the general expression for the differential of a curvilineal area is $\int ydx$ (Chap. III.), where x = abscissa AP, and y = ordinate Pp; if we represent X any function of x by the ordinate Pp, the integral $\int Xdx$ will be represented by the area $AapP$. Now the expression Xdx may be considered either as the differential of the area $AapP$, or of the area BpP, and therefore the ordinate which limits the area at its commencement is indeterminate. If, however, x be supposed to increase from any determinate magnitude $x = AP = a$ to $x = AS = b$, the integral $\int Xdx$ will increase from the area $AapP$ to $AasS$, and the difference of these areas $= PpsS$, which is entirely determinate, and is the same whether we suppose the areas to commence from the point A or the point B.



Since the area $PpsS$ is manifestly comprised between the sum of the inscribed rectangles pQ, qR, rS, &c., and the sum of the circumscribed rectangles Pq, Qr, Rs, &c., it is evident, if we make $Pp = A$, $Qq = B$, &c., and $PQ = QR = \text{\&c.} = \alpha$, that the curvilineal area is comprised between the values,

$$A\alpha + B\alpha + C\alpha + \text{\&c.} \quad \text{and} \quad B\alpha + C\alpha + D\alpha + \text{\&c.}$$

The difference between these two values is evidently equal to the rectangle pz, and this difference may be made as small as we please by di-

minishing the distance of the ordinates. When this distance becomes infinitely small, the sum of these products becomes ultimately equal to the curvilineal area $PpsS$; and it was on this account that the function from which the differential was derived was called its integral.

CHAP. II.—INTEGRATION OF DIFFERENTIAL EQUATIONS.

193. It has been shown in art. 63, that when any equation is given between x a variable quantity and y a function of x, we may deduce the equation that expresses the relation between dx and dy. We are now to consider how we may return from the *differential equation* to the equation from which it was derived, and which is usually called its *integral* or *primitive equation*.

194. As any primitive equation and its differential equation are both true together, it follows that, by means of the two equations, we may eliminate any one of the constant quantities, and thus deduce a differential equation, in which one of the constant quantities contained in the primitive shall not be found.

If, for example, we take the primitive equation $y + ax + b = 0$, we shall have, in taking its differential $dy + adx = 0$, an equation in which b is not found; if, however, it be required to find an equation in which a shall be wanting, we have only to eliminate a from the two equations

$$y + ax + b = 0, \qquad dy + adx = 0,$$

and the resulting equation $ydx - xdy + bdx = 0$ may likewise be considered as the differential equation of the primitive $y + ax + b = 0$.

It appears, from what we have stated, that a differential equation may contain one constant quantity less than its primitive equation contains; and, on the contrary, when any differential equation is given, its primitive equation may contain one constant quantity more than the differential equation.

195. Any equation which expresses the relation between dy and dx is called a differential equation of the *first order*; an equation which expresses the relation between d^2y, dy and dx, is called a differential equation of the second order, and so on.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

196. When it is required to find the primitive equation corresponding to any differential equation of the first order, we may endeavour to separate the variable quantities, that is, to bring the equation to the form $Xdx + Ydy = 0$, where X is a function of x alone, and Y a function of y alone. When we have arrived at this equation, the terms Xdx, Ydy, may be integrated by the methods already explained in the first chapter, and we have $\int Xdx + \int Ydy = C$, C being an arbitrary constant.

Ex.—Let $mydx + nxdy = 0$.

Dividing the terms of this equation by xy, it becomes $\frac{mdx}{x} + \frac{ndy}{y} = 0$; and taking the integrals, we have $m \log x + n \log y = \text{const.} = \log c$, since the arbitrary constant may be considered as a logarithm. Hence

$$\log x^m + \log y^n = \log (x^m y^n) = \log c; \text{ and } x^m y^n = c.$$

197. Prop. I.—*To separate the variable quantities in homogeneous equations.*

When the sum of the exponents of the variable quantities x and y is the same in each term, the equation is said to be *homogeneous*. In this case we may always separate the variables by assuming $y = xz$. Thus, if we take the example

$$xdx + ydy = nydx,$$

then making $y = xz$ and $dy = zdx + xdz$, we have

$$xdx + xz\,(zdx + xdz) = nxzdx;$$

or, $$(1 - nz + z^2)\,dx + xzdz = 0;$$

$$\therefore \frac{dx}{x} + \frac{zdz}{1 - nz + z^2} = 0. \text{ Hence, integrating,}$$

$$\log x + \int \frac{zdz}{1 - nz + z^2} = C.$$

The form of the integral $\int \frac{zdz}{1 - nz + z^2}$ will depend upon the value of n. If n be > 2, it will be a logarithmic function, and if n be < 2, it will be expressed by means of a circle.

198. As a second example, we may take the differential equation

$$xdy - ydx = dx\,\sqrt{(x^2 + y^2)}.$$

Assume, as before, $y = xz$, $dy = xdz + zdx$; then the equation becomes

$$dx\,\sqrt{(1 + z^2)} - xdz = 0; \quad \text{or } \frac{dx}{x} - \frac{dz}{\sqrt{(1 + z^2)}} = 0.$$

Hence, taking the integrals,

$$\log x - \log (z + \sqrt{1 + z^2}) = \text{const.} = \log c;$$

$$\therefore \frac{x}{z + \sqrt{(1 + z^2)}} = x(\sqrt{1 + z^2} - z) = \sqrt{x^2 + y^2} - y = c.$$

We have, therefore, finally, $x^2 = c^2 + 2cy$.

199. Prop. II.—*It is required to integrate the equation*

$$(a + mx + ny)\,dx + (b + px + qy)\,dy = 0.$$

Assume $x = t + \alpha$, $y = u + \beta$, then $dx = dt$, and $dy = du$. Hence

$$(a + m\alpha + n\beta + mt + nu)\,dt + (b + p\alpha + q\beta + pt + qu)\,du = 0,$$

and assuming $a + m\alpha + n\beta = 0$, $b + p\alpha + q\beta = 0$, since we have two indeterminate quantities α and β, we shall then have

$$(mt + nu)\,dt + (pt + qu)\,du = 0,$$

which is an homogeneous equation, and therefore may be integrated by the last article.

200. This transformation will not apply when $mq - np = 0$, because then the values of α and β would be infinite. But in this case we have $q = \frac{np}{m}$, and therefore $px + qy = \frac{p}{m}(mx + ny)$; hence the proposed equation becomes

$$adx + bdy + (mx + ny)\left(dx + \frac{p}{m}\,dy\right) = 0.$$

Assume now, $mx + ny = z$, then $dy = \frac{dz}{n} - \frac{mdx}{n}$;

and these values being substituted in the preceding equation, we obtain, after proper reduction,

$$dx + \frac{(bm + pz)\,dz}{anm - bm^2 + (nn - pm)z} = 0,$$

an equation which may easily be integrated by the rules in the last chapter.

201. Prop. III.—*To integrate the equation* $dy + Pydx = Qdx$, *when* P *and* Q *denote any functions of* x.

Assume $y = vz$, then $dy = zdv + vdz$; substitute these values in the proposed equation, and it becomes

$$zdv + v(dz + Pzdx) = Qdx.$$

Now, as we have only one equation between the two indeterminate functions v and z, namely $y = vz$, we may assume

$$vdz + Pvzdx = 0, \text{ and consequently } zdv = Qdx.$$

From the first of these equations we have $\frac{dz}{z} + Pdx = 0$, therefore $\log z = -\int Pdx = -R$ by substitution. Hence, by the nature of logarithms (Alg. art. 391), $z = e^{-R}$. It is not necessary to add a constant quantity in this place, as it will be sufficient to add it at the end of the operation. From the second equation we have

$$dv = \frac{Qdx}{z} = e^{R}\,Qdx;\quad \therefore\quad v = \int e^{R}\,Qdx + C, \text{ and}$$

$$y = vz = e^{-R}\left(\int e^{R}\,Qdx + C\right)$$

The general equation $dy + Pydz = Qdx$, which involves the first power only of y and dy, has been called a *linear equation of the first order*; it has also, with more propriety, been called a *differential equation of the first degree and the first order*.

202. The most general form that can be given to a differential equation of the first order, and consisting of three terms only, is

$$au^m z^n dz + bu^p z^q du = cu^r z^s du.$$

To give this equation a more simple form, let all its terms be divided by $u^m z^s$, it then becomes

$$az^{n-s}dz + bu^{p-m}z^{q-s}du = cu^{r-m}du.$$

Suppose now, $y = z^{n-s+1}$, $x = u^{p-m+1}$, then

$$dy = (n-s+1)z^{n-s}dz, \quad dx = (p-m+1)u^{p-m}du,$$

and it is manifest that

z^{q-s} is of the form y^{α}, and $u^{r-m}du$ of the form $x^{\beta}dx$.

Substituting these different values above, and reducing, we obtain finally an equation of this form,

$$dy + b'y^{\alpha}dx = c'x^{\beta}dx.$$

203. We have already considered the case when $\alpha = 1$, in art. 201. When $\alpha = 2$, we get the equation

$$dy + by^2dx = cx^{\beta}dx,$$

first considered by *Riccati*, an Italian mathematician, whose name it bears. He only succeeded, however, in separating the variables when $\beta = -2$, or $\beta = \frac{-4i}{2i \pm 1}$, where i denotes any positive integer.

INTEGRATION OF FUNCTIONS WHICH FULFIL THE CONDITIONS OF INTEGRABILITY.

204. An equation of the form $Mdx + Ndy = 0$, does not always arise from differentiating a function of two variables; for, after differentiation, it may be multiplied or divided by any function; or this equation may be supposed to arise from the elimination of an arbitrary constant between the primitive equation and its immediate differential (art. 191). If, for example, we had the equation $xdy - ydx = 0$, it could not be immediately produced by taking the differential of any function of x and y, but if we divide the equation by x^2, so as to give it the form $\frac{xdy - ydx}{x^2} = 0$, it becomes a complete differential, namely, that of the fraction $\frac{y}{x}$. Hence

$$\frac{y}{x} + c = 0, \quad \text{or} \quad y + cx = 0$$

is the primitive equation. The equation $xdy - ydx = 0$ may also be supposed to arise from eliminating c between the equation $y + cx = 0$ and its differential $dy + cdx = 0$.

When a differential equation may be obtained immediately by the process of differentiation, it is called a *complete differential.*

205. **Prop. IV.**—*To determine when an equation is a complete differential.*

Let $u = f(x, y)$ be any equation between x and y, and let $du = Mdx + Ndy$, then $\frac{du}{dx} = M$, $\frac{du}{dy} = N$. Now it appears, from art. 59, that

$$\frac{dM}{dy} = \frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx} = \frac{dN}{dx}.$$

Hence, if the condition $\frac{dM}{dy} = \frac{dN}{dx}$ be satisfied, the function $Mdx + Ndy$ is a complete differential.

In the example $xdy - ydx = 0$, $M = -y$, $N = x$, therefore, $\frac{dM}{dy} = -1$, $\frac{dN}{dx} = 1$, hence the equation is not a complete differential. But if we take $\frac{xdy - ydx}{x^2} = 0$, $M = -\frac{y}{x^2}$, $N = \frac{x}{x^2} = \frac{1}{x}$, therefore, $\frac{dM}{dy} = -\frac{1}{x^2} = \frac{dN}{dx}$, and, consequently, this equation is a complete differential.

206. **Prop. V.**—*To integrate an equation which is a complete differential.*

Since, in the equation $du = Mdx + Ndy$, the function Mdx has been deduced from u, by considering x as variable and y constant, it follows that all the terms of u which contain only y must have disappeared. Hence, if we put Y to denote these terms, we shall have

$$u = \int Mdx + Y,$$

the integral $\int Mdx$ being taken with respect to x only as variable. The function Y may be determined by comparing the differential of $\int Mdx + Y$ with the given differential $Mdx + Ndy$.

Ex. 1.

207. To integrate the differential $du = \frac{ydx - xdy}{x^2 + y^2}$.

This expression being reduced to the form $Mdx + Ndy$, gives

$$M = \frac{y}{x^2 + y^2}, \quad N = \frac{-x}{x^2 + y^2}, \quad \therefore \frac{dM}{dy} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{dN}{dx},$$

and, consequently, this function is a complete differential. We have, therefore,

$$\int Mdx = \int \frac{ydx}{x^2 + y^2} = \tan^{-1}\frac{x}{y}.$$

Hence $$u = \tan^{-1}\frac{x}{y} + Y.$$

Differentiating this expression upon the supposition that both x and y are variable, we shall find

$$du = \frac{ydx - xdy}{x^2 + y^2} + dY.$$

And, comparing this with the proposed differential, we have $dY = 0$, and $Y = C$, a constant quantity.

Ex. 2.

208. To integrate the differential

$$du = dx\sqrt{(a^2 + y^2)} + dy\frac{a^2 + xy + 2y^2}{\sqrt{(a^2 + y^2)}}.$$

Here $M = \sqrt{(a^2 + y^2)}$, $N = \dfrac{a^2 + xy + 2y^2}{\sqrt{(a^2 + y^2)}}$, $\dfrac{dM}{dy} = \dfrac{y}{\sqrt{(a^2 + y^2)}} = \dfrac{dN}{dx}$,
and, consequently, this expression is a complete differential. Hence

$$\int Mdx = \int dx\sqrt{(a^2 + y^2)} = x\sqrt{(a^2 + y^2)},$$
$$u = x\sqrt{(a^2 + y^2)} + Y.$$

To determine Y, we take the differential of this expression, considering both x and y variable, and we obtain

$$du = dx\sqrt{(a^2 + y^2)} + \frac{xydy}{\sqrt{(a^2 + y^2)}} + dY.$$

And, comparing this with the proposed differential,

$$du = dx\sqrt{(a^2 + y^2)} + dy\frac{a^2 + xy + 2y^2}{\sqrt{(a^2 + y^2)}},$$

we find $\quad dY = \dfrac{(a^2 + 2y^2)\,dy}{\sqrt{(a^2 + y^2)}}$, and, consequently,

$$Y = \int\frac{(a^2 + 2y^2)\,dy}{\sqrt{(a^2 + y^2)}} = y\sqrt{(a^2 + y^2)}.$$

Hence the required integral is

$$u = x\sqrt{(a^2 + y^2)} + y\sqrt{(a^2 + y^2)} = (x + y)\sqrt{a^2 + y^2}.$$

209. It appears, from what we have stated in art. 204, that when the equation $Mdx + Ndy = 0$ does not satisfy the condition of integrability, some factor has disappeared, which, if it were known and restored, would render this expression a complete differential. A general method, however, of determining this factor is attended with such difficulties that very little progress has been made in this way, except in some particular cases, whose solutions had been previously obtained by more simple and direct means.

EQUATIONS OF THE FIRST ORDER, IN WHICH THE DIFFERENTIALS EXCEED THE FIRST DEGREE.

210. When a differential equation involves the second or higher powers of dx and dy, such as the equation

$$dy^3 + Pdy^2dx + Qdydx^2 + Rdx^3 = 0,$$

which may be put under the form,

$$\left(\frac{dy}{dx}\right)^3 + P\left(\frac{dy}{dx}\right)^2 + Q\frac{dy}{dx} + R = 0\ldots\ldots(1),$$

we may, by the theory of algebraic equations, deduce from it as many values of $\frac{dy}{dx}$ as there are units in its highest exponent (Alg. art. 262). Representing these roots by p, p', p'', we shall have the equations

$$\frac{dy}{dx} - p = 0, \quad \frac{dy}{dx} - p' = 0, \quad \frac{dy}{dx} - p'' = 0,$$

which may all be treated by the preceding methods, since the differentials in each do not exceed the first degree. Let the integrals of these equations be represented by

$$L = 0, \qquad M = 0, \qquad N = 0,$$

then each of these integral may evidently be considered as the integral of the proposed equation; for if $L = 0$, we have $dL = \frac{dy}{dx} - p = 0$, and, consequently, equation $(1) = 0$. Also, the product of all these equations, $LMN = 0$, may be considered as the integral of the given equation. For if $LMN = 0$, one of its factors, for example L, must necessarily $= 0$. We have, likewise,

$$dL \,.\, MN + dM \,.\, LN + dN \,.\, LM = 0,$$

which reduces itself to the single term $dL \,.\, MN = 0$ Hence $dL = 0$, or $\frac{dy}{dx} - p = 0$, and, consequently, the proposed equation will be verified by the equation $LMN = 0$.

Ex. —To integrate the equation $dy^2 - a^2dx^2 = 0$.

This equation may be decomposed into two others,

$$dy + adx = 0, \qquad dy - adx = 0,$$

and the integrals of these equations are

$$y + ax + c = 0, \qquad y - ax + c' = 0,$$

from either of which the differential equation $dy^2 - a^2dx^2 = 0$ may be derived. It may also be deduced from their product

$$(y + ax + c)\,(y - ax + c') = 0.$$

211. When the proposed equation contains only one of the two variable quantities x and y, for example x, we may, by the resolution of equations, obtain $\frac{dy}{dx} = X$ a function x, from whence $y = \int Xdx$; but if it be more easy to resolve the equation with respect to x than with respect to $\frac{dy}{dx}$, which we shall denote by p, then we shall have $x = P$, some function of p. And, since $dy = pdx$, therefore $y = \int pdx = px - \int xdp = Pp - \int Pdp$. The relation between x and y is now to be found by eliminating p from the two equations

$$x = P, \qquad y = Pp - \int Pdp.$$

Ex.—To integrate the equation $xdx + ady = b\sqrt{(dx^2 + dy^2)}$.

Substituting p for $\frac{dy}{dx}$, this equation becomes $x + ap = b\sqrt{(1+p^2)}$, therefore,

$$x = b\sqrt{(1+p^2)} - ap = P,$$

$$y = bp\sqrt{(1+p^2)} - \tfrac{1}{2}ap^2 - b\int dp\sqrt{(1+p^2)}.$$

And, when p is eliminated from these two equations, we shall obtain the required integral.

212. PROP. VI.—*To integrate the equation* $y = px + P$; P *being a function of* p *only.*

Differentiating this equation, we have

$$dy = pdx + \left(x + \frac{dP}{dp}\right) dp = 0;$$

and, since $dy = pdx$, there remains $\left(x + \frac{dP}{dp}\right) dp = 0$; therefore,

$$x + \frac{dP}{dp} = 0, \quad \text{and} \quad dp = 0.$$

If we eliminate p between the first of these and the proposed equation, we obtain a primitive equation which will satisfy the given equation, but it contains no arbitrary constant. The second factor $dp = 0$ being integrated, gives $p = \frac{dy}{dx} = c$, therefore, $dy = cdx$ and $y = cx + c'$. The constants c and c' are not both arbitrary, for by making in the proposed equation $p = c$, we have $y = cx + C$, C being the same function of c that P is of p. Hence it follows, that $c' = C$, and the integral of the proposed equation, therefore is $y = cx + C$.

213. *Ex.*—Let $ydx - xdy = n\sqrt{(dx^2 + dy^2)}$, or $y = px + n\sqrt{1+p^2}$.

By differentiating, we have $dy = pdx + xdp + \frac{npdp}{\sqrt{(1+p^2)}}$; and, since $dy = pdx$, there remains $xdp + \frac{npdp}{\sqrt{(1+p^2)}} = 0$, which may be decomposed into the two factors

$$x + \frac{np}{\sqrt{(1+p^2)}} = 0, \quad \text{and} \quad dp = 0.$$

The second factor gives $p = c$, and, consequently,

$$y = cx + n\sqrt{(1+c^2)} \ldots\ldots\ldots\ldots\ldots (1)$$

is the complete integral. The first factor gives $p = \frac{x}{\sqrt{(n^2 - x^2)}}$, $\sqrt{(1+p^2)} = -\frac{np}{x} = \frac{-n}{\sqrt{(n^2 - x^2)}}$, and substituting in the proposed equation, we obtain

$$y^2 + x^2 = n^2.$$

This equation involves no arbitrary constant, nor can it be derived

from the complete integral (1), by giving any particular value to the arbitrary constant c; it has, therefore, received the name of a *particular solution.*

Scholium.

214. In the differential equation which we have given in the last article, we obtained a solution in which there is no arbitrary constant, and which has a form entirely different from that of the complete integral, nor can it be derived from it by giving any particular value to the arbitrary constant. These *particular solutions* were for some time considered a sort of paradox in the Integral Calculus, until Lagrange, Poisson, and others, succeeded in elucidating some of the difficulties connected with this subject. Lagrange's theory may be briefly stated thus:

215. When we have a differential equation $V = 0$, it may be supposed to arise from eliminating the arbitrary constant between the primitive equation $u = f(x, y, c) = 0$, and its immediate differential $du = Mdx + Ndy = 0$. Now, whatever value be given to c, it is evident that we shall have the same equation $V = 0$ resulting from this elimination, even if c be considered as a function of x and y. Let us, then, suppose c to be variable, we shall now have $du = Mdx + Ndy + Cdc = 0$; but if $Cdc = 0$, we shall still have $du = Mdx + Ndy = 0$, and, therefore, the proposed equation $V = 0$ will also result from the elimination of c between the two equations $u = 0$, $du = 0$, in the same manner as before. Now, if $Cdc = 0$, we may either have $dc = 0$, and, therefore, $c = \text{const.}$, and the complete integral $f(x, y, c) = 0$ will remain the same; or we may have $C = 0$. If, from the equation $C = 0$, we derive a value of $c = \varphi(x, y) = \varphi$, a function of x and y, and substitute this value in $u = f(x, y, c) = 0$, the form of the function will be changed, and yet its differential $du = Mdx + Ndy = 0$ being the same, the equation resulting from the elimination of φ between the two equations $f(x, y, \varphi) = 0$ and $Mdx + Ndy = 0$ must be $V = 0$, the same as in the first case; and consequently the integral $u = f(x, y, \varphi)$ will satisfy the differential equation $V = 0$.

216. To apply these remarks to the equation given in art. 213, we have the complete integral

$$u = y - cx - n\sqrt{(1 + c^2)} = 0,$$

taking the differential, and making c vary at the same time with x and y, we obtain

$$du = dy - cdx - \left(x + \frac{nc}{\sqrt{(1 + c^2)}}\right) dc = 0;$$

hence $\quad -C = x + \dfrac{nc}{\sqrt{(1 + c^2)}}$. Making $C = 0$, we find $c = \dfrac{x}{\sqrt{(n^2 - x^2)}}$. This value of c changes the equation

$$y - cx = n\sqrt{(1 + c^2)} \quad \text{into} \quad x^2 + y^2 = n^2,$$

which is the particular solution given in art. 213.

217. As another example, let $u = x^2 - 2cy - a^2 - c^2 = 0$, be a primitive equation whose differential, after the elimination of c, is

$$\frac{dy^2}{dx^2}(x^2 - a^2) - 2xy\frac{dy}{dx} = x^2.$$

If we differentiate u, making c also variable, we shall find $C = c + y$; hence $y + c = 0$, or $c = -y$; and, therefore, u becomes

$$x^2 + y^2 - a^2 = 0,$$

which is the particular solution in this example.

218. The theory of *particular solutions* may be elucidated from geometrical considerations. It will be seen, that the first of these equations arises from the solution of the problem, *To find a curve such that all the perpendiculars, let fall from a given point* C *upon the tangents of this curve, shall be equal.* Now, it is evident that the circle whose radius is n, and centre C, will satisfy the conditions of the question. This circle having $x^2 + y^2 = n^2$ for its equation is the particular solution found in art. 213; but every straight line situated so that its least distance from the given point C shall be $= n$, will also resolve the problem; and, as an infinite number of straight lines may be drawn so as to satisfy this condition, it follows that it is in the equation comprehending all these lines, that we are to look for the complete integral of the above differential equation, which is, in fact,

$$y - cx = n\sqrt{(1 + c^2)}.$$

INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE SECOND AND HIGHER ORDERS.

219. Whatever difficulties occur in finding the integral of a differential equation of the first order, it will easily be conceived that they must be greater and more numerous when we have to consider differential equations of the second and higher orders; and it is only in a small number of very limited equations that mathematicians have succeeded in effecting their solution.

220. Prop. VII.—*To find the integral of* $d^2y = Xdx^2$, X *being a function of* x.

Since dx is constant, and $\frac{d^2y}{dx} = Xdx$, we have $\frac{dy}{dx} = \int Xdx + C$.

Let the integral $\int Xdx = P$, then $\frac{dy}{dx} = P + C$ and $dy = Pdx + Cdx$, and, taking the integrals a second time,

$$y = \int Pdx + Cx + C',$$

where C' denotes a second indeterminate constant quantity.

Ex.—Let $d^2y - axdx^2 = 0$.

We have then $\frac{dy}{dx} = \int axdx = \frac{1}{2}ax^2 + C$,

and $\qquad y = \int dx(\frac{1}{2}ax^2 + C) = \frac{1}{6}ax^3 + Cx + C'$.

The integral of $d^3y = Xdx^3, \ldots d^ny = Xdx^n$ may be found in the same manner.

221. Prop. VIII.—*To determine the integral of an equation involving only* $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, *and constant quantities.*

If in this equation we make $\frac{dy}{dx} = p$, and, consequently, $\frac{d^2y}{dx^2} = \frac{dp}{dx}$; then such an equation may be generally expressed thus: $\frac{dp}{dx} = P$, P being a function of p. Hence we have $dx = \frac{dp}{P}$ and $x = \int \frac{dp}{P} + C$. Also, $dy = pdx = \frac{pdp}{P}$, therefore, $y = \int \frac{pdp}{P} + C'$. If, then, we can eliminate p between the two equations

$$x = \int \frac{dp}{P} + C, \qquad y = \int \frac{pdp}{P} + C',$$

we shall have the integral required in terms of x and y.

222. *Ex.*—Let $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxd^2y} = a$.

By putting pdx for dy and $dpdx$ for d^2y, this equation becomes $\frac{(1 + p^2)^{\frac{3}{2}}dx}{dp} = a$; from which we deduce

$$dx = \frac{adp}{(1 + p^2)^{\frac{3}{2}}}, \qquad dy = pdx = \frac{apdp}{(1 + p^2)^{\frac{3}{2}}};$$

$$\therefore \quad x = C + \frac{ap}{\sqrt{(1 + p^2)}}, \qquad y = C' - \frac{a}{\sqrt{(1 + p^2)}}.$$

Eliminating p we obtain $(x - C)^2 + (y - C')^2 = a^2$.

This differential equation is evidently formed by putting the general expression for the radius of curvature equal to a constant quantity, and, as we should expect, the integral is the equation to a circle of which this constant quantity is the radius.

223. Prop. IX.—*To integrate the equation* $d^2y = Ydx^2$, Y *being a function of* y.

Making, as before, $dy = pdx$, and $d^2y = dpdx$, we obtain $\frac{dp}{dx} = Y$, and multiplying by dy, $pdp = Ydy$. Hence, we have, by integration,

$$p^2 = C + 2\int Ydy, \quad \text{and } x = \int \frac{dy}{p} = \int \frac{dy}{\sqrt{(C + 2\int Ydy)}}.$$

Ex.—Let $a^2d^2y + ydx^2 = 0$.

Here $a^2pdp = -ydy$, $\quad \therefore \; a^2p^2 = \text{const.} - y^2 = C^2 - y^2$.

$$\text{Hence } x = \int \frac{dy}{p} = \int \frac{ady}{\sqrt{(C^2 - y^2)}} = a\sin^{-1}\frac{y}{C} + C',$$

$$\text{or } \quad \frac{y}{C} = \sin\frac{x - C'}{a} = \cos\frac{C'}{a}\sin\frac{x}{a} - \sin\frac{C'}{a}\cos\frac{x}{a};$$

therefore y is of the form $c \sin \dfrac{x}{a} + c' \cos \dfrac{x}{a}$, c and c' being considered as two arbitrary constants.

224. PROP. X.—*To integrate an equation which involves* $\dfrac{d^2y}{dy^2}, \dfrac{dy}{dx}$, *and the independent variable* x.

This equation may be transformed immediately to a differential equation of the first order, by the substitution of pdx and $dpdx$ instead of dy and d^2y. If we can find the integral of this equation, and thence the value of p in terms of x, we shall have the value of y from the equation $y = \int pdx$; or, if the value of x be given in terms of p, we shall have $y = \int pdx = px - \int xdp$.

225. *Ex.*—Let $\dfrac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxd^2y} = X$ any function of x.

This equation becomes $\dfrac{(1 + p^2)^{\frac{3}{2}}dx}{dp} = X$, or $\dfrac{dx}{X} = \dfrac{dp}{(1 + p^2)^{\frac{3}{2}}}$;

$$\therefore \int \frac{dx}{X} + C = \frac{p}{\sqrt{(1 + p^2)}}.$$

If we represent $\int \dfrac{dx}{X} + C$ by V, then $p = \dfrac{V}{\sqrt{(1 - V^2)}}$, and

$$y = \int pdx + C' = \int \frac{Vdx}{\sqrt{(1 - V^2)}} + C'.$$

226. PROP. XI.—*To integrate an equation involving* $\dfrac{d^2y}{dx^2}, \dfrac{dy}{dx}$, *and* y.

If we substitute for d^2y its value $dpdx$, and then eliminate dx from this equation by means of the equation $dy = pdx$, the result will be an equation of the first order, containing only p, dp, and dy. When this last equation can be integrated, if p can be found in terms of y, we shall get x from the formula $x = \int \dfrac{dy}{p}$; or, if y can be obtained more easily in terms of p, we shall find x from the formula $x = \dfrac{y}{p} + \int \dfrac{ydp}{p^2}$.

227. *Ex.*—Let $\dfrac{d^2y}{dx^2} + A\dfrac{dy}{dx} + By = 0$, or $\dfrac{dp}{dx} + Ap + By = 0$.

(1). Multiplying this equation by $pdx = dy$, we have

$$pdp + Apdy + Bydy = 0.$$

As this is a homogeneous equation with respect to p and y, the variables may be separated by making $p = uy$, and $dp = ydu + udy$; we then find

$$\frac{dy}{y} = \frac{-udu}{u^2 + Au + B} = \frac{-udu}{(u - a)(u - b)},$$

a and b being the roots of the equation $u^2 + Au + B = 0$. Also,

$$dx = \frac{dy}{p} = \frac{dy}{uy} = \frac{-du}{(u-a)(u-b)};$$

$$\therefore \quad \frac{dy}{y} - adx = \frac{-du}{u-b}, \quad \frac{dy}{y} - bdx = \frac{-du}{u-a}.$$

Hence, $\log y - ax = \log \dfrac{c}{u-b}$, $\log y - bx = \log \dfrac{c'}{u-a}$;

$$\therefore \quad u - b = \frac{c}{y} e^{ax}, \quad u - a = \frac{c'}{y} e^{bx};$$

Subtracting the second equation from the first, we get

$(a-b)\,y = ce^{ax} - c'e^{bx}$; which may be put under the form

$$y = Ce^{ax} + De^{bx},$$

in which equation C and D may be considered as two arbitrary constants.

(2). If the roots a and b are imaginary, they will be of the form $a = \alpha + \beta\sqrt{-1}$, $b = \alpha - \beta\sqrt{-1}$, and, therefore, this equation becomes

$$y = e^{\alpha x}(Ce^{\beta x\sqrt{-1}} + De^{-\beta x\sqrt{-1}}).$$

Substituting for $e^{\pm\beta x\sqrt{-1}}$ the expression $\cos \beta x \pm \sqrt{-1} \sin \beta x$, and putting $C + D = C'$, $(C+D)\sqrt{-1} = D'$, we obtain

$$y = e^{\alpha x}(C' \cos \beta x + D' \sin \beta x).$$

(3). When $a = b$, then $\dfrac{dy}{y} = \dfrac{-udu}{(u-a)^2} = \dfrac{-(u-a)du}{(u-a)^2} - \dfrac{adu}{(u-a)^2}$;

$$\therefore \quad \log y = \log \frac{c}{u-a} + \frac{a}{u-a}.$$

Also $dx = \dfrac{-du}{(u-a)^2}$, and $x + c' = \dfrac{1}{u-a}$.

Substituting for $u - a$ in the preceding equation, we have

$$\log y = \log (cx + cc') + a(x + c'), \text{ consequently}$$

$$y = c\,(x + c')\,e^{a(x+c')} = (Cx + D)\,e^{ax}.$$

228. PROP. XII.—*To integrate the equation* $d^2y + Pdydx + Qydx^2 = 0$, *P and Q being functions of x only.*

Assume $y = e^{\int tdx}$, t being a new variable; we have, then,

$$dy = tdxe^{\int tdx}, \quad d^2y = (t^2dx^2 + dtdx)\,e^{\int tdx}.$$

Substituting these values in the proposed equation, and dividing by the common factor $e^{\int tdx}$, we obtain

$$t^2dx^2 + dt\,dx + Ptdx^2 + Qdx^2 = 0;$$

$$\therefore \quad dt + (t^2 + Pt + Q)\,dx = 0.$$

When the coefficients P and Q are constant, and equal to A and B, this equation is the same as that which we have integrated in the last article.

METHOD OF RESOLVING DIFFERENTIAL EQUATIONS BY APPROXIMATION.

229. When we cannot by any known methods find the integral of a differential equation in finite terms, we must endeavour to resolve it by approximation, that is, we must express the value of y in terms of x, by means of a series.

When the form of the series is known, we may determine the coefficients by substituting the series and its differential instead of y and dy in the proposed equation. If, for example, we had the equation

$$dy + ydx - mx^n dx = 0,$$

we may assume $y = Ax^\alpha + Bx^{\alpha+1} + Cx^{\alpha+2} + \&c.$

then $dy = \alpha Ax^{\alpha-1}dx + (\alpha + 1)Bx^\alpha dx + (\alpha + 2) Cx^{\alpha+1} dx + \&c.$

Substituting the values of y and dy in the equation above, and dividing the whole by dx, it becomes

$$\left.\begin{array}{llll} \alpha Ax^{\alpha-1} & + (\alpha + 1) Bx^\alpha & + (\alpha + 2) Cx^{\alpha+1} & + \&c. \\ - mx^n & + \quad Ax^\alpha & + \quad Bx^{\alpha+1} & + \&c. \end{array}\right\} = 0.$$

This equation becomes identical if we assume $\alpha - 1 = n$, or $\alpha = n + 1$ and

$$A = \frac{m}{\alpha}, \; B = \frac{-m}{\alpha(\alpha + 1)}, \; C = \frac{m}{\alpha(\alpha + 1)(\alpha + 2)}, \; D = \&c.$$

Hence we have

$$y = m\left\{\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{(n + 1)(n + 2)} + \frac{x^{n+3}}{(n + 1)(n + 2)(n + 3)} - \&c.\right\}.$$

This value of y is incomplete, since it contains no arbitrary constant; we may, however, obtain a value of y that shall have the requisite generality by proceeding as follows.

230. Let us suppose we know that $x = a$, when $y = b$. Assume $x = a + t$, and $y = b + u$, then it is manifest, that if the value of u be found by a series involving t, all the terms of the series ought to vanish when $t = 0$. By this transformation, the equation $dy + ydx - mx^n dx = 0$ becomes

$$du + (b + u)\, dt - m\,(a + t)^n\, dt = 0.$$

Assume now, $\quad u = At^\alpha + Bt^{\alpha+1} + Ct^{\alpha+2} + \&c.;$

then, proceeding as before, we find

$$\left.\begin{array}{llll} \alpha At^{\alpha-1} & + (\alpha + 1) Bt^\alpha & + (\alpha + 2) Ct^{\alpha+1} & + \&c. \\ + b & + \quad At^\alpha & + \quad Bt^{\alpha+1} & + \&c. \\ - ma^n & - \quad mna^{n-1}t & - \frac{1}{2} mn(n-1)a^{n-2}t^2 & - \&c. \end{array}\right\} = 0.$$

It is necessary in this equation to assume $\alpha - 1 = 0$, or $\alpha = 1$, and hence we find

$$A = ma^n - b, \quad B = \tfrac{1}{2}(mna^{n-1} - ma^n + b),$$

$$C = \tfrac{1}{6}\Big(mn(n - 1)\, a^{n-2} - mna^{n-1} + ma^n - b\Big), \quad D = \&c.$$

If we now substitute $x - a$ and $y - b$ for t and u respectively, the result will have all the generality that belongs to a primitive equation expressing the relation between x and y.

231. The same process which we have described is applicable to equations of higher orders. The most general is that where we assume for y a series in which both the exponents and the coefficients are undetermined. The following example will be sufficient to illustrate this.

Ex.—Let $d^2y + ax^ny\,dx^2 = 0.$

Assume $y = Ax^\alpha + Bx^{\alpha+\delta} + Cx^{\alpha+2\delta} + \&c.$

and suppose δ to be positive, or the series of exponents to be an increasing one. We have then

$$ax^nydx^2 = adx^2\,[Ax^{\alpha+n} + Bx^{\alpha+\delta+n} + \&c.]$$

$$d^2y = dx^2[\alpha(\alpha-1)Ax^{\alpha-2} + (\alpha+\delta)(\alpha+\delta-1)Bx^{\alpha+\delta-2} + \&c.]$$

It is not possible to give to α such a value that the two terms containing $x^{\alpha+n}$ and $x^{\alpha-2}$ shall correspond, except in the particular case when $n = -2$. But if the term containing $x^{\alpha-2}$ be made to vanish (which may be done either by making $\alpha = 0$, or $\alpha = 1$), we may compare the terms involving $x^{\alpha+n}$ and $x^{\alpha+\delta-2}$. Making the exponents of these terms equal, we have $\alpha + n = \alpha + \delta - 2$; and, therefore, $\delta = n + 2$.

Hence, substituting the values $\alpha = 0$, $\delta = n + 2$; and also, $\alpha = 1$, $\delta = n + 2$, in the preceding equations; and afterwards determining the coefficients A, B, C, &c. in the usual manner, we obtain for the value of y the two following series:

$$A - \frac{aAx^{n+2}}{(n+1)(n+2)} + \frac{a^2Ax^{2n+4}}{(n+1)(n+2)(2n+3)(2n+4)} - \&c.$$

$$Ax - \frac{aAx^{n+3}}{(n+2)(n+3)} + \frac{a^2Ax^{2n+5}}{(n+2)(n+3)(2n+4)(2n+5)} - \&c.$$

These two series are only particular values of y, since they contain each but one arbitrary constant A. This differential equation, however, is of such a nature that, from two particular values of y, we may deduce its general value; for let us denote these values by y' and y'', then, as each of them must satisfy the differential equation, we have

$$C(d^2y' + ax^ny'dx^2) = 0, \quad C'(d^2y'' + ax^ny''dx^2) = 0,$$

C and C' denoting two arbitrary constants; and, consequently, their sum must be equal to nothing, independently of the values of C and C'; that is,

$$(Cd^2y' + C'd^2y'') + ax^ndx^2(Cy' + C'y'') = 0.$$

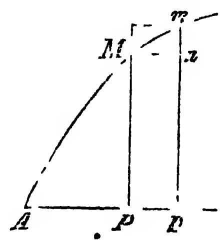
But the very same result will be obtained if we substitute $Cy' + C'y''$ instead of y in the given equation; therefore, $Cy' + C'y''$ is also a value of y, and, as it involves two arbitrary constants C and C', it possesses all the generality of which the value of y is susceptible. Hence it follows,

that if C be put instead of A in one of the two series above, and C' instead of A in the other series, the sum of the two series will be a general expression for the value of y.

CHAP. III.—APPLICATION OF THE INTEGRAL CALCULUS TO GEOMETRY.

TO FIND THE AREAS OF CURVES.

232. Prop. I.—*It is required to find a general expression for the area* APM, AP *being the abscissa of the curve* AM, *and* PM *its ordinate.*



Let $AP = x$, $PM = y$, and the area $APM = u$; also, let Pp the increment of $x = h$, then will the ordinate $pm = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1 \cdot 2} + \&c.$ (art. 93). And because u is evidently a function of x and y, or a function of x only, since y is a function of x; when x becomes $x + h$, u will be changed into the area Apm, which, therefore, by Taylo's theorem, is equal to

$$u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1 \cdot 2} + \frac{d^3u}{dx^3}\frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$\therefore \text{ area } PpmM = \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1 \cdot 2} + \frac{d^3u}{dx^3}\frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

Now the area $PpmM$ is always comprised between the values of the rectangles Pn and Pm, or between $PM \times Pp$ and $pm \times Pp$, that is, the value of the series $\frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1 \cdot 2} + \&c.$, is always contained between the values of the two functions

$$yh \quad \text{and} \quad yh + \frac{dy}{dx}\frac{h^2}{1} + \frac{d^2y}{dx^2}\frac{h^3}{1 \cdot 2} + \&c.;$$

and since these functions have the same first term yh, it follows, from art. 52, that this term is equal to the first term of the other series, or that

$$\frac{du}{dx}h = yh, \quad \text{or} \quad du = ydx, \quad \text{and} \quad u = \int ydx.$$

Hence, to find the area of any curve, we must either substitute for y its value in terms of x, or for dx its value in terms of y and dy, and then find the integral by the methods already explained. This is called the *method of quadratures.*

Examples.

233. *Ex.* 1.—To find the area of a curve of the parabolic kind whose equation is $px = y^n$.

Here we have $pdx = ny^{n-1}dy$, and, therefore,

$$\int ydx = \int \frac{ny^n\,dy}{p} = \frac{ny^{n+1}}{(n+1)p} + \text{const.}$$

If we suppose the area to commence at A; where $x = 0$, $y = 0$, then $u = 0$ and $y = 0$ at the same time; therefore, constant $= 0$, and

$$u = \frac{ny^{n+1}}{(n+1)p} = \frac{n}{n+1}xy.$$

If $n = 2$, the curve is the common parabola, and the area is equal to two-thirds of the circumscribing parallelogram.

*234. *Ex.* 2.—To find the area of a circle.

If $AP = x$, $PM = y$, and the radius of the circle $= a$, then $y = \sqrt{(2ax - x^2)}$ and $\int ydx = \int dx\sqrt{(2ax - x^2)}$.

The integral of this expression can only be obtained in a series, and it will be found equal to

$$2x\sqrt{2ax}\left(\frac{1}{3} - \frac{1}{2}\,\frac{x}{5\,.\,2a} - \frac{1\,.\,1}{2\,.\,4}\,\frac{x^2}{7\,.\,4a^2} - \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\,\frac{x^3}{9\,.\,8a^3} - \&c.\right)$$

235. *Ex.* 3.—To find the area of an ellipse.

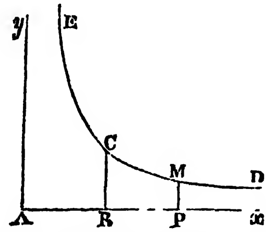
If a be a semitransverse, and b the semiconjugate axis, and v the area of a circle whose radius is a and abscissa $= x$ the abscissa of the ellipse, then will the area of the ellipse $= \dfrac{b}{a}v$.

236. *Ex.* 4.—To find the area of the common hyperbola.

If $a =$ semitransverse axis, $b =$ semiconjugate axis, $x =$ abscissa measured from the centre, then the area is equal to

$$\frac{b}{2a}x\sqrt{(x^2 - a^2)} - \frac{ab}{2}\log\frac{x + \sqrt{(x^2 - a^2)}}{a}.$$

237. *Ex.* 5.—To find the area of a curve of the hyperbolic species between the asymptotes.



The general equation to this curve is $y = \dfrac{p}{x^n}$, therefore, $\int ydx = \int \dfrac{pdx}{x^n}$

$$= \int px^{-n}\,dx = \frac{px^{1-n}}{1-n} + \text{const.}$$

To get the value of the area $BCMP$ taken from $x = AB = a$ to $x = AP = b$, we must successively make $x = a$ and $x = b$ in the preceding expression, and then subtract the first result from the second. We shall thus have the

$$\text{area } BCMP = \frac{p}{1-n}(b^{1-n} - a^{1-n}).$$

(1). If n be < 1, and $a = 0$, or the point B be supposed to coincide with the point A, the space $BCMP$ will be changed into $AyEMP$; and the term $a^{1-n} = 0$; therefore, the area $AyEMP = \frac{pb^{1-n}}{1-n}$. If a be of a determinate magnitude, and b be infinite, we shall get the area $BCDx$, which is therefore infinite.

(2). If n be > 1, the area $BCMP$ becomes equal to $\frac{p}{n-1}\left(\frac{1}{a^{n-1}} - \frac{1}{b^{n-1}}\right)$. When, therefore, $a = 0$, or B coincides with A, the area $AyEMP = \frac{p}{n-1}\left(\frac{1}{0} - \frac{1}{b^{n-1}}\right)$, and, consequently, it is infinite. If a be finite and b infinite, the area $BCDx$ $= \frac{p}{n-1}\left(\frac{1}{a^{n-1}} - \frac{1}{\infty}\right) = \frac{p}{n-1}\frac{1}{a^{n-1}}$, and is therefore finite.

(3). If $n = 1$, the curve is the common hyperbola; and it is called the equilateral hyperbola when the angle at A is a right angle.

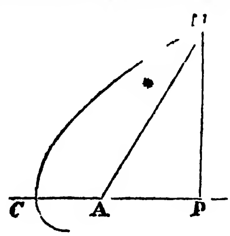
In this case the expression for the area $= \int \frac{pdx}{x} = p \log x +$ const., and taken from $x = a$ to $x = b$, the area $BCMP = p(\log b - \log a)$ $= p \log \frac{b}{a}$. The asymptotic spaces are both infinite in this case; for this expression is manifestly infinite, either when $a = 0$, or $b = \infty$.

Let $AB = BC = 1$, then $p = AB \times BC = 1$, therefore, the area $BCMP = p \log \frac{b}{a} = \log b =$ Naperian logarithm of AP. It was from the consideration of this property that these logarithms were originally called *hyperbolic logarithms*. But any other system of logarithms whose modulus is < 1, may be found from the area of the hyperbola, by varying the angle between the asymptotes: thus, if the angle be 25° 44′, the area will represent the common system of logarithms.

238. *Ex*. 6.—To find the area of a cycloid.

The whole area of the cycloid is equal to three times the area of the generating circle.

239. Prop. II.—*To find an expression for the area of a polar curve.*



Let CM be any polar curve, and A the pole. Let $AM = u$, $\angle CAM = \theta$. Draw MP perpendicular to CAP, and put $AP = x$, $PM = y$; also, put the area $CAM = v$, area $CPM = w$. Now we have

area $CAM = CPM - APM$, or $v = w - \frac{1}{2}xy$;

$\therefore dv = dw - d.\frac{1}{2}xy = ydx - \frac{1}{2}(xdy + ydx)$
$= \frac{1}{2}(ydx - xdy)$.

But $x = u\cos MAP = -u\cos\theta$, $dx = -du.\cos\theta + ud\theta.\sin\theta$,
$y = u\sin MAP = u\sin\theta$, $dy = du.\sin\theta + ud\theta.\cos\theta$.

Substituting these values in the preceding equation, we obtain

$$dv = \tfrac{1}{2}u^2 d\theta, \qquad \text{and} \quad v = \int \tfrac{1}{2}u^2 d\theta:$$

240. *Ex.* 7.—To find the area of the spiral whose equation is $u = a\theta^n$.

$$\text{area} = \int \frac{u^2 d\theta}{2} = \int \frac{a^2\theta^{2n} d\theta}{2} = \frac{a^2\theta^{2n+1}}{4n+2}.$$

This integral does not require a constant quantity to be added, as the area vanishes at the same time with θ.

In the spiral of Archimedes $n = 1$, therefore the area $= \frac{1}{6}a^2\theta^3$.

In the hyperbolic spiral $n = -1$, therefore the area $ACM = -\frac{a^2}{2\theta}$ + const. The area of this curve, which makes an infinite number of revolutions round the point A, is infinite when $\theta = 0$. We must, therefore, take this integral between two given values of θ, $\theta = b$, and $\theta' = c$, and the corresponding area will become $\frac{a^2}{2}\left(\frac{1}{b} - \frac{1}{c}\right)$.

241. *Ex.* 8.—To find the area of the logarithmic spiral.

Here $\theta = \log u$ and $d\theta = \frac{du}{u}$. Hence

$$\int \tfrac{1}{2} u^2 d = \int \tfrac{1}{2} u du = \tfrac{1}{4}u^2.$$

We suppress the constant quantity, because the area is nothing when $u = 0$.

TO FIND THE LENGTHS OF CURVES.

242. The differential of the arc of a curve when referred to rectangular co-ordinates is expressed by $\sqrt{dx^2 + dy^2}$, (art. 98). If, therefore, from the equation of the curve, we find the value of dy in terms of dx and x, or the value of dx in terms of dy and y, and substitute either of them in the preceding formula, we shall obtain a differential the integral of which will be the length of the curve. This is called the *rectification* of curves.

243. *Ex.* 1.—To find the length of the common parabola.

Let $AP = x$, $PM = y$, arc $AM = s$; then the equation of the curve being $y^2 = 2px$, we have $dx = \frac{ydy}{p}$, and

$$s = \int \sqrt{(dx^2 + dy^2)} = \int \sqrt{\left(\frac{y^2dy^2}{p^2} + dy^2\right)} = \int \frac{dy}{p}\sqrt{(p^2 + y^2)}.$$

To integrate this expression we have

$$\int dy\sqrt{(y^2 + p^2)} = \int \frac{y^2 dy}{\sqrt{(y^2 + p^2)}} + \int \frac{p^2 dy}{\sqrt{(y^2 + p^2)}}; \text{ also,}$$

$$\int \frac{y^2 dy}{\sqrt{(y^2 + p^2)}} = \int y \times \frac{ydy}{\sqrt{(y^2 + p^2)}} = y\sqrt{(y^2 + p^2)} - \int dy\sqrt{(y^2 + p^2)}.$$

Substituting this in the preceding equation, and transposing, we get

$$2\int dy\sqrt{(y^2+p^2)} = y\sqrt{(y^2+p^2)} + \int\frac{p^2dy}{\sqrt{(y^2+p^2)}}$$

$$= y\sqrt{(y^2+p^2)} + p^2\log\left(y+\sqrt{y^2+p^2}\right);$$

$$\therefore \quad s = \frac{y\sqrt{(y^2+p^2)}}{2p} + \frac{p}{2}\log\left(y+\sqrt{y^2+p^2}\right) + C.$$

To determine the value of C, we must consider that when $y = 0$, then $s = 0$, and, therefore,

$$0 = \tfrac{1}{2}p\log p + C.$$

Subtracting this equation from the preceding one, we get

$$s = \frac{y\sqrt{(y^2+p^2)}}{2p} + \frac{p}{2}\log\frac{y+\sqrt{(y^2+p^2)}}{p}.$$

244. *Ex* 2.—To find the length of the arc of an ellipse.

To simplify the calculation, let the semitransverse axis $AC = 1$, the excentricity $= e$, and the semiconjugate axis $b = \sqrt{(1-e^2)}$. Also, let $AP = x$, $PM = y$, and the arc $AM = s$. Then, from the equation of the ellipse

$$y = \sqrt{1-e^2}\sqrt{(1-x^2)}, \quad \therefore dy = \frac{-\sqrt{(1-e^2)}\,xdx}{\sqrt{(1-x^2)}},$$

$$\therefore \quad s = \int\sqrt{(dx^2+dy^2)} = \int\frac{dx\sqrt{(1-e^2x^2)}}{\sqrt{(1-x^2)}}.$$

The integral of this expression can only be obtained in the form of a series; and it has been already given in this form in art. 189.

If, in this series, we suppose $x = 1$, and therefore $A =$ a quadrant $= \frac{1}{2}\pi$, we shall have the value of the elliptic quadrant equal to

$$\frac{\pi}{2}\left(1 - \frac{e^2}{2.2} + \frac{1.3}{2.2.1.4}\frac{e^4}{} - \frac{1.3.3.5}{2.2.4.1.6.6}e^6 + \&c.\right),$$

a series which converges very rapidly when e is small.

245. *Ex.* 3.—To find the length of the logarithmic spiral.

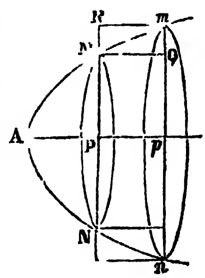
The equation to this curve is $\theta = \log u$, and the differential of the arc of a polar curve $= \sqrt{(u^2d\theta^2 + du^2)}$, which in this case, therefore, $= du\sqrt{2}$. Hence

$$s = u\sqrt{2} + C = u\sqrt{2},$$

if we suppose the arc to commence from the origin of the radius vectors. Hence, although an infinite number of revolutions be made between the pole and any given point of the curve, yet they include an arc of finite length, which is equal to the diagonal of the square described on the radius vector.

TO FIND THE CONTENT OF SOLIDS.

246. PROP. III.—*To find an expression for the differential of a solid of revolution.*



Let AMN be a portion of a solid generated by the revolution of the curve AM about the line AP, taken in the plane of the curve as an axis. Let MN, mn, be two planes perpendicular to the axis AP, cutting the plane APM in the lines PM, pm. Draw MQ, mR, parallel to AP. Put $AP = x$, $PM = y$, and the content of the solid $AMN = u$; also, let Pp the increment of $x = h$; then will the ordinate $pm = y + ph + qh^2 + \&c. = y + k$ (art. 93). And because u is evidently a function of x, when x becomes $x + h$, u will be changed into

$$u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3u}{dx^3}\,.\frac{h^3}{1\,.\,2\,.\,3} + \&c.,$$

which, therefore, is equal to the solid content Amn. Hence we have

$$MNnm = \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3u}{dx^3}\,\frac{h^3}{1\,.\,2\,.\,3} + \&c. . (1)$$

Now, the content of the solid $MNnm$ is always comprised between the content of the cylinder NQ, having the circle MN for its base, and altitude Pp, and the cylinder nR having the same altitude and the circle mn for its base, that is, between $\pi\,.\,PM^2 \times Pp$ and $\pi\,.\,pm^2 \times Pp$, or between the expressions

$$\pi y^2 h \quad \text{and} \quad \pi h(y + k)^2.$$

And as these functions have manifestly the same first term $\pi y^2 h$, it follows, from art. 52, that this term is equal to the first term of series (1), or that

$$\frac{du}{dx}h = \pi y^2 h, \quad \text{or } u = \pi \int y^2 dx.$$

Examples.

1. The content of a paraboloid is equal to half that of a cylinder having the same base and altitude.

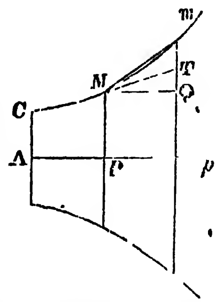
2. The content of a parabolic spindle, about a double ordinate to the axis, is equal to $\frac{8}{15}$ of the circumscribing cylinder.

3. The content of a sphere is equal to $\frac{2}{3}$ of the circumscribing cylinder.

4. The content of a spheroid is also equal to $\frac{2}{3}$ of the circumscribing cylinder.

TO FIND THE SURFACES OF SOLIDS.

247. Prop. IV.—*To find an expression for the differential of the surface of a solid of revolution.*



The same construction and the same notation being used as in the last article, it is evident that the surface described by the arc Mm, is $>$ surface described by the chord Mm and $<$ the sum of the two surfaces described by MT and Tm. Now, the surface described by the chord Mm is manifestly the surface of the frustum of a cone, and this, by Geometry (prop. 127), is equal to $\frac{1}{2} Mm\,(2\pi.PM + 2\pi.pm) = \pi.\,Mm\,(PM + pm)$. Also, the surface described by $MT = \pi.\,MT\,(PM + pT)$, and the area of the ring described by $Tm = \pi(pm^2 - pT^2) = \pi.\,Tm\,(pM + pT)$. Hence the surface described by the arc Mm is always contained between the values

$$\pi.Mm(Pm+pm), \text{ and } \pi.MT(PM+pT) + \pi.Tm(pm+pT).$$

And because $MQ = h$, $Qm = ph + qh^2 + \&c. = k$,

$$QT = ph, \text{ and } TM = qh^2 + rh^3 + \&c.;$$

therefore $Mm = \sqrt{(h^2 + k^2)}$, $MT = \sqrt{(h^2 + p^2h^2)} = h\sqrt{(1+p^2)}$. Hence, substituting these values in the preceding expressions, we shall have

$$\pi.\,Mm\,(PM + pm) = \pi\sqrt{(h^2 + k^2)}\,(2y + k),$$

and $$\pi.\,MT(PM + pT) + \pi.\,Tm\,(pm + pT)$$

$$= \pi h\sqrt{(1 + p^2)}\,(2y + ph) + \pi(qh^2 + \&c.)\,(2y + 2ph + \&c.).$$

Now, it is evident, if we substitute for k its value $ph + qh^2 + \&c.$, and expand these two functions according to powers of h, that the first term in each is $2\pi yh\sqrt{(1 + p^2)}$.

Also, because the surface described by CM is manifestly a function of x, if we put it $= u$, we shall have, by Taylor's theorem, the surface described by Mm

$$= \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

Hence it may be proved, in the same manner as in the last article, that

$$\frac{du}{dx}\frac{h}{1} = 2\pi yh\sqrt{(1 + p^2)}; \quad \therefore u = \int 2\pi y\sqrt{(dx^2 + dy^2)}.$$

Examples.

248. *Ex.* 1.—To find the surface of the segment of a sphere.
Let $AP = x$, $PM = y$, radius $CM = a$, then

$$y = \sqrt{(2ax - x^2)}, \quad dy = \frac{(a - x)dx}{\sqrt{(2ax - x^2)}}. \quad \text{Hence}$$

$$dx^2 + dy^2 = dx^2 + \frac{(a - x)^2 dx^2}{2ax - x^2} = \frac{a^2 dx^2}{2ax - x^2} = \frac{a^2 dx^2}{y^2};$$

$$\therefore \quad \int 2\pi y \sqrt{(dx^2 + dy^2)} = \int 2\pi a dx = 2\pi a x.$$

Hence it appears that the surface of a segment of a sphere is equal to the circumference of a great circle multiplied into the height of the segment; and also that the whole surface of the sphere $= 4 \,.\, \pi a^2 =$ four times the area of one of its great circles.

Ex. 2.—The surface of a right cone is equal to the circumference of the base multiplied by half the slant side.

Ex. 3.—The surface of a paraboloid is equal to $\frac{2\pi}{3p}\left[(y^2 + p^2)^{\frac{3}{2}} - p^3 \right]$.

Ex. 4.—The whole surface generated by the revolution of a cycloid round its base $= \frac{32}{3}\pi a^2$; and the surface generated by the revolution of the cycloid round its axis $8\pi a^2(\pi - \frac{4}{3})$.

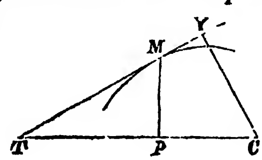
GEOMETRICAL PROBLEMS PRODUCING DIFFERENTIAL EQUATIONS.

249. PROBLEM I.—*To find the equation to a curve in which the subtangent is a given function of the abscissa.*

Let X denote the given function of x the abscissa; then (art. 95), the subtangent is equal to $\frac{ydx}{dy}$, and therefore by the question $\frac{ydx}{dy} = X$ or $\frac{dy}{y} = \frac{dx}{X}$, and $\log y = \int \frac{dx}{X}$.

If the subtangent $= a$, a constant quantity, then $\log y = \int \frac{dx}{a} = \frac{x}{a} + \text{const.} = \frac{x}{a} + \log c$, therefore $\log \frac{y}{c} = \frac{x}{a}$, which is the equation to the logarithmic curve.

250. PROBLEM II.—*To find a curve such that all the perpendiculars let fall from a given point upon the tangents of this curve shall be equal.*



Let C be the origin of co-ordinates, $CP = x$, $PM = y$; then, because PT is measured in the same direction with x, $PT = -\frac{ydx}{dy}$; also let CY the perpendicular on the tangent $= n$. We have then

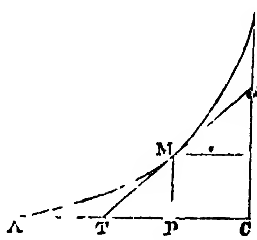
$$CT : CY :: TM : PM, \text{ that is (art. 95),}$$

$$x - \frac{ydx}{dy} : n :: y\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} : y, \text{ and consequently}$$

$$xdy - ydx = n\sqrt{(dx^2 + dy^2)}.$$

The complete integral of this equation is $y = cx + n\sqrt{(1 + c^2)}$, and a particular solution is $x^2 + y^2 = n^2$. The first is the equation to a straight line whose least distance from the point C is $= n$, and the second is the equation to a circle. (See articles 213, 218.)

251. Problem III.—*Let* CB *be perpendicular to* AC, *and* A *a given point in one of the lines* AC; *let the line* ST *be drawn cutting off equal segments* AT, CS; *it is required to find the nature of the curve to which* ST *is always a tangent.*



Let M be the point in which the tangent ST meets the curve, draw MP perpendicular to AC, and MQ perpendicular to BC. Put $AC = a$, $AP = x$, $PM = y$; then $CP = a - x$, and $PT = \frac{ydx}{dy}$. And because $TP : PM :: QM : QS$, therefore $QS = (a - x)\frac{dy}{dx}$. Hence

$y + (a - x)\frac{dy}{dx} = CS = AT = x - \frac{ydx}{dy}$. As this equation may be proved to be of the form $y = px + P$, its integral will be most easily found, by first taking its differential (art. 212); thus, making dx constant, we have

$$dy + (a - x)\frac{d^2y}{dx} - dx\frac{dy}{dx} = dx - \frac{dy^2dx - ydxd^2y}{dy^2};$$

$$\therefore\ (a - x)\frac{d^2y}{dx} = \frac{ydxd^2y}{dy^2}, \text{ and } \frac{dy}{\sqrt{y}} = \frac{dx}{\sqrt{(a-x)}};$$

integrating, we find $\sqrt{y} = C - \sqrt{(a - x)}$;

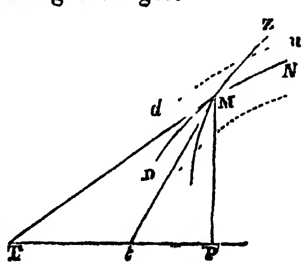
but when $x = 0$, then $y = 0$, therefore $C = \sqrt{a}$, and

$$\sqrt{y} = \sqrt{a} - \sqrt{(a - x)}, \quad \text{or } x = 2\sqrt{ay - y^2},$$

which equation belongs to the common parabola.

The same result would have been obtained by eliminating p between the equations $x + \frac{dP}{dp} = 0$ and $y = px + P$. (See art. 212).

252. Problem IV.—*To determine the nature of a curve which shall intersect all other curves of a given species at a given angle.*



By curves of a given species are meant curves like a series of parabolas, which are obtained by giving to the parameter all possible values, and which have the same vertex, and the same axis measured in the same direction.

Let DN, dn, &c. be the curves so intersected, and MZ the curve which cuts them. Through any point M draw the tangents MT, Mt to the two curves MN,

MZ, then the angle TMt, from the conditions of the question, must be equal to a given angle. Let x, y be the co-ordinates of the curve MZ, and x', y' those of the curve DN; also let a = trigonometrical tangent of the angle TMt. Because the angle $TMt = PtM - PTM$, and the tangents of the angles PtM, PTM are $\frac{dy}{dx}$, $\frac{dy'}{dx'}$, respectively (art. 93), therefore

$$\tan \angle TMt = a = \left(\frac{dy}{dx} - \frac{dy'}{dx'}\right) \div \left(1 + \frac{dy}{dx}\frac{dy'}{dx'}\right) \text{ (Trig., art. 78);}$$

$$\therefore a\left(1 + \frac{dy}{dx}\frac{dy'}{dx'}\right) = \frac{dy}{dx} - \frac{dy'}{dx'} \ldots . (1)$$

If, now, we find the value of $\frac{dy'}{dx'}$ in terms of x', y', and the parameter p, and write x and y instead of x' and y', since at the point M they are respectively equal to each other, we shall have an equation involving p. Eliminating p by means of the equation to the curve DN, we shall have a result which will include all their intersections with MZ, and will consequently be the equation to this curve.

If, for example, DN, dn, &c., be a series of curves of the parabolic kind, whose equation is $y'^n = px'^m$, we find $\frac{dy'}{dx'} = \frac{mpx'^{m-1}}{ny'^{n-1}}$. We may eliminate p from this expression before we substitute it in equation (1), and we shall then obtain $\frac{dy'}{dx'} = \frac{my'}{nx'} = \frac{my}{nx}$. Substituting this value in equation (1), and multiplying by $nxdx$, it becomes

$$a(nxdx + mydy) + mydx - nxdy = 0.$$

This equation, being homogeneous, may be integrated by the method explained in art. 197. If the curves be supposed to cut each other at right angles, then a is infinite; dividing this equation therefore by a, the part $\frac{mydx - nxdy}{a}$ vanishes; therefore, $nxdx + mydy = 0$, and its integral

$$nx^2 + my^2 = c, \text{ a constant quantity.}$$

This equation shows that the curve is an ellipse, the centre of which is in the common vertex of all the parabolas.

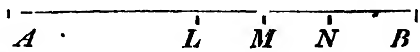
This problem is celebrated by the name of the *Problem of Trajectories*; it was originally proposed by Leibnitz, as a challenge to the English mathematicians, and resolved by Newton the day on which he received it.

CHAP. IV.—APPLICATION OF THE INTEGRAL CALCULUS TO MECHANICS.

FUNDAMENTAL EQUATIONS OF MOTION.

253. Prop. I.—*To investigate a general equation between the space and the velocity, when the velocity is variable.*

When a body passes over equal spaces in equal times, the velocity is said to be uniform, and is measured by the space described in a unit of time. Let a body be supposed to describe the line AB uniformly; and let $AM = s$, $AN = s'$, $t =$ time of describing AM, $t' =$ do. AN, $v =$ the uniform velocity, then will



$$s' - s = v(t' - t); \qquad \text{and} \quad v = \frac{s' - s}{t' - t}.$$

If the velocity is variable, it is no longer measured by the space actually described in a unit of time, but by that which would have been described if the motion had continued uniform from that point, or had ceased to increase or decrease. Hence, in this case, $\frac{s' - s}{t' - t}$ is not equal to v, but it will evidently more nearly approximate to the value of v, the smaller $s' - s$ and $t' - t$ be taken; and, if these be indefinitely small, the ultimate limit of the quotient $\frac{s' - s}{t' - t}$ will be the exact value of the velocity v. But (art. 17),

$$\text{limit of } \frac{s' - s}{t' - t} = \frac{ds}{dt}; \qquad \therefore \quad v = \frac{ds}{dt}.$$

As this is one of the fundamental equations of motion, we shall establish it on the same principles as we have adopted in the previous parts of the Differential Calculus.

Let $t' - t = \tau$; and let LM also be the space described in the previous time τ. Now, since s is evidently a function of t, we may put $s = \varphi(t)$: we have then, by Taylor's theorem,

$$MN = \varphi(t + \tau) - \varphi(t) = \frac{ds}{dt}\tau + \frac{d^2s}{dt^2}\frac{\tau^2}{1.2} + \&c.$$

$$LM = \varphi(t) - \varphi(t - \tau) = \frac{ds}{dt}\tau - \frac{d^2s}{dt^2}\frac{\tau^2}{1.2} + \&c.$$

Suppose the motion in LN to be continually accelerated, then it is evident that the space $v\tau$, described with the velocity at M continued uniform during the time τ, will be less than the space MN described with a continually accelerated motion, since the velocity at every instant of time is less in the former case than in the latter. And, for a similar reason, the space $v\tau$ is greater than the space LM, since the velocity at

M is greater than the velocity at every point in LM. Hence, since the value of $v\tau$ is always comprised between the two series,

$$\frac{ds}{dt}\tau + \frac{d^2s}{dt^2}\frac{\tau^2}{1.2} + \&c., \quad \text{and} \quad \frac{ds}{dt}\tau - \frac{d^2s}{dt^2}\frac{\tau^2}{1.2} + \&c.,$$

and these series have the same first term $\frac{ds}{dt}\tau$, it follows, from art. 52, that

$$v\tau = \frac{ds}{dt}\tau; \quad \text{and} \;\therefore\; v = \frac{ds}{dt}.$$

And in the same manner may the proposition be proved when the motion is continually retarded.

254. Prop. II.—*To investigate the general equation between any variable force and the velocity.*

The accelerating force is measured by the velocity uniformly generated in a second of time; and, by the second law of motion, if the force continues to act upon the body, the additional velocity communicated to the body in each succeeding second will be the same as if the body were at rest. Hence, if f be the accelerating force, v the velocity acquired at the end of t seconds, and v' at the end of $(t + \tau)$ seconds;

$$v' - v = f\tau; \quad \text{and} \quad f = \frac{v' - v}{\tau}.$$

If the force be variable, it can no longer be measured by the quotient of the increment of the velocity divided by the time in which it is generated; since the continual accessions of velocity in equal times are unequal. But, if we suppose the time to be diminished indefinitely, the force in this case will approximate to a constant force; and, therefore, by taking the ultimate limit of this quotient, we obtain the exact measure of the accelerating force. But the

$$\text{limit of } \frac{v' - v}{\tau} = \frac{dv}{dt}; \quad \text{and} \;\therefore\; f = \frac{dv}{dt}.$$

This theorem may also be proved from Taylor's theorem, in the same manner as the last.

Let $v = \psi(t)$, the velocity acquired at the end of t seconds; $v' = \psi(t + \tau)$, and $v_{,} = \psi(t - \tau)$, the velocities acquired at the end of $t + \tau$ and $t - \tau$ seconds; then

$$v' - v = \psi(t + \tau) - \psi(t) = \frac{dv}{dt}\tau + \frac{d^2v}{dt^2}\frac{\tau^2}{1.2} + \&c.$$

$$v - v_{,} = \psi(t) - \psi(t - \tau) = \frac{dv}{dt}\tau - \frac{d^2v}{dt^2}\frac{\tau^2}{1.2} + \&c.$$

Now, if the accelerating force f continued uniform from the end of the time t, the increment of velocity during the time τ would be $f\tau$. But it is evident that $f\tau$ will be less than $v' - v$, and greater than $v - v_{,}$, if the accelerating force continually increase in intensity, and, therefore, $f\tau$ will be comprised between the two series,

$$\frac{dv}{dt}\tau + \frac{d^2v}{dt^2}.\frac{\tau^2}{1.2} + \&c.; \quad \text{and} \quad \frac{dv}{dt}\tau - \frac{d^2v}{dt^2}\frac{\tau^2}{1.2} + \&c.$$

And, since these two series have the same first term $\frac{dv}{dt}\tau$, it follows, from art. 52, that

$$f\tau = \frac{dv}{dt}\tau; \quad \text{and} \quad \therefore \quad f = \frac{dv}{dt}.$$

And in the same manner may it be proved, if the intensity of the force continually diminishes.

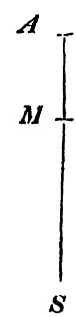
255. *Cor.* 1.—We have also $f = \frac{dv}{dt} = \frac{d.\frac{ds}{dt}}{dt}$, and if dt be constant

$$f = \frac{d^2s}{dt^2}.$$

256. *Cor.* 2.—Since $vdt = ds$, and $dv = fdt$, we obtain by multiplying these two equations together,

$$vdv = fds.$$

We shall now apply these formulæ to a few problems on rectilineal motion.



257. PROBLEM. I.—*A body falls from rest from a given point* A *towards a centre of force* S, *whose intensity varies inversely as the square of the distance* SM; *to determine the velocity at any point* M, *and the time of describing* AM.

Let $SA = a$, $AM = s$, $SM = x$; therefore, $x = a - s$. Let v be the velocity at M, and t the time of describing AM. Also, let μ be the magnitude of the force at the distance 1; then $\frac{\mu}{x^2}$ will be the magnitude of the force at the distance x.

(1). To find the velocity, we have

$$vdv = fds = \frac{\mu}{x^2} \times -dx, \text{ and integrating}$$

$$\frac{v^2}{2} = \frac{\mu}{x} + C. \quad \text{And when } x = a,\ v = 0; \quad \therefore \quad 0 = \frac{\mu}{a} + C.$$

Subtracting the last equation from the preceding one,

$$v^2 = \frac{2\mu}{x} - \frac{2\mu}{a} = \frac{2\mu}{a}\,\frac{a - x}{x}.$$

(2). To find the time,

$$dt = \frac{ds}{v} = \sqrt{\frac{a}{2\mu}}\,\frac{-x^{\frac{1}{2}}dx}{\sqrt{(a-x)}} = -\sqrt{\frac{a}{2\mu}}\,\frac{xdx}{\sqrt{(ax - x^2)}};$$

$$\therefore \quad t = \sqrt{\frac{a}{2\mu}}\left[\sqrt{(ax - x^2)} - \frac{a}{2}\,\text{vers}^{-1}\frac{2x}{a}\right] + C.$$

And when $x = a$, $t = 0$, therefore

$$0 = -\tfrac{1}{2}a \text{ vers}^{-1}(2) + C = -\tfrac{1}{2}a\pi + C.$$

Subtracting the last equation from the preceding one,

$$t = \sqrt{\frac{a}{2\mu}}\left[\sqrt{(ax - x^2)} + \frac{a}{2}\left(\pi - \text{vers}^{-1}\frac{2x}{a}\right)\right].$$

When the body arrives at S, $x = 0$; therefore, the time of falling from A to $S = \frac{\pi}{\sqrt{}}\left(\frac{a}{2}\right)^{\frac{3}{2}}$.

Cor.—The velocity acquired in falling from an infinite distance to the point M is equal to $\frac{2\mu}{x}\left(1 - \frac{x}{a}\right) = \frac{2\mu}{x}$; since a is infinite.

258. Problem. II.—*To find the velocity and time when the force varies directly as the distance SM.*

Adopting the same notation as in the last proposition,

$$f = \mu x \text{; and } vdv = -\mu x dx; \therefore v^2 = \mu(a^2 - x^2).$$

Also $dt = \frac{ds}{v} = \frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2 - x^2)}}$; $\therefore t = \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}$;

the constant being 0, since $t = 0$, when $x = a$.

When the body arrives at S, $t = \frac{\pi}{2\sqrt{\mu}}$.

Cor.—The times of falling from different distances to the centre of force are all equal.

259. *Scholium.*—The attraction of the earth on external bodies varies inversely as the square of the distance from its centre, supposing it to be a sphere. And the attraction on any bodies within the earth varies directly as the distance from the centre.

260. Problem. III.—*To determine the vertical motion of a heavy body at the surface of the earth; the force of gravity being supposed constant, and the resistance of the air to be proportional to the square of the velocity.*

It appears from Mechanics (art. 307) that when a body moves through the atmosphere, the resistance of the fluid is nearly proportional to the square of the velocity. If the body's motion be perpendicularly downwards, it will be accelerated by gravity and retarded by the resistance of the air; and the whole force acting on the body will be the difference of gravity to the resistance. But if the motion of the body be perpendicularly upwards, it will be retarded by the sum of these two forces. Hence we shall have two different solutions for the motion of the body in ascent and descent.

(1). *For the descent.* Let R represent the resistance of the air measured as a pressure in pounds avoirdupois; then, by the third law of motion, the retarding force of the air will be to the force of gravity (g)

as $\dfrac{R}{\text{wt. of the body}}$ to 1, and therefore the retarding force of the air $=$ $\dfrac{R \times g}{w}$. Suppose this equal to kv^2, then will k be a constant quantity dependent on the density and other circumstances of the fluid, and also on the weight and force of the body. We have then,

$$f = g - kv^2, \quad ds = \frac{vdv}{f} = \frac{vdv}{g - kv^2};$$

$$\therefore s = C - \frac{1}{2k} \log (g - kv^2);$$

and if the body falls from a state of rest, $v = 0$, when $s = 0$,

$\therefore \ 0 = C - \dfrac{1}{2k} \log g$; subtracting this equation from the one above;

$$s = \frac{1}{2k} \log \frac{g}{g - kv^2}.$$

To find the time:

$$dt = \frac{dv}{f} = \frac{dv}{g - kv^2} = \frac{1}{2\sqrt{g}} \left[\frac{dv}{\sqrt{g} + v\sqrt{k}} + \frac{dv}{\sqrt{g} - v\sqrt{k}}\right]$$

and integrating from $t = 0$, $v = 0$;

$$t = \frac{1}{2\sqrt{gk}} \log \left(\frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}}\right).$$

In passing from logarithms to numbers,

$$\frac{\sqrt{g} - v\sqrt{k}}{\sqrt{g} + v\sqrt{k}} = e^{-2t\sqrt{gk}}.$$

As t increases, the exponential in the second number of this equation diminishes; and when t is very great, the exponential in the second number of this equation becomes extremely small, therefore this equation will become

$$\sqrt{g} - v\sqrt{k} = 0; \text{ or } v = \sqrt{\frac{k}{g}}.$$

This is called the *terminal* velocity. It is the limit to which the velocity continually approaches; so that, after a certain time, the motion is nearly uniform. The same result also may be obtained from putting

$f = g - kv^2 = 0$, for then the body will be no longer accelerated.

(2). *For the ascent.* In this case $f = -(g + kv^2)$;

$$\therefore ds = -\frac{vdv}{g + kv^2} \text{ and } s = C - \frac{1}{2k} \log (g + kv^2).$$

Suppose V to be velocity of projection, or $v = V$, when $s = 0$; then

$= C - \dfrac{1}{2k} \log (g + kV^2)$, subtracting this from the equation above,

$$s = \frac{1}{2k} \log \frac{g + kV^2}{g + kv^2}$$

If $v = 0$, we shall have the whole height ascended

$$= \frac{1}{2k} \log \left(1 + \frac{kV^2}{g}\right).$$

To find the time:

$$dt = -\frac{dv}{g + kv^2}; \quad \therefore t = C - \frac{1}{\sqrt{gk}} \tan^{-1}\left(v \sqrt{\frac{k}{g}}\right);$$

and when $t = 0, v = V$, therefore $0 = C - \frac{1}{\sqrt{gk}} \tan^{-1}\left(V \sqrt{\frac{k}{g}}\right)$.

Eliminating C from these equations,

$$t = \frac{1}{\sqrt{gk}} \left\{\tan^{-1}\left(V \sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(v \sqrt{\frac{k}{g}}\right)\right\}.$$

When $v = 0$, the body is at its greatest height, therefore

$$\text{whole time of ascent} = \frac{1}{\sqrt{gk}} \tan^{-1}\left(V \sqrt{\frac{k}{g}}\right).$$

Examples for Practice.

1. A body falls from rest from a given point, towards a force which is constant; to find the velocity acquired at any point, and the time of motion.

2. The force varies inversely as the cube of the distance from the centre of force; to find the velocity at any point, and the time of motion.

$$\text{Ans.} \quad v = \mu \frac{\sqrt{(a^2 - x^2)}}{ax}; \quad t = a \frac{\sqrt{(a^2 - x^2)}}{\sqrt{\mu}}.$$

3. The force varies inversely as the square root of the distance; to find the velocity at any point, and the time of motion.

$$\text{Ans.} \quad v = 2\sqrt{\mu}(\sqrt{a} - \sqrt{x})^{\frac{1}{2}}; \quad t = \frac{2}{3\sqrt{\mu}}(\sqrt{x} + 2\sqrt{a})(\sqrt{a} - \sqrt{x})^{\frac{1}{2}}.$$

4. The force varies inversely as the nth power of the distance; to find the velocity at any point.

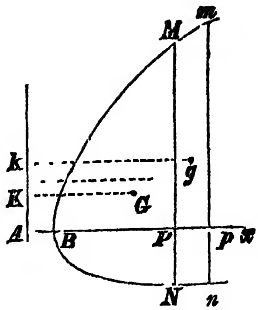
$$\text{Ans.} \quad v = \sqrt{\left[\frac{2\mu}{n-1} \frac{a^{n-1} - x^{n-1}}{a^{n-1}x^{n-1}}\right]}.$$

THE CENTRE OF GRAVITY.

261. Prop. III.—*To obtain a general formula for finding the centre of gravity of any body.*

Let MBN be any body, Ax the axis of x, and let the body be cut by a plane MPN perpendicular to Ax. Let G, G', g be the cen-

tres of gravity of the bodies MBN, mBn, and $MNnm$, and suppose Gk, $G'k'$, gk to be drawn perpendicular to a plane passing through the origin A parallel to the plane MPN.



Let $AP = x$, $Ap = x + h = x'$; the mass $MBN = M$, $mBn = M'$, and $MNnm = m$. Also let $Gk = x_1$, $G'k' = x'_1$, and $gk = k$. Now we may suppose MBN, mBn, $MNnm$ to be collected at their respective centres of gravity G, G', g; and we have, from Mechanics (art. 71),

$$M'x'_1 = Mx_1 + mk;$$

or, $$M'x'_1 - Mx_1 = mk.$$

Because M is a function of x, by Taylor's theorem,

$$M' = M + \frac{dM}{dx}\frac{h}{1} + \frac{d^2M}{dx^2}\frac{h^2}{1.2} + \&c.$$

$$\therefore\quad m = \frac{dM}{dx}\frac{h}{1} + \frac{d^2M}{dx^2}\frac{h^2}{1.2} + \&c.$$

Now the value of k is evidently comprised between x and $x + h$. Hence the value of mk is comprised between mx and $m(x + h)$, or between the two series,

$$(1)\ \ldots x\left[\frac{dM}{dx}\frac{h}{1} + \frac{d^2M}{dx^2}\frac{h^2}{1.2} + \&c.\right], \text{ and}$$

$$(x+h)\left[\frac{dM}{dx}\frac{h}{1} + \frac{d^2M}{dx^2}\frac{h^2}{1.2} + \&c.\right].$$

Let $Mx_1 = u$ a function of x; then will $M'x'_1 = u'$ the same function of $x + h$; therefore, by Taylor's theorem,

$$M'x'_1 - Mx = \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c.$$

And since the value of this function is always contained between the values of the two series (1), and these series have the same first term $x\dfrac{dM}{dx}h$, it follows from art. 52, that

$$\frac{du}{dx}h = x\frac{dM}{dx}h, \quad \therefore\ du = xdM; \text{ and } Mx_1 = u = \int xdM.$$

$$\text{Hence}\quad x_1 = \frac{\int xdM}{M}.$$

262. Case I.—*To find the centre of gravity of a symmetrical area.*

Let MBN be a curvilineal area of which Ax is a diameter bisecting all the lines MN at right angles. Then if we suppose this area to be of uniform density, it is evident that the weight or quantity of any part will be as the area of that part. Let $AP = x$, $PM = PN = y$, then the differential of the area $= 2ydx$; hence $dM = 2ydx$, and, therefore,

$$x_1 = \frac{\int x\,.\,dM}{M} = \frac{\int x\,.\,2ydx}{\int 2ydx} = \frac{\int xydx}{\int ydx}.$$

263. CASE. II.—*To find the centre of gravity of a curve line.*

When the body is a curve line of uniform density, let ds be the differential of its length, then dM will be proportional to ds. Hence

$$x_1 = \frac{\int x dM}{\int dM} = \frac{\int x ds}{\int ds}; \text{ and } y_1 = \frac{\int y ds}{\int ds},$$

where $ds = \sqrt{(dx^2 + dy^2)}$.

If the curve be symmetrical with respect to Ax, the centre of gravity will manifestly be in this line.

264. CASE III.—*To find the centre of gravity of a surface of revolution.*

If s be the length of the curve, $2\pi y ds$ is the differential of the surface, and, as before, this is proportional to dM. Also the centre of gravity will be manifestly in the axis of revolution. Hence

$$x_1 = \frac{\int x dM}{\int dM} = \frac{\int xy ds}{\int y ds}.$$

265. CASE IV.—*To find the centre of gravity of a solid of revolution.*

The differential of the solid $= \pi y^2 dx$, and therefore

$$x_1 = \frac{\int x dM}{\int dM} = \frac{\int xy^2 dx}{\int y^2 dx}.$$

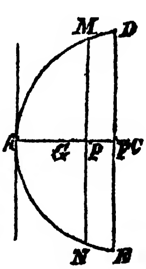
Also the centre of gravity will be in the axis of revolution.

For other surfaces and solids, and for curves not lying in one plane, or curves of double curvature, we must refer the student to Whewell, Pratt, Poisson, &c.

266. *Scholium.*—We may in the same manner apply Taylor's theorem to investigate the general equations on the moment of inertia, the centres of oscillation, percussion, &c.; but as these may always be more readily found by taking the infinitely small elements of these functions, or the limits to which the increments may be reduced—and as we have already proved that these limits, or indefinitely small elements, are always the same as the second term in the expansion of Taylor's theorem—we shall not in future go through a formal demonstration, but leave this to the consideration of the student himself.

Ex. 1

267. *To find the centre of gravity of a circular segment.*



Let $AP = x$, $PM = y$, the radius of the circle $= a$, we have then (vol. ii. p. 18) $y = \sqrt{2ax - x^2}$,

$$\therefore AG = \frac{\int xy dx}{\int y dx} = \frac{\int x dx \sqrt{2ax - x^2}}{\int dx \sqrt{2ax - x^2}}.$$

Now $\int dx \sqrt{2ax - x^2}$ (taken from $x = 0$) $=$ area AMP. Also,

$$\int x dx \sqrt{2ax - x^2} = -\int (adx - xdx) \sqrt{2ax - x^2} + \int adx \sqrt{2ax - x^2}$$

$$= -\tfrac{1}{3}(2ax - x^2)^{\frac{3}{2}} + a \times \text{area } AMP;$$

$$\therefore \quad AG = -\frac{\frac{1}{3}(2ax - x^2)^{\frac{3}{2}}}{\text{area } AMP} + a = -\frac{y^3}{3 \text{ area } AMP} + a.$$

$$\text{Hence } CG = \frac{y^3}{3 \text{ area } AMP}, \text{ and when } x = AF,$$

$$CG = \frac{DF^3}{3 \text{ area } ADF}.$$

Ex. 2.

268. *To find the centre of gravity of a circular arc* DAE.

Retaining the same notation, $AG = \frac{\int x ds}{s}$. Let ϕ = the angle ACM or the corresponding arc whose radius is 1. Then $\frac{y}{a} = \sin \phi$, and

$$\frac{dy}{a} = d\phi \cos \phi = \frac{ds}{a} \frac{a - x}{a}.$$

Hence $ady = ads - xds$, or $xds = ads - ady$,

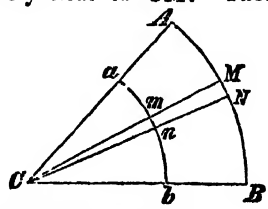
$$\therefore \quad AG = \frac{\int xds}{s} = \frac{\int (ads - ady)}{s} = \frac{as - ay}{s} = a - \frac{ay}{s}.$$

Hence, $CG = \frac{ay}{s}$; and when $x = AF$, $CG = a \frac{\text{chord } DE}{\text{arc } DAE}$.

Ex. 3.

269. *To find the centre of gravity of the sector of a circle* CAB.

Draw any radius CM, and CN indefinitely near to CM. Then the sector CMN may be ultimately considered as a triangle, whose centre of gravity is in m, at $\frac{2}{3}$ of the distance CM. Suppose the sector CAB to be divided into an infinite number of equal sectors, each of these to be collected at its centre of gravity, these will manifestly form the arc ab, and the centre of gravity of this arc will be the centre of gravity of the sector CAB. But, by the last example,



$$CG = Ca \frac{\text{chord } ab}{\text{arc } ab} = \tfrac{2}{3} CA \frac{\text{chord } AB}{\text{arc } AB}.$$

This also might have been obtained by dividing the sector into a segment and a triangle, and finding the centre of gravity of each of these separately.

Ex. 4.

270. *To find the centre of gravity of the surface of any spherical segment.* (See the figure to Ex. 1.)

Retaining the same notation, and putting φ for the arc corresponding to AM, and having a radius $= 1$, we have

$$\frac{x}{a} = \text{vers}\,\varphi\,;\quad \text{hence,}\ \frac{dx}{a} = d\varphi \sin\varphi = \frac{ds}{a}\ \frac{y}{a}, \text{ and } yds = adx,$$

$$\therefore\quad AG = \frac{\int xyds}{yds} = \frac{\int axdx}{\int adx} = \tfrac{1}{2}x.$$

Examples for Practice.

1. To find the centre of gravity of the area DAE, when the curve is a parabola. Ans. $AG = \frac{3}{5}AF$.

2. When the curve is of the parabolic species, whose equation is $y^{m+n} = a^m x^n$. Ans. $AG = \dfrac{m+2n}{2m+3n}AF$.

3. When the area is a semicircle. Ans. $CG = \dfrac{4}{3\pi}AC$.

4. When the area is the segment of an ellipse, whose semi-axis $CA = a$, and the other semi-axis $= b$. Ans. $CG = \dfrac{a^2}{b^2}\,\dfrac{DF^3}{3\ \text{area}\ ADF}$.

5. When the area is the segment of a hyperbola, whose semi-axis $CA = a$, and the other semi-axis $= b$. Ans. $CG = \dfrac{a^2}{b^2}\,\dfrac{DF^3}{3\,\text{area}\,ADF}$.

6. To find the centre of gravity of the semi-circumference of a circle. Ans. $CG = \dfrac{2\,\text{radius}}{\pi}$.

7. To find the centre of gravity of the surface of a cone. Ans. $VG = \frac{2}{3}$ dist. from vertex V to the centre of the base.

8. To find the centre of gravity of a hemisphere. Ans. $AG = \frac{5}{8}$ radius.

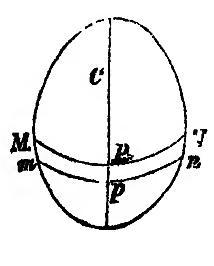
9. To find the centre of gravity of a paraboloid. Ans. $AG = \frac{2}{3}AF$.

10. To find the centre of gravity of a frustum of a paraboloid, of which the radii of the two ends are a, b, and the length of the axis h.

Ans. Dist. from the end whose radius is $a = \dfrac{h}{3}\cdot\dfrac{a^2+2b^2}{a^2+b^2}$.

MOMENT OF INERTIA, CENTRE OF OSCILLATION, &c.

271. Prop. IV.—*To obtain a general formula for finding the moment of inertia of any body.*



Let CMN be any body revolving about an axis passing through C perpendicular to the plane MN. Let $CP = r$, $Cp = r'$; and let $M =$ mass included within the distance r from the axis; $M' =$ do. within the distance r': $u =$ moment of inertia of the mass M, or the sum of each particle multiplied by the square of its distance from the axis: $u' =$ moment of inertia of the mass M'; and $m = M' - M$, the mass $MNnm$ included between the two distances r and r'. We have then,

$u' - u$ = moment of inertia of the mass m which is evidently $> mr^2$ and $< mr'^2$;

$$\therefore \quad \frac{u' - u}{m} \text{ or } \frac{u' - u}{M' - M} > r^2 \text{ and } < r'^2.$$

Now, when Pp is diminished indefinitely $\frac{u' - u}{M' - M}$ becomes ultimately $= \frac{du}{dM}$, and r' is ultimately $= r$. Hence,

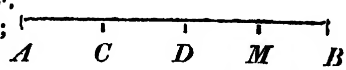
$$\frac{du}{dM} = r^2, \quad \therefore \quad du = r^2 dM; \quad \text{and} \quad u = \int r^2 dM.$$

In determining the moment of inertia of lines and planes, they are supposed, as in finding their centre of gravity, to be made up of particles of matter uniformly diffused over them.

272. Ex. 1.—*To find the moment of inertia of the straight line* AB, *revolving about an axis perpendicular to it at* C.

Let $AD = DB = a$, $CD = b$ $CM = r$.

In this case dM is proportional to dr; therefore,



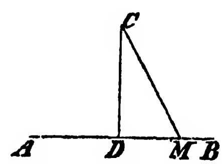
$$\int r^2 dM = \int r^2 dr = \tfrac{1}{3} r^3.$$

And this integral is to be taken from $r = -(a - b)$ to $r = a + b$;

$$\therefore \text{ moment of inertia } = \tfrac{1}{3}\left[(a + b)^3 + (a - b)^3\right] = 2a\left(b^2 + \tfrac{1}{3}a^2\right).$$

273. Ex. 2.—*To find the moment of inertia of a line vibrating lengthways in its own plane, the two extremities* A *and* B *being equally distant from* C *the centre of suspension.*

Let $AD = DB = a$, $CD = b$, $CM = r$, $DM = x$.



In this case

dM is proportional to dx, and $r^2 = b^2 + x^2$;

$$\therefore \int r^2 dM = \int dx\,(b^2 + x^2) = b^2 x + \tfrac{1}{3}x^3.$$

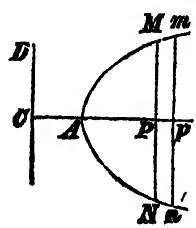
And this integral is to be taken from

$$x = -a \text{ to } x = +a; \text{ therefore,}$$

$$\text{moment of inertia } = 2a\left(b^2 + \tfrac{1}{3}a^2\right).$$

274. Case I.—*To find the moment of inertia of a symmetrical area or curve line vibrating flatways, or in a direction perpendicular to its plane.*

(1). Let CD be the axis of rotation situated in the plane AMN. Let $CP = x$, $PM = PN = y$; then the moment of inertia of the elementary area $MNnm$ is evidently $x^2 \times 2ydx$; therefore, the moment of inertia of the area AMN is



$$\int 2x^2 y\,dx.$$

(2). Also, the moment of inertia of the arc MAN is

$$\int 2x^2 ds.$$

275. CASE II.—*To find the moment of inertia of a symmetrical area or curve line vibrating edgeways in its own plane.*

(1). The moment of inertia of the straight line MN vibrating lengthways

$$= (x^2 + \tfrac{1}{3}y^2) \times 2y \quad \text{(art. 273)};$$

$\therefore$ moment of inertia of the elementary area $MNnm = (x^2 + \frac{1}{3}y^2) \times 2ydx$.

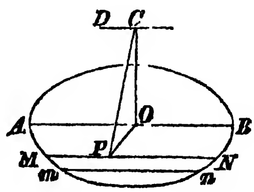
Hence, moment of inertia of the area $AMN = \int 2ydx(x^2 + \frac{1}{3}y^2)$.

(2). The moment of inertia of the elementary arc $Mm = r^2 ds = (x^2 + y^2)ds$. Hence, taking both branches of the curve,

$$\text{moment of inertia} = \int 2ds(x^2 + y^2).$$

276. Ex. 3.—*To find the moment of inertia of a circle, having its plane parallel to the axis of rotation* CD.

Let the diameter AB be parallel to CD; and let OC be drawn from the centre O perpendicular to CD. Draw any chord MN parallel to AB, and OP perpendicular to MN. Let $OA = a$, $OC = b$, $OP = x$, $PM = y$; then, since every point in the line MN is equally distant from the axis CD, the



$$\text{moment of inertia of } MN = 2y \cdot CP^2 = 2y(b^2 + x^2);$$

$\therefore$ moment of inertia of the elementary area $MNnm = 2ydx(b^2 + x^2)$.

Hence, the moment of inertia of the circle $AMNB$ is equal to

$$\int 2ydx\,(b^2 + x^2) = b^2\int 2ydx + \int 2x^2dx\,\sqrt{(a^2 - x^2)}.$$

Now, $\int 2ydx =$ area of the circle $AMNB = \pi a^2$, taken from $x = -a$ to $x = +a$.

Also, $\int 2x^2dx\sqrt{(a^2 - x^2)} = -\frac{1}{2}x(a^2 - x^2)^{\frac{3}{2}} + \frac{1}{2}a^2\int dx\sqrt{(a^2 - x^2)}^{\frac{1}{2}}$;

and, taking the integral from $x = -a$ to $x = +a$, the term $-\frac{1}{2}x(a^2 - x^2)^{\frac{3}{2}}$ vanishes, $\therefore \int 2x^2dx^2\sqrt{(a^2 - x^2)} = \frac{1}{4}a^2\int 2dx(a^2 - x^2)^{\frac{1}{2}} = \frac{1}{4}a^2 \times \pi a^2$ taken from $x = -a$ to $x = +a$.

Hence the moment of inertia of the circle $= \pi a^2(b^2 + \frac{1}{4}a^2)$.

277. Ex. 4.—*To find the moment of inertia of the circumference of the circle* AMNB.

This is evidently the integral of $2a\,(b^2 + x^2)$, and will be found to be $2\pi a(b^2 + \frac{1}{2}a^2)$.

But we may find the moment of inertia in the following manner. Since u, the moment of inertia of the circle AMN, is $\pi a^2b^2 + \frac{1}{4}\pi a^4$, considering a variable, we have $du = 2\pi a\,(b^2 + \frac{1}{2}a^2)\,da$. This is manifestly the moment of inertia of an elementary annulus contained by the circumference AMN, and having a breadth $= da$. Hence it is evident that the

$$\text{moment of inertia of the circumference } AMN = \frac{du}{da} = 2\pi a(b^2 + \tfrac{1}{2}a^2).$$

278. Case III.—*To find the moment of inertia of a solid and a surface of revolution, whose axis* CP *is perpendicular to the axis of rotation* CD. (See fig. to Case I.)

(1). Putting $CP = x$, $PM = y$ as before; the moment of inertia of the circle generated by the revolution of MP (art. 276) is $\pi y^2(x^2 + \frac{1}{4}y^2)$; and, therefore, the moment of inertia of the elementary solid $MNnm$ is $\pi y^2 dx(x^2 + \frac{1}{4}y^2)$. Hence,

moment of inertia of solid $AMN = \int \pi y^2 dx(x^2 + \frac{1}{4}y^2)$.

(2). We shall find, in the same manner, from art. 277, the moment of inertia of the surface generated by the curve $AM = \int 2\pi y ds(x^2 + \frac{1}{2}y^2)$.

279. Case IV.—*To find the moment of inertia of a solid and a surface of revolution about its own axis.*

(1). The moment of inertia of a circle revolving in its own plane about its centre is $\pi a^2 \times \frac{1}{2}a^2$; and, therefore, the moment of inertia of the elementary solid $MNnm$ is $\pi y^2 dx \times \frac{1}{2}y^2$

Hence, moment of inertia of the solid $AMN = \int \frac{1}{2}\pi y^4 dx$.

(2). The moment of inertia of the circumference generated by the point M is $2\pi y \times y^2$; and, therefore, the moment of inertia of the elementary surface generated by MN is $2\pi y ds \times y^2$. Hence,

moment of inertia of the surface $AMN = \int 2\pi y^3 ds$.

Examples for Practice.

To find the moment of inertia in the following figures:—

1. A circle revolving in its own plane about its centre. Ans. $\frac{1}{2}Ma^2$.

2. A circle revolving in its own plane about any other point at a distance b from the centre. Ans. $M(b^2 + \frac{1}{2}a^2)$.

3. The circumference of a circle revolving in its own plane at a distance b from the centre. Ans. $M(b^2 + a^2)$.

4. A rectangle, whose length and breadth are $2a$, $2b$, about a point in the axis at a distance c from its centre Ans. $M(c^2 + \frac{1}{3}a^2 + \frac{1}{3}b^2$.

5. A cone revolving about its axis (rad. of base $= a$). Ans. $\frac{3}{10}Ma^2$.

6. A sphere about a diameter. Ans. $\frac{2}{5}Ma^2$.

7. A parallelopided about an axis. Ans. $\frac{1}{3}M(a^2 + b^2)$.

8. A cylinder about its axis. Ans. $\frac{1}{2}Ma^2$.

9. A sphere about an axis at a distance b from its centre. Ans. $M(b^2 + \frac{2}{5}a^2)$.

10. A paraboloid about its axis. Ans. $\frac{1}{3}Ma^2$.

11. A right cone revolving about an axis passing through the vertex perpendicular to the axis of the figure. Ans. $\frac{3}{5}M(b^2 + \frac{1}{4}a^2)$.

12. A paraboloid about an axis passing through the vertex perpendicular to the axis of the figure. Ans. $M(\frac{1}{3}b^2 + \frac{1}{6}a^2)$.

280. Prop. V.—*To find the centres of gyration and oscillation in given figures.*

We have, from Mechanics, art. 321,

$$Mk^2 = \text{moment of inertia; therefore, } k = \sqrt{\left[\frac{\int r^2 dM}{M}\right]}.$$

Also, art. 325, $k^2 = CO \times CG$; therefore, $CO = \frac{k^2}{CG}$.

Ex. 1. In a straight line vibrating about its extremity, $CO = \frac{2}{3}a$.

2. In a circle vibrating in its own plane, $CO = h + \frac{a^2}{2h}$.

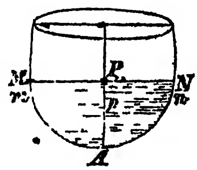
3. In a circle vibrating about an axis in its own plane,

$$CO = h + \frac{a^2}{4h}.$$

4. In a sphere, $CO = h + \frac{2a^2}{5h}$.

THE EMPTYING OF VESSELS THROUGH SMALL ORIFICES.

281. Prop. VI.—*To investigate a formula for the time of emptying any vessel through a small orifice.*



Let PM be the surface of the descending fluid. Put $AP = x$, $PM = y$, $k =$ the area of the transverse section at the vena contracta, and $K =$ the area of the descending surface. The velocity of the fluid at the vena contracta $= \sqrt{2gx}$; therefore, the quantity of fluid discharged in the indefinitely small time dt is equal to $kdt\sqrt{2gx}$. Also, if we suppose the surface of the fluid to descend from PM to pm in the time dt; then $Pp = -dx$ ultimately; and the content Mn is ultimately $= -Kdx$; but this is equal to the quantity of fluid discharged, therefore

$$-Kdx = kdt\sqrt{2gx}; \quad \text{and} \quad dt = \frac{-Kdx}{k\sqrt{2gx}}.$$

From this differential equation, the value of t may be found, when K is given in terms of x.

282. *Cor.*—If MAm be a solid of revolution round the vertical axis AP, then $K = \pi y^2$; therefore

$$dt = \frac{-\pi y^2 dx}{k\sqrt{2gx}}.$$

Ex. 1.—The time in which a cylinder will empty itself is $\frac{2\pi a^2\sqrt{h}}{k\sqrt{g}}$.

2. The time in which an inverted cone will empty itself by a small orifice at the bottom is $\frac{2\pi a^2\sqrt{h}}{5k\sqrt{g}}$.

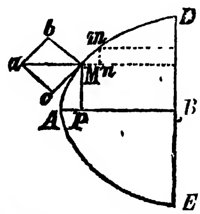
3. The time in which an inverted paraboloid will empty itself by a small orifice at the vertex is $\frac{4\pi ph^{\frac{3}{2}}}{3k\sqrt{g}}$.

4. A sphere is filled with water; to compare the times of emptying the upper and lower hemispheres by a small orifice at the bottom.

Ans. As $8\sqrt{2} - 7$ to 7.

THE RESISTANCE OF FLUIDS.

283. PROP. VII.—*To investigate a general expression for the resistance of any plane figure or solid of revolution* ADE *moving in a fluid in the direction of its axis.*



(1). Let Mm be an indefinitely small arc, which may ultimately be considered as an inclined plane. Let Ma represent the force with which Mm strikes against any particle at M. Then, if the figure be constructed as in Hydrodynamics (art 449), the whole force against a particle at M : effective force

$$:: Ma^2 : Mb^2 :: 1 : \frac{dy^2}{ds^2}.$$

Also, the number of particles which impinge against Mm is proportional to mn or dy. Hence the effective force of resistance against Mm is as $\frac{dy^2}{ds^2}dy$; and, therefore, the whole effective force of resistance against the curve ADE is proportional to

$$\int \frac{dy^2}{ds^2} dy = \int \frac{dy^3}{dx^2 + dy^2}.$$

(2). If the figure be a solid generated by the revolution of the curve AD about the axis AB, the number of particles impinging against the annulus generated by Mm is as $2\pi y \times dy$; and the force of each particle being as $\frac{dy^2}{ds^2}$, the effective force on the surface is proportional to

$$\int \frac{y dy^3}{ds^2}.$$

Ex. 1.—If DAE be a semicircle, the resistance on DAE is $\frac{2}{3}$ of the resistance on the diameter DE.

2. If DAE be a cone, the resistance on the base DE is to the resistance on the surface DAE as AD^2 to BD^2.

3. The resistance on the surface of a hemisphere is half the resistance on the base DE.

4. If AD be a cycloid, the resistance against this solid, generated by the revolution of AD about its base AB, is to the resistance against the circle generated by BD as 1 to 3.

ASTRONOMY.

CHAP. I.—MOTIONS OF THE HEAVENLY BODIES.

1. ASTRONOMY is that science which treats of the motions of the heavenly bodies, and explains the laws by which those motions are regulated.

2. When we look at the heavens on a clear night, all the stars seem to be at the same distance from us, and, therefore, appear to be situated in the surface of a sphere, having the eye for its centre. If we continue to look for some time at this hemisphere, we shall perceive that its appearance is changing every instant. New stars are continually rising in the east, whilst others, in the meantime, are setting in the west. Those stars that, in the beginning of the evening, were first seen above the eastern horizon, will be found, in a few hours, to be situated towards the south, where they have their highest elevation, and may be traced moving gradually westward, until at last they sink altogether under the horizon. If we look to the north, we perceive that many stars in that quarter never set at all; but appear to move round a fixed point, as a centre, in circles of different magnitudes. A little more attentive observation will convince us that every star appears to describe a circle, or portion of a circle, in the heavens, as far as its course lies above the horizon; and that the motions of all the heavenly bodies are exactly the same as if they were situated in a hollow spherical dome, which turned uniformly round an axis passing through the eye of the spectator and the fixed point in the north called the pole.

3. Before we attempt to explain the cause of these apparent motions, it will be necessary to form some idea of the figure and size of the earth; and to ascertain whether we are at rest or in motion; as the apparent motions of all surrounding objects are considerably modified or changed by the real motion of the observer himself. Thus, when a person is seated in a carriage, and travelling rapidly along a highway, all the trees and houses appear to be moving in an opposite direction. When a person is placed in the cabin of a vessel, which is moving with considerable velocity through the water, the same appearance takes place; and, if the water be very smooth, the spectator is hardly conscious of his own motion. It is only in consequence of the jerks and inequalities of the motion that we are made to feel that we are in reality not at rest. The earth, then, may be in motion, although we ourselves are unconscious of its motion.

4. The first rude notion which any one forms of the figure of the

earth is that of a flat surface of indefinite extent in all directions. We soon, however, learn to correct this idea, when we perceive the sun rising apparently out of the earth in the morning, and setting or sinking within the earth in the evening, as it is not probable that the sun has actually passed through the substance of the earth. It must, therefore, have passed under the earth, from the western part of the horizon, where it set, to the east, where it rose again in the morning. The same things may be observed with respect to the moon and stars, which set and rise again in all points of the visible horizon. We conclude, therefore, that the earth is limited, not only in a horizontal direction, but also immediately under us, so that the sun, moon, and stars, can pass freely without interruption.

Again, when we observe any object, such as a ship, moving from us at a distance on the surface of the sea, we first lose sight of the hull, or lower part of the ship, afterwards the lower parts of the masts, and at length the whole disappears; and yet it is evident, from the distinctness with which the last portion of the sail is seen, that we only lose sight of the vessel in consequence of the roundness of the earth, or a portion of its opaqueness being interposed between the vessel and ourselves.

Lastly, when the earth is situated directly between the sun and the moon, the shadow of the earth falls upon the moon, and causes an eclipse; and it is observed that this shadow is always circular, whatever may be the position of the sun and moon with respect to the earth.

From these facts it is inferred that the earth is an opaque body, nearly spherical; and it will afterwards be seen that its circumference is about 25,000 miles, and its diameter a little more than 7900 miles.

5. The diurnal motions of the heavenly bodies may either be produced by a real revolution of the heavens from east to west, or by the rotation of the earth on an axis in the opposite direction from west to east. The reality of the diurnal revolution of the heavens is liable to very serious objections, as it supposes that a circular motion is common to an immense number of bodies, far distant, and entirely detached from one another; and that this motion is so regulated that their revolutions are all performed in the same time, and in planes parallel to one another. The impossibility of explaining this upon any physical or mechanical principles, renders it extremely improbable that the heavens should revolve round the earth But the rotation of the earth about an axis is perfectly consistent with analogy and the laws of mechanics; as we find that the sun and many of the planets, much larger than the earth, revolve round an axis: and the simplicity with which this accounts for the diurnal motions of all the heavenly bodies, justifies us in adopting this as the true explanation. It will be seen that this hypothesis becomes absolutely certain from other considerations, which we shall afterwards notice.

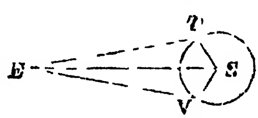
6. Besides the diurnal motions of the heavenly bodies, which are common to them all, there are several which have motions peculiar to themselves. The sun, which is the great source of light and heat, is obviously much farther to the south during winter than during summer; it does not, therefore, keep the same station in the heavens, nor describe the same circle every day. The moon not only changes her form, diminishes and increases but if we observe the stars near which she is

situated one evening, the next evening we shall find her considerably to the eastward of them; and every day she removes to a still greater distance, until, in about 27 days, she makes a complete tour of the heavens, and approaches them from the west. There are, besides, several stars which are continually changing their places; sometimes they move to the westward, sometimes to the eastward, and sometimes they appear stationary, but, upon the whole, their motion is towards the east. These stars are called *planets.** There are some other bodies, having apparent proper motions, called *comets.*† But the greater number of the heavenly bodies are stationary, and retain nearly the same relative distance from each other, and are, therefore, called *fixed stars.*

THE SUN.

7. If we observe any large stars to the east a little before sunrise, we shall find that they rise earlier every day, with respect to the sun. Those stars which are situated nearly in the path of the sun, and which set soon after it, in a few days are lost altogether in the sun's rays, and afterwards make their appearance in the east before sunrise. The sun then moves towards them in a direction contrary to its diurnal motion. It was by observations of this kind that the ancients ascertained that it described a great circle in the heavens in about $365\frac{1}{4}$ days.

8. When the sun's motion in the heavens is determined by observations on the meridian from day to day, we find that it is not uniform, but is continually retarded from January to July, and accelerated from July to January. If, at the same time, the apparent diameter of the sun be observed with the heliometer (an instrument that measures small angles with great exactness), it is found to be greatest on the 31st of December, when the angular motion is greatest, and least on the 1st of July, when the angular motion is least. Now, it is manifest, that, in the case of the sun, the apparent diameter must be inversely proportional to its distance from us. For, if E be the earth, S the centre of the sun, ET, EV two tangents to the surface of the sun, then will the apparent semidiameter of the sun be represented by the angle $SET = \delta$. We



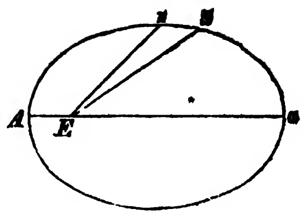
have then $\sin\delta = \frac{ST}{ES}$; and because δ is a small angle not exceeding $16\frac{1}{4}$ minutes, δ will not differ sensibly from $\sin\delta$. Hence,

$$\delta = \frac{ST}{ES} = \frac{\text{rad. of the sun}}{\text{dist. of the sun}}.$$

Now it appears, from observation, that the sun's angular motion in its orbit is proportional to the square of its apparent diameter, and, therefore, inversely proportional to the square of its distance. Let ASa be the sun's orbit, E the earth, and S the sun. Suppose the sun to move over the space Ss in one day, then will the radius vector ES be transferred to the situation Es, having described the small sector SEs. Now,

* From πλανήτης, a wanderer. † From κόμη, a bush of hair.

the angle SEs being very small, the sector ESs may be considered as a triangle, and



$$\text{area} = \tfrac{1}{2}ES \times Es \times \sin SEs$$
$$= \tfrac{1}{2}\varrho^2 \sin\alpha \text{ (Mens., prob. 2).}$$

Let ϱ' be the radius vector of the sun in any other position of its orbit, and α' its angular velocity: we have then, from observation,

$$\varrho^2 : \varrho'^2 :: \alpha' : \alpha :. \sin\alpha' : \sin\alpha,$$

since α and α' are very small; and, therefore, $\frac{1}{2}\varrho^2 \sin\alpha = \frac{1}{2}\varrho'^2 \sin\alpha'$. Hence it appears that *equal areas are described by the radius vector in equal times;* and, in any times whatever, the areas are proportional to the times.

9. On comparing the sun's diameter, measured from time to time with its angular distance u from that point in its orbit when its diameter is least, Kepler discovered, after much labour and calculation, that, if its greatest distance be represented by $1 + e$, and its least distance by $1 - e$, then the radius vector at any other point would be expressed thus, $\rho = \dfrac{A}{1 - e\cos u}$; but this is the polar equation to an ellipse (prop. 22), where A is equal to the semiparameter. Hence it follows, that the apparent orbit described by the sun is an ellipse, with the earth in one of its foci; and that if the semimajor axis or mean distance $= 1$, the excentricity is equal to $0{\cdot}01679$.

10. Although we are thus enabled to compare the relative distances of the sun from the earth in different parts of its orbit, we have hitherto had no means of ascertaining its real distance, or the actual magnitude of the sun itself. This may be done nearly upon the same principles as we determine the distance of any inaccessible object in Trigonometry, by taking a considerable distance of the earth's surface, whose length is known, as a base line, and observing the change of the sun's situation in the heavens arising from this cause. We shall consider this more fully when we treat of Parallax; in the meantime we may observe, that from this, and other methods of a more refined nature, the sun's horizontal parallax, or the angle which the radius of the earth subtends at the centre of the sun, is found to be about $8''{\cdot}6$; from whence it follows, that if a = mean distance of the sun from the earth, and r = earth's radius, $a \times \sin 8''{\cdot}6 = r$; if, therefore, $r = 3956$ miles, a will be found equal to about 95 millions of miles. Also, since the sun's diameter at this distance subtends an angle of $32'\ 3''$, its real diameter must be 882,000 miles. Hence it may be easily shown that its bulk is to that of the earth in proportion of 1384472 to 1.

11. Although we have seen that the sun apparently describes an ellipse about the earth, situated in one of its foci, it does not necessarily follow that the sun is really in motion, as the same phenomena may be as readily explained by supposing the earth to move round the sun. When we reflect that the sun is above a million times greater than the earth, and, also, that all the planets describe elliptic orbits round the sun,

it is extremely probable that the earth also, like the planets, moves about the sun. This argument derives additional force from physical astronomy, where it is proved that the earth is subject to the same laws as the planets; and the phenomena of the aberration of light are proved in the most rigorous manner to arise from the velocity of light combined with the motion of the earth.

12. We shall, therefore, assume that the earth describes an elliptic orbit about the sun, situated in one of its foci, and that the *species* or relative dimensions of this ellipse are given from observation. The calculation of the sun's place at any time, therefore, is reduced to the problem of drawing a line through the focus of a given ellipse, so as to cut off an area equal to a given area. We shall now proceed to define a variety of terms which are generally used in astronomy to ascertain the positions of the heavenly bodies; and we may observe, that they are all dependent either, 1st, On the plane of the equator; 2nd, On the plane of the ecliptic; or, 3rd, On the plane of the horizon.

DEFINITIONS.

13. That line about which the earth revolves from west to east in 24 sidereal hours, is called the *axis of the earth;* and if it be produced to the heavens, it is called the *axis of the heavens.*

14. The *celestial concave,* or surface, in which the fixed stars and all the heavenly bodies are seen, is an imaginary spherical surface, supposed to be at such a distance from the spectator, that all lines drawn to the same point in it from different parts of the earth, make no sensible angle there, and, therefore, may be considered parallel.

The *fixed stars* are generally considered to be at such a distance from the earth, that lines drawn to any of these stars from opposite points of the earth's orbit have no sensible inclination.

15. The two points where the axis of the earth meets its surface are called the *poles of the earth;* and the points where this axis produced appears to meet the concave of the heavens, are called the *poles of the heavens.*

16. If a plane pass through the centre of the earth perpendicular to its axis, the intersection of this plane with the surface of the earth is called the *terretsrial equator;* and if it be produced on all sides, its intersection with the celestial concave is called the *celestial equator.* This circle is also called by astronomers the *equinoctial.*

17. If a plane pass through the axis of the earth and any place on its surface, the intersection of this plane with the surface of the earth is called the *terrestrial meridian* of that place; and if the plane be produced on all sides to the sphere of the heavens, it marks out the *celestial meridian.*

18. If at any place of the earth's surface the *direction* of gravity (that is, the direction of the plumb line) be produced upwards, the point in which it cuts the celestial sphere is called the *zenith* of the place; and if the same line be produced downwards, the point in which it cuts the heavens on the opposite side is called the *nadir.*

19. If a plane perpendicular to this line pass through the place itself,

it is called the *sensible horizon ;* and if a plane parallel to the former pass through the centre of the earth, and be produced to the heavens, it is called the *rational horizon.*

20. The *latitude* of any place on the surface of the earth is the angle which a vertical line (or the direction of gravity) makes with the plane of the equator. The intersection of any plane parallel to the equator with the earth's surface is called a *parallel of latitude.* The *longitude* of a place on the earth's surface is the inclination of its meridian to that of some given station, such as Greenwich.

21. *Vertical circles,* or *circles of altitude,* are great circles passing through the zenith and nadir, or great circles perpendicular to the horizon.

22. The *altitude* of a heavenly body is the arc of a circle of altitude intercepted between the place of the body in the celestial concave and the horizon. The *azimuth* is the arc of the horizon intercepted between the north or south point and the circle of altitude passing through the place of the body. A circle of altitude passing through the east or west point is called the *prime vertical.*

23. The great circle which the sun appears to describe in the heavens is called the *ecliptic.* It is inclined to the equinoctial at an angle of about 23° 28′; and the intersections of these two great circles are called the first point of Aries and the first point of Libra.

24. *Circles of declination* are great circles perpendicular to the celestial equator; and *parallels of declination* are small circles parallel to the celestial equator.

25. The *declination* of a heavenly body is the arc of a circle of declination intercepted between its place in the celestial concave and the celestial equator. The *right ascension* of a heavenly body is the arc of the celestial equator intercepted between the first point of Aries and the circle of declination passing through the body. The right ascension is always measured from west to east, entirely round the circle.

26. *Circles of latitude,* in the celestial concave, are great circles perpendicular to the ecliptic. *Parallels of latitude* are small circles parallel to the ecliptic.

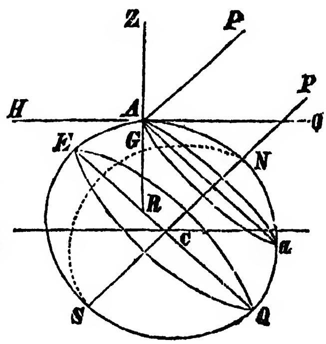
27. The *latitude* of a heavenly body is the arc of a circle of latitude intercepted between its place in the celestial concave and the ecliptic. The *longitude* of a heavenly body is the arc of the ecliptic intercepted between the first point of Aries and the circle of latitude passing through the body; it is measured from west to east, entirely round th circle.

28. The ecliptic is divided into 12 equal parts, called signs, each containing 30 degrees. They begin at the first point of Aries, where the sun, in its motion in the ecliptic, passes from the south to the north of the equator. Their names and characters are as follow :—

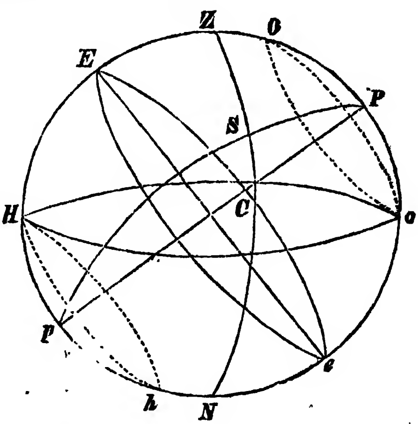
Northern.				Southern.			
1. Aries	♈	4. Cancer	♋	7. Libra	♎	10. Capricornus	♑
2. Taurus	♉	5. Leo	♌	8. Scorpio	♏	11. Aquarius	♒
3. Gemini	♊	6. Virgo	♍	9. Sagittarius	♐	12. Pisces	♓

These signs take their names from twelve constellations which formerly occupied the same place in the heavens; but they are now separated from each other by an interval of about 30 degrees, in consequence of the precession of the equinoxes. The zone, which lies 8° on each side of the ecliptic, is called the *zodiac*. In this zone the moon and all the older planets appear to make their revolutions among the fixed stars.

29. We shall now proceed to illustrate these definitions by reference to a figure.



Let C be the centre of the earth, which is not exactly a sphere, but more nearly an oblate spheroid of small excentricity, generated by the revolution of an ellipse about its minor axis NCS: then N, S are its poles; the great circle EQ, perpendicular to NS, is the equator; the small circle Aa is a parallel of latitude: the line AP, parallel to SN, is the direction of the elevated pole of the heavens, and the vertical line AZ that of the zenith. The angle ARE is the latitude of A; this is evidently equal to the distance of the zenith from the celestial equator, and also is equal to the angle PAO, the elevation of the pole. $NAES$ is the meridian of A, and NGS the meridian of some fixed station G, or the first meridian; and the angle contained between these two planes is the longitude of A. If the plane HO be perpendicular to AZ, it will be A's sensible horizon.



30. Also, if $PEpe$ represent the celestial concave, C the position of a spectator on the surface of the earth, the magnitude of which may be neglected in observing the fixed stars; then will Z be the zenith, N the nadir, Ho the celestial horizon, P, p the poles of the heavens, Po the altitude of the pole, which, by the last article, is equal to the latitude of the place. Also, if S be any heavenly body, ZSN will be a circle of altitude, and PSp a circle of declination. The small circles Hh, Oo, parallel to the equator, are parallels of declinaion. Now, a sttar on the meridian at E will come to the horizon in the west at a point 90° distant from H; it will then descend to e, and afterwards rise again in the east at a point 90° distant from H and o, and will then ascend to E. Thus, the star will rise due east and set due west; and as the great circles Ho, Ee bisect each other, it will be as long above the horizon as it is below it. A star situated at O will descend to o, and afterwards rise to O, and will

manifestly never set. A star at *H* will describe the circle *Hh* in its diurnal revolution, and will never rise. It is evident, therefore, that those stars which are in the surface *PoO* are always above the horizon, and never set. Those stars which are between the parallels *Oo* and *Hh* describe circles of diurnal revolution which are cut by the horizon, and, therefore, rise and set every day. And, lastly, all the stars which are situated between the circle *Hh* and the pole *p*, are continually below the horizon, and never rise. It is manifest likewise, that all the circles of diurnal motion, except the equator, are cut into two unequal portions by the horizon, and, therefore, the times above and below the horizon are unequal.

THE MOON.

31. The moon, next to the sun, is the most remarkable of all the heavenly bodies. It appears to advance, like the sun, among the stars in a direction contrary to the general diurnal motion of the heavens, but much more rapidly; and after $27\frac{1}{3}$ days it makes a complete revolution of the heavens.

32. From observations of the moon's apparent diameter, similar to those which were made with respect to the sun, Kepler ascertained that the orbit of the moon is an ellipse, having the earth in one of its foci. Its radius vector describes equal areas in equal times, and its angular motion is inversely proportional to the square of its distance from the earth.

33. The distance of the moon may be determined from its horizontal parallax. The moon's parallax, like the sun's, may be found by observations at remote geographical stations, or by means of occultations; from which, also, its apparent diameter is most easily and correctly found. From such observations it results, that the moon's distance is about 60 semidiameters of the earth, or about 240,000 miles; and that the excentricity of its orbit is about 0·05484 of the mean distance.

34. When the moon is opposite to the sun, or is due south at midnight, her appearance is a complete luminous circle, and she is then said to be full. About 7 days afterwards, when the distance between the sun and moon is nearly a quadrant, her disk has the appearance of a semicircle, and she is called half moon. When she is nearer to the sun than this distance, she has the form of a crescent, with its convex side always turned towards the sun.

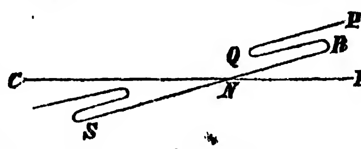
All the phases of the moon are easily explained, on the supposition that the moon is an opaque spherical body, which moves in an orbit round the earth, whilst it receives its light from the sun. At full, she has the same phase turned to the earth that is turned towards the sun; it appears, therefore, to be completely illuminated. At new moon, the dark side is turned towards the earth, and, therefore, she is invisible; and, in other positions, part of the dark side and part of the illumined side being turned towards the earth, she assumes all the different appearances which we see during a complete lunar month.

THE PLANETS.

35. Besides the sun and moon, which have an apparent motion independent of the diurnal revolution of the heavens, there are several stars

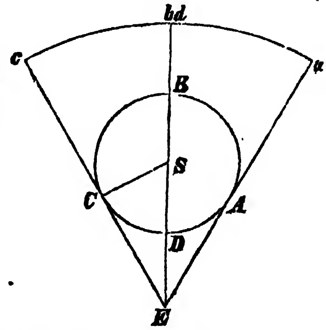
which change their relative situations among the rest. Four of these, *Venus, Mars, Jupiter*, and *Saturn*, are remarkably large and brilliant. *Mercury* is also visible to the naked eye as a large star, but from its proximity to the sun is seldom seen. A sixth, *Uranus* (which has also been called *Georgium Sidus* and *Herschel*) is scarcely visible without a telescope; and six others, *Ceres, Pallas, Vesta, Juno, Astræa*, and a planet lately discovered, can never be seen by the naked eye.

36. The apparent motions of the planets are much more irregular than those of the sun and moon. Sometimes they advance rapidly, like the sun and moon, in a direction contrary to the diurnal motion of the heavens; afterwards their motion becomes continually retarded, and at last their apparent motion ceases altogether, and they become stationary. They then reverse their motion, and move back upon their former course, first with a continually increasing motion, then diminishing, until the reversed or retrograde motion ceases altogether, and they become stationary again; after which their motion is again progressive from west to east. On the whole, however, the amount of the direct motion much exceeds that of the retrograde, and by this excess the planet gradually advances from west to east. Thus, if *EC* represent a portion of the ecliptic, the track of the planets in the heavens will be represented by the irregular figure *PQRS*, the motion from *P* to *Q* being direct, at *Q* stationary, from *Q* to *R* retrograde, at *R* stationary, from *R* to *S* direct, and so on.



37. All these apparent irregularities may easily be explained, on the supposition that the planets do not move round the earth as their centre of motion, but round some other body, such as the sun. Two of these bodies, Mercury and Venus, evidently always accompany the sun. Venus, for example, never recedes farther from it than 46°, and is alternately an evening or a morning star, as she is east or west of the sun. When her disk is viewed through a telescope, her phases vary precisely in the same manner as those of the moon. She gradually increases from a thin crescent to a complete circle, and again diminishes, until at last she becomes altogether invisible.

38. From these observations it is evident that Venus is an opaque spherical body, shining only by the reflected light of the sun, and that her orbit surrounds the sun, but excludes the earth. Thus, if *S* represent the place of the sun, *E* the earth, and Venus be supposed to describe an orbit *ABCD* round the sun, in the order of the letters; when she is at *C* she is at her greatest angular distance to the east of the sun, and the appearance of her disk is nearly a semicircle; as she approaches to *D*, more of her dark side is turned towards the earth, and she has the appearance of a crescent. At *D* she is generally not visible, but some-



times she is seen like a dark spot passing over the body of the sun. From *D* to *A* she is a morning star, and the crescent is turned the contrary way, with its convexity still towards the sun. At *A*, her greatest elongation from the sun, her disk is again nearly a semicircle, and from *A* to *B* the illumined part of the disk gradually increases, whilst her diameter diminishes. At *B* she turns her illumined disk towards the earth, and, consequently, her appearance, as seen through the telescope, is a complete circle, like that of the full moon. If the earth were stationary at *E*, Venus, in moving from *A* to *B* and *C* in her orbit, would appear to move from *a* to *b* and *c*, among the fixed stars, in the same direction as she actually moves in her orbit, and, therefore, her motion is said to be direct; but from *C* to *D* and *A* her motion among the fixed stars would be from *c* to *d* and *a*, contrary to her former direction, and, therefore, it is said to be retrograde.

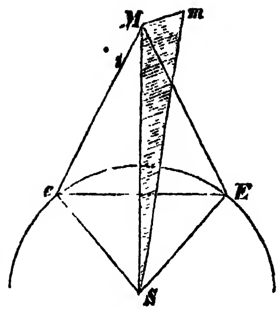
39. If the orbit *ABCD* be nearly a circle, the angle *SCE* will be nearly a right angle; and, as the angle *SEC* is known from observation, we can immediately determine the proportion of *SC* to *SE*. By calculating this value of *SC*, when Venus is in different parts of its orbit, we find that it is not always the same; and that the orbit of Venus is, in fact, an ellipse, with the sun in its focus.

40. Mercury, like Venus, accompanies the sun, and never recedes from it farther than 28° 48′. Its orbit is very elliptical, as is evident from the inequality of its greatest elongations from the sun, when observed at different times, which vary from 16° 12′ to 28° 48′.

MARS.

41. The two planets, Mercury and Venus, appear to accompany the sun, like moons or satellites, and they never go farther than a certain angular distance from the sun. But the remaining planets go to all possible distances, and are seen in opposition to the sun as well as in conjunction. We will begin with Mars.

42. If this planet be observed several times in the same node, or when it is in the plane of the ecliptic, the intervals elapsed will be observed to be always the same, whether the motion of the planet, at the moment of such passage, be direct or retrograde, swift or slow. As the situation of the planet in its node proves that it is really at that time in the plane of the ecliptic, we are enabled from thence very easily to determine the time of the revolution of Mars in its orbit, whatever be the body round which it revolves. When Mars is in opposition to the sun, its apparent diameter is about 18″; and when it is in conjunction, or when seen nearly in the same direction with the sun, its apparent diameter does not exceed 4″. If the phases of Mars be observed, it appears perfectly round, both in opposition and conjunction; and in the intermediate positions, it has a gibbous or an oval appearance, the breadth of the enlightened part being never less than $\frac{7}{8}$ths of the whole. From these circumstances, it is evident that the orbit of Mars includes both the sun and the earth within it; and Kepler has proved that this orbit is an ellipse, with the sun in one of its foci, nearly in the following manner.

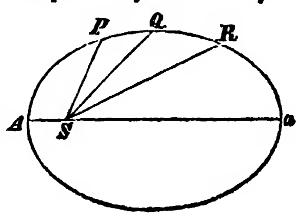


43. Let the planet Mars be observed at M, when the earth is situated at E; and in 686 days after, when Mars has made an entire revolution, and returned to the point M, let the earth be at e. Then, from the time elapsed, and the construction of the earth's orbit, SE, Se, and the angle ESe, are given, and, therefore, Ee, and the angles SEe, SeE, can easily be found. Also, the angles SEM, SeM are known from observation; therefore, the angles EeM, eEM, and the base Ee, are given, to find the sides EM, eM. Hence SM may be found from the triangle ESM, and also the angle ESM; and, consequently, the situation of Mars, with respect to the sun, is obtained.

In the preceding construction we have supposed Mars to be in the ecliptic; if the planet be situated without this plane, we must conceive M to be the orthographic projection of the planet's place on the plane of the ecliptic.

44. The distance of a planet from the sun, in different parts of its orbit, being determined in this manner, it appears that these orbits are ellipses, having the sun in their common focus; and that the angular motions of a planet round the sun are inversely as the squares of their distances from the sun, so that the sectors described by the radius vector are proportional to the times. These two important propositions are generally known by the name of the *First and Second Laws of Kepler.*

45. When we have determined three radius vector of an ellipse, and the angles between them, the ellipse may be constructed; and hence the major axis, the excentricities, &c. of the planetary orbits may be computed. Thus, let SP, SQ, SR, be three radius vectors of the ellipse Aa, found as above, which put equal to ϱ, ϱ', ϱ'', respectively; then the angles PSQ, PSR are also known from observation; put these equal to α, β: also, put the transverse axis $= 2a$, the excentricity $= ae$, and the angle $ASP = \varphi$. We have, then, from the ellipse (prop. 22),



$$\varrho = \frac{a(1-e^2)}{1+e\cos\varphi},\ \varrho' = \frac{a(1-e^2)}{1+e\cos(\varphi+\alpha)};\ \varrho'' = \frac{a(1-e^2)}{1+e\cos(\varphi+\beta)},$$

from which three equations we may easily determine the three unknown quantities, a, e, and φ.

46. When the mean distances of the planets are compared, and also their periodical times, it is found that *the squares of the periodical times are as the cubes of the distances.* This is known by the name of the *Third Law of Kepler.*

47. From the preceding observations we shall now give a general account of the solar system. The sun (☉) is an immense body, above a million times bigger than the earth, and occupies very nearly the centre of the system. Thirteen primary planets revolve about the sun in

elliptic orbits, the sun being situated in one of the foci. The names and order of these planets are:

1. Mercury	☿	5. Vesta	⚶	10. Jupiter	♃
2. Venus	♀	6. Juno	⚵	11. Saturn	♄
3. Earth	⊖	7. Ceres	⚳	12. Uranus	♅
4. Mars	♂	8. Pallas	⚴	13. Leverrier's planet.	
		9. Astræa			

The earth is attended by one moon or satellite (☾), Jupiter by four satellites, Saturn by seven and a ring, and Uranus by six satellites.

48. The planets Mercury, Venus, Mars, Jupiter, and Saturn, have been known from the earliest ages in which astronomy has been cultivated. Uranus was discovered by Sir William Herschel in 1781. The four small planets, Ceres, Juno, Pallas, and Vesta, were all discovered in the beginning of this century; they are extremely small, and revolve in orbits at nearly the same distance from the sun. In December, 1845, Hencke discovered a fifth planet belonging to the same group; and in September, 1846, a new planet was discovered situated beyond Uranus.

49. It is a remarkable fact that Kepler had predicted the discovery of a planet between Mars and Jupiter. He observed that the distances of the planets from the sun, beginning with the nearest, formed a series, each term of which was nearly double the preceding one; but, between Mars and Jupiter, a term was wanting. The discovery of Uranus, which followed also the same law, directed the attention of astronomers to this inquiry; and, in 1789, Baron von Zach published the elements of the orbit of the planet which *ought* to be found between Mars and Jupiter. After the discovery of Ceres in 1801, Professor Bode, of Berlin, formed the following scale of the planets' distances from the sun:—

Mercury	$4 = 4$	Jupiter	$52 = 4 + 3.\ 2^4$
Venus.........	$7 = 4 + 3.\ 2^0$	Saturn.......	$100 = 4 + 3.\ 2^5$
Earth	$10 = 4 + 3.\ 2^1$	Uranus	$196 = 4 + 3.\ 2^6$
Mars	$16 = 4 + 3.\ 2^2$	New planet....	$388 = 4 + 3.\ 2^7$
Ceres, Pallas, Juno, Vesta, Astræa ..	$28 = 4 + 3.\ 2^3$	....	$772 = 4 + 3.\ 2^8$

The five planets between Mars and Jupiter are so extremely small, that they have been supposed to be the fragments of some greater planet, which has burst by an explosion, and that more such fragments may hereafter be discovered.

50. The discovery of the planet beyond Uranus was also predicted, and is justly considered as one of the greatest triumphs of astronomical science. In comparing the calculated with the observed places of Uranus, a difference appeared which could only be explained by the attraction of a planet situated beyond Uranus. M. Leverrier, in France, and Mr. Adams, at Cambridge, both undertook, independent of each other, to calculate its place; and the situation of the new planet, discovered by M. Galle in 1846, is nearly the same as that indicated by the theory.

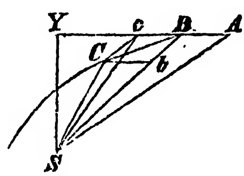
51. All the larger planets revolve in planes which are inclined to the plane of the ecliptic at very small angles, and they all revolve round the sun in the same direction with the earth, that is, from west to east.

52. The satellites observe the same laws in their revolutions about the primary planets, as these do with respect to the sun; that is, they describe ellipses with the primary planet in one of the foci—they describe equal areas in equal times, and the squares of the periodic times are as the cubes of the mean distances. All their orbits, also, are inclined, at very small angles, to the orbit of the primary planet; and all the satellites move from west to east, except those of Uranus, which are a remarkable exception to the general rule; for these move nearly in one plane, perpendicular to the orbit of the planet.

53. Besides the satellites, Saturn is surrounded by a circular ring, concentric with itself, which, being seen obliquely, is of an elliptic form, more or less elongated, according to the obliquity under which it is seen. The ring revolves on an axis at right angles to its own plane, in the same time with the planet itself; and it is remarkable, that if a satellite, at the mean distance of the middle of the ring, revolved about Saturn, the time of its revolution would be the same as that of the ring.

We shall now proceed to explain, as briefly as possible, the very important conclusions which are derived from the three laws of Kepler.

54. Prop. I.—*When a body, acted on by forces tending to a fixed point, has a projectile motion impressed on it in any direction not passing through that point, it will move in a curve line situated in one plane; and the radius vector, drawn from the centre of force, will describe areas proportional to the times.*



Suppose the body to describe AB uniformly in a unit of time, then, by the first law of motion (Mech., art. 227), if no force were to act on the body, it would in the next unit of time go on to c, in the same straight line, describing Bc equal to AB. But when the body comes to B, let a force tending to the centre S act on it, and by a single impulse cause it to describe Bb in a unit of time, if this force alone acted on it. Complete the parallelogram $BcCb$, and join SC, Sc. Because the body would describe Bc in consequence of the original motion, and Bb from the attractive force at S, by the composition of motion (Mech., art. 229) the body will describe BC, the diagonal of the parallelogram. Also, since Cc is parallel to SB, the area $SBC = SBc$, and, therefore, is $= SAB$, because $AB = Bc$.

In like manner, if an impulsive force act on the body at C in the direction CS, the body will describe an area, in the next unit of time, equal to the areas in the two preceding units; and so on successively.

Suppose, now, the unit of time to be diminished, and the number of units to be increased, indefinitely; the areas described in these units will still be equal to each other. Also, the polygon $ABCD$ ... will ultimately become a curve line, and the force which was supposed to act by impulses at B, C, D, &c., will be a continuous force acting at every point of the curve. And, because equal areas are described in all equal times, it is manifest that in different times the areas will be proportional to the times.

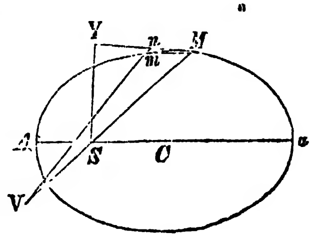
55. *Cor.* 1.—Conversely, if a body move in a curve, so that the radius

vector drawn to a fixed point describes areas proportional to the times, it is urged by a central force tending to that point.

56. *Cor.* 2.—The velocity at any point A is inversely as the perpendicular $SY(p)$ drawn from S to the tangent AY. For, since the area $SAB = A = \frac{1}{2}AB \times SY = \frac{1}{2}vt \times p$, therefore $v = \frac{2A}{pt}$. And, because A is proportional to t, the velocity is inversely as the perpendicular on the tangent.

57. *Cor.* 3.—Hence it follows, from the second of Kepler's laws, that the primary planets are all urged by forces tending towards the sun, and that the secondary planets are urged by forces towards their primary.

58. Prop. II.—*If a body describe an ellipse, and is urged by a force tending to the focus of that ellipse, the intensity of the force is inversely as the square of the distance.*



Let a body describe the ellipse MA, about a centre of force S, situated in its focus. Let v be the velocity of the body at M, and t the time of describing an indefinitely small arc Mm. Draw SY perpendicular to the tangent at M; also draw mn parallel to SM, and suppose a circle to be drawn through the points M, m, and a third point indefinitely near to them; this will ultimately be the circle of curvature (Ellipse, art. 121). Let MS be produced to meet this circle in V, MV will be the chord of curvature. Join Mm, Vm. The angle $mMn = mVM$ (Geom., prop. 48), and the angle $Mmn = mMV$, because mn is parallel to MV, therefore the triangles VMm, Mmn are similar, and $MV : Mm :: Mm : mn$, consequently,

$$mn \,.\, MV = Mm^2 = Mn^2 \times \frac{MV^2}{mV^2}.$$

But, since the force through the indefinitely small space mn may be considered constant, we have $mn = \frac{1}{2}Ft^2$ (Mech., art. 254); also, Mn ultimately $= vt$, and $\frac{MV}{mV} = 1$ ultimately: hence, putting $MV = c$, and the area described in a unit of time $= A$, we have

$$\tfrac{1}{2}Ft^2 \,.\, c = v^2t^2, \quad \therefore \quad F = \frac{2v^2}{c} = \frac{8A^2}{cp^2} \text{ (art. 54).}$$

But (Ellipse, art. 124), $c = \dfrac{2(a^2 - e^2x^2)}{a}$;

and (Ellipse, art. 94), $p = b\sqrt{\left(\dfrac{a - ex}{a + ex}\right)}$;

consequently, $cp^2 = \dfrac{2b^2}{a}(a - ex)^2 = 2a(1 - e^2)r^2$ (Ell., art. 78);

$$\therefore \quad F = \frac{4A^2}{a(1 - e^2)} \, \frac{1}{r^2}.$$

59. *Cor.* 1.—Hence it appears that the forces which make planets describe ellipses, having the sun in their common focus, are inversely as the squares of the distances from the centre of the sun.

60. *Cor.* 2.—If μ be the force at the distance 1, then will $\frac{\mu}{r^2}$ be the force at the distance r; hence

$$\mu = \frac{4A^2}{a(1-e^2)}; \quad \text{and} \quad A = \tfrac{1}{2}\sqrt{\mu . a(1-e^2)}.$$

61. PROP. III.—*If several bodies revolve in ellipses about the same centre of force, varying inversely as the square of the distance, the squares of the times of revolution will be as the cubes of the major axes.*

For the whole area of the ellipse $= \pi ab = \pi a^2 \sqrt{(1-e^2)}$.

Also, the time of revolution $= \dfrac{\text{area of ellipse}}{A} = \dfrac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}$.

If, therefore, T and T' be the times of the revolution of two planets in elliptic orbits whose major axes are $2a$, $2a'$;

$$T : T' :: a^{\frac{3}{2}} : a'^{\frac{3}{2}}, \quad \text{or,} \quad T^2 : T'^2 :: a^3 : a'^3.$$

Thus the third law of Kepler is derived from the action of a centripetal force, which tends towards the sun, and varies inversely as the squares of the planet's distance.

62. "Of all the laws to which induction from pure observation has ever conducted man, this *third law of Kepler* may justly be regarded as the most remarkable, and the most pregnant with important consequences. When we contemplate the constituents of the planetary system from the point of view which this relation affords us, it is no longer mere analogy which strikes us—no longer a general resemblance among them, as individuals independent of each other, and circulating about the sun, each according to its own peculiar nature, and connected with it by its own peculiar tie. The resemblance is now perceived to be a true *family* likeness; they are bound up in one chain—interwoven in one web of mutual relation and harmonious agreement—subjected to one pervading influence, which extends from the centre to the farthest limits of that great system, of which all of them, the earth included, must henceforth be regarded as members." *

COMETS.

63. *Comets* are luminous bodies which have motions peculiar to themselves, and which appear to be a collection of vapour, having a nucleus in its centre, in general not very distinctly defined. These bodies do not remain long visible; some appearing only for a few days, and others only for a few months. The most remarkable phenomenon, which makes them objects of attention to all mankind, is the tail of light which they often exhibit. When approaching the sun, a nebulous tail is seen to issue from them in a direction opposite to the sun: this, after having increased, again decreases as they retire from the sun, until

* Sir J. Herschel.

it disappears. The stars are not only visible through the tail, but have, in some instances, been seen through the nucleus, or the central part of the comet.

64. It is now established that the motions of comets are regulated by the same dynamical laws as those of the planets; and that, in general, they describe ellipses round the sun, situated in one of the foci. In a few cases, they have been ascertained to move in hyperbolas, and, consequently, when they have once passed their perihelion, they run off into infinite space, never more to return. There are three comets, whose orbits have now been determined and given in tables. The first is *Halley's* comet, the most remarkable of all; its period is 75 or 76 years; it appeared last in 1835. The second of these is the comet of *Encke*; it revolves in an ellipse of great excentricity, inclined at an angle of about 13° 22′ to the plane of the ecliptic in about $3\frac{1}{3}$ yerrs. The third is the comet of *Biela*. It is a very small comet, without a tail, or without any solid nucleus whatever. The time of its revolution is about $6\frac{3}{4}$ years, and its orbit nearly intersects that of the earth. An interesting question has arisen from observing the periodic times of Encke's comet. It appears that these periods are continually diminishing, and, consequently, its major axis is becoming continually less. This effect is precisely the same as would be produced by a very rare resisting medium, and, consequently, this explanation is generally received by astronomers. It will, therefore, either fall ultimately into the sun, or be dissipated altogether in space, before it reaches this centre of force.

FIXED STARS.

65. Besides the heavenly bodies which we have described, there is an immense number of objects called *fixed stars*. These are said to be fixed, because they always preserve the same situation (or very nearly so) with respect to each other. They are so far distant, that the diameter of the earth's orbit has been supposed to subtend no sensible angle as seen from them. Later observations, however, appear to show that some of these stars have a small sensible annual parallax.

Several stars, which appear single to the naked eye, when examined with a good telescope, appear to be formed of two or three stars, very near each other; such are Castor, α Herculis, the pole star, &c. Some of these double stars appear of different colours. The larger star in α Herculis is red, the smaller blue; ϵ Lyræ is composed of four stars, three white and one red; γ Andromeda is double, the larger reddish white, the smaller a fine sky-blue.

66. Considerable changes have taken place among the fixed stars. Several stars have disappeared, and new ones have appeared. Some stars also change their lustre periodically; the most remarkable of them is Algol, or β Persei. When brightest, it is of the second, and when least, of the fourth magnitude. Its period is about 2 days 21 hours; it changes from the second to the fourth magnitude in $3\frac{1}{2}$ hours, and back again in the same time; and so remains during the rest of the time. These are called *periodical* stars.

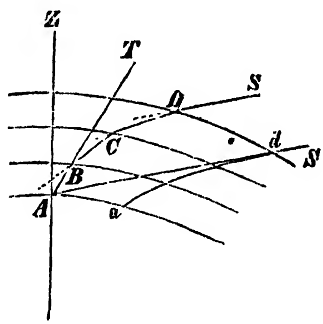
67. Although the fixed stars have been supposed to have no proper motion with respect to each other, yet later and more correct observa-

tions have shown that many of the double stars revolve about each other in regular elliptic orbits, thus proving that the same dynamical laws which govern the solar system extend throughout all space. One of the most remarkable of them is γ Virginis. The major semiaxis of the ellipse is $12''\cdot09$, the excentricity 0·83; and its period of revolution 629 years. It is a bright star, of the fourth magnitude, and its component stars are almost exactly equal. The period of the two stars in Castor is 253 years; and the period of ξ Ursæ is $58\frac{1}{3}$ years. These have been named *binary stars*.

CHAP. II.—VARIOUS CORRECTIONS.

REFRACTION.

68. When a ray of light passes through a medium of uniform density, it proceeds in a straight line, without any deviation; but if it passes obliquely from one medium to another of different density, its direction is changed, and the deviation from its former course is called, in astronomy, its *refraction*.

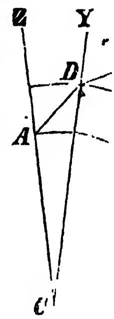


69. Suppose, now, a spectator placed on the earth's surface at A. A star situated at S, if there were no atmosphere, would be seen in the direction AS; but, in consequence of the air which surrounds the earth, the light, which proceeds in the direction SA, will be bent out of its course at d, and never reach the spectator at A. Conceive the atmosphere to be divided into several successive strata, by boundaries which are concentric with the earth's surface; then, if we suppose the densities in each of these strata to be uniform, a ray of light which proceeds in the direction SD will be deflected at D, in the direction DC; at C, again, it will be deflected in the direction CB ...... and it will at length enter the eye at A, in the direction BA; thus, the star will be seen in the direction ABT, instead of AS. If, now, we suppose the number of strata to be increased, and their thickness to be diminished indefinitely, this will represent the true state of the atmosphere, in which the density of the medium continually varies. Also the polygonal figure $DCBA$ will ultimately become a curvilineal figure, and the ray of light SD will enter the eye in the direction of AT, a tangent to the curve AD.

70. We have already seen, in Hydrostatics (art. 399), that the density of the air is affected not only by the weight of the superincumbent atmosphere, but also by its temperature. The refracting power of the atmosphere is likewise considerably affected by the moisture contained

in it, and, as neither the temperature nor the quantity of vapour are subjected to any known laws, it is difficult to determine the exact amount of refraction, particularly when the bodies are situated near the horizon. It is seldom, however, that the astronomer is obliged to make observations on altitudes less than 10 degrees.

71. It is proved, in works of optics, that a ray of light passing through several media, with parallel surfaces, will be as much refracted altogether as if it passed at once into the last medium. In the case of the earth's atmosphere, the surfaces of the different strata of air are not exactly parallel to one another at the different points D, C, B, &c., through which the ray passes, yet the approximation will in general be sufficiently correct, if we suppose the whole atmosphere to be compressed into a single stratum of the same uniform density as at the earth's surface.



Suppose this homogeneous atmosphere to extend to D, and let a ray of light SD be refracted at D in the straight line DA. Let the angle $ZAD = z$, $ADC = y$, $YDS = y + \rho$, then the amount of refraction is equal to ρ. Also, let the radius of the earth $CA = r$, the height of the homogeneous atmosphere $= h$, and therefore $CD = r + h$. Now, from the principles of optics,

$\sin YDS : \sin CDA :: \sin(y+\rho) : \sin y :: n : 1$, a given ratio;

$\therefore \quad n \sin y = \sin(y+\rho) = \sin y + \rho \cos y$, nearly,

since ϱ is always a very small quantity; hence $\varrho = (n-1)\dfrac{}{\cos y}$

But $\sin y : \sin z :: CA : CD :: r : r + h$;

$$\therefore \quad \sin y = \frac{r}{r+h} \sin z = \left(1 - \frac{h}{r}\right) \sin z = (1 - \beta) \sin z, \text{ nearly,}$$

$$\text{and} \quad \cos y = \sqrt{1 - (1-\beta)^2 \sin^2 z} = \sqrt{(\cos^2 z + 2\beta \sin^2 z)}$$
$$= \cos z\,(1 + \beta \tan^2 z), \text{ nearly,}$$

putting $\dfrac{h}{r} = \beta$, and neglecting all powers of β higher than the first.

$$\text{Hence } \varrho = (n-1)\,\frac{(1-\beta)\sin z}{\cos z\,(1+\beta \tan^2 z)} = (n-1)(1-\beta)\tan z\,(1-\beta \tan^2 z).$$

In this equation ϱ is determined in parts of the radius. If ϱ be estimated in seconds, the equation becomes, by reduction,

$$\varrho'' = \frac{n-1}{\sin 1''}\,(\tan z - \beta \tan z \sec^2 z).$$

72. When the height of the barometer is 29·6 inches, and the temperature $= 50°$, $\dfrac{n-1}{\sin 1''}$ has been found, by numerous experiments, $= 57{\cdot}82''$. Also, at the same altitude, the refraction is nearly propor-

tional to the density of the air, or is proportional to $\frac{p}{k(1+\alpha\tau)}$ (Hydr., art. 397); where p is the pressure on a unit of surface, or is equal to the height of the barometer, k is a constant quantity, $\alpha=\frac{1}{480}=\cdot002083$, and $\tau = t^\circ - 32^\circ$. Hence, if ϱ be the refraction when the height of the barometer is b inches, and the temperature $= t^\circ$, we shall have

$$\varrho : 57{\cdot}82'' \times (\tan z - \beta \tan z \sec^2 z) :: \frac{b}{1+\alpha(t-32)} : \frac{29{\cdot}6}{1+18\alpha},$$

$$\varrho = 57{\cdot}82'' \times \frac{b}{29{\cdot}6} \quad \frac{1{\cdot}0375\,(\tan z - \beta \tan z \sec^2 z)}{1+\cdot002083\,(t^\circ - 32^\circ)},$$

where $\beta = \cdot00128$.

*73. The absolute quantity of refraction may be found by observing the greatest and least altitudes of a circumpolar star. The sum of these altitudes, diminished by the sum of the refractions, is equal to twice the altitude of the pole: from whence, if the altitude of the pole be otherwise known, the sum of the refractions will be had. Hence, as the law of refraction is known from the preceding article, and the sum of the refractions is given, the refraction due to each altitude may easily be determined. These refractions, being calculated at all altitudes from the horizon to the zenith, are arranged in tables, and they are always to be subtracted from the observed altitudes.

74. It appears, then, from what has been stated, that the refraction is nothing in the zenith, that it increases from the zenith to the horizon nearly in the ratio of the tangent of the zenith's distance; that, at the altitude of 45°, the refraction is 57″·82, and in the horizon it is 33′. If a spectator were situated at the centre of the earth, there would be no refraction, because all the rays of light would pass through the air in a direction perpendicular to the different strata.

75. The atmosphere not only refracts light, but it also reflects it, which is the cause of twilight, and a considerable portion of daylight. The light which comes from the sun, when it is a little below the horizon, passes to the upper regions of the atmosphere, and is there partially reflected, and thus produces that inferior species of illumination called *twilight*. From repeated observations, it is found that some twilight can always be observed when the sun is less than 18° below the horizon.

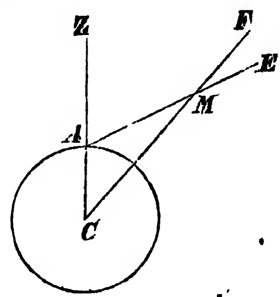
76. From the effects of refraction, the sun and full moon appear of an oval figure when they are in the horizon. The refraction of the upper limb being five minutes less than that of the lower limb, the vertical is so much less than the horizontal diameter.

PARALLAX.

77. As the place of an observer on the surface of the earth is continually changing, in consequence of its rotation about its axis, while its centre is unaffected by this motion, astronomers have agreed to refer the situations of all the heavenly bodies to the earth's centre. It is necessary, therefore, to calculate the effect arising from this supposed change in the position of the observer.

78. Def.—The parallax of any object in the heavens is the difference of its angular position as it would be seen from the centre of the earth, and as seen from its surface.

79. Prop. I.—*To find the effect of parallax.*



Let C be the centre of the earth, which we will first suppose to be a sphere, A the situation of an observer on its surface, M any object in the heavens, such as the moon. Produce CA to Z, the observer's zenith. From A the moon is seen in the direction AME, and its zenith distance is the angle ZAM; from C it would be seen in the direction CMF, and its zenith distance would be the angle ZCM; the difference of these angles, AMC, is called the parallax of M.

Let $CA = r$, $CM = d$, angle $ZAM = z$, angle $AMC = p$, then

$$CM : CA :: \sin CAM : \sin CMA \text{ or } \sin p;$$

$$\therefore \quad \sin p = \frac{CA}{CM} \cdot \sin CAM = \frac{r}{d} \sin z.$$

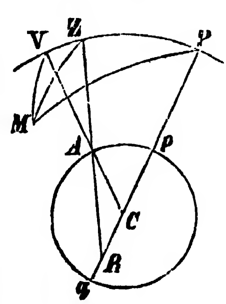
Hence it follows, if r and d be given, that the parallax is the greatest when $z = 90°$, or when the body is in the horizon. If P be put for the horizontal parallax, $\quad \sin P = \frac{r}{d}, \quad$ therefore

$$\sin p = \sin P \sin z, \quad \text{or,} \quad p = P \sin z,$$

without any sensible error.

It is evident that the effect of parallax is in the vertical plane ZCM, or in a plane which passes through the heavenly body, the observer, and the centre of the earth.

80. We have here supposed the earth to be a sphere, in which case, the radius of the earth, CA produced, will pass through the observer's zenith. In all the heavenly bodies, except the moon, this supposition will not produce any sensible error in the result, for the greatest parallax (namely, that of Mars) does not exceed 22″. But when it is necessary to make very correct observations on the moon, whose greatest parallax is 61′ 32″, we must take into consideration the spheroidal figure of the earth.



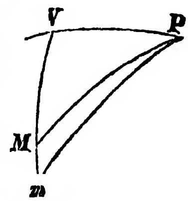
81. Let C be the centre of the earth, p its pole, pA a meridian passing through A, the place of the observer; then pAq is very nearly an ellipse, whose minor axis is pq. Draw the vertical line AR, and produce RA, CA, Cp, to meet the heavens in V, Z, P.

The true zenith of A is Z, but V is the point in the heavens in which a heavenly body has no parallax, because the points C, A, V are in the same straight line. Also, the parallax of any body in the heavens will take place in the plane $CAVM$, and not in the vertical plane AZM;

and these planes never coincide with each other, except when the body is on the meridian.

The latitude of the point A is the complement of PZ. If, then, PV be used for the colatitude, instead of PZ, and VM be found from the triangle VPM, all the preceding formulæ will be true for the spheroidal figure of the earth. The point V is called the *geocentric* zenith, and the complement of PV the *geocentric*, and sometimes the *reduced*, latitude.

82. Prop. II.—*To find the effect of parallax in right ascension.*



Let V be the geocentric zenith, P the pole of the heavens, m the situation of any heavenly body as seen from a place on the surface of the earth, and M its situation as seen from the centre: then the arc Mm is the effect of parallax, and is in the circle VM. Draw the circles of declination PM, Pm; then the angle MPm is the effect of parallax in right ascension, and $Pm - PM$ is its effect in declination.

Let the angle $VPM = h$, $VPm = h'$, $MPm = h' - h = \alpha$, $VP = 90° - l$, the geocentric colatitude, $PM = 90° - d$, $Pm = 90° - d'$, $Pm - PM = d - d' = \delta$. We have, then,

$$\begin{array}{llll} \sin MPm & : \sin PmM & :: \sin Mm \text{ or } \sin P \sin Vm & : \sin PM, \\ \sin PmM & : \sin VPm & :: \sin VP & : \sin Vm \\ \\ \sin MPm & : \sin VPm & :: \sin P \sin V & : \sin PM. \end{array}$$

Hence we have, by forming an equation, and substituting α and P for $\sin \alpha$ and $\sin P$,

$$\alpha = \frac{P \cos l \sin (h + \alpha)}{\cos d}.$$

83. Prop. III.—*To find the effect of parallax in declination.*

We have, from Trigonometry (art. 134),

$$\cos VP \cos VPM = \cot PM \sin VP - \sin VPM \cot V,$$

$$\cos VP \cos VPm = \cot Pm \sin VP - \sin VPm \cot V.$$

Multiplying the first equation by $\sin VPm$, and the second by $\sin VPM$, and substituting, we get

$$\sin l \cos h \sin h' = \tan d \cos l \sin h' - \sin h \sin h' \cot V.$$

$$\sin l \cos h' \sin h = \tan d' \cos l \sin h - \sin h \sin h' \cot V.$$

Subtracting the last equation from the one above it,

$$\sin l \sin (h' - h) = \cos l (\tan d \sin h' - \tan d' \sin h);$$

$$\therefore \quad \tan d = \sin \alpha \frac{\tan l}{\sin h'} + \tan d' \frac{\sin h}{\sin h'},$$

$$\tan d - \tan d' = \sin \alpha \frac{\tan l}{\sin h'} - \tan d' \left(1 - \frac{\sin h}{\sin h'}\right).$$

But $\tan d - \tan d' = \dfrac{\sin d}{\cos d} - \dfrac{\sin d'}{\cos d'} = \dfrac{\sin(d-d')}{\cos d \cos d'} = \dfrac{\sin \delta}{\cos d \cos d'}$;

also, $1 - \dfrac{\sin h}{\sin h'} = \dfrac{\sin h' - \sin h}{\sin h'} = \dfrac{2\sin \frac{1}{2}\alpha \cos(h + \frac{1}{2}\alpha)}{\sin h'}$.

Making these substitutions above, we get

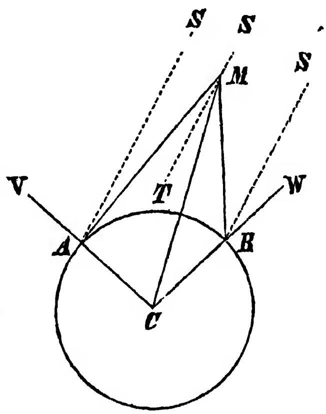
$$\frac{\sin \delta}{\cos d \cos d'} = \frac{\sin \alpha \tan l}{\sin h'} - \frac{2\sin \frac{1}{2}\alpha \cos(h + \frac{1}{2}\alpha)}{\sin h'} \tan d'.$$

And because $\sin \delta = \delta$, $\sin \alpha = \alpha = 2\sin \frac{1}{2}\alpha = P \dfrac{\cos l \sin h'}{\cos d}$ very nearly,

$$\frac{\delta}{\cos d \cos d'} = P\frac{\sin l}{\cos d} - P\frac{\cos l \tan d' \cos(h + \frac{1}{2}\alpha)}{\cos d}$$

$$\therefore \quad \delta = P \sin l \cos d' - P \cos l \sin d' \cos(h + \tfrac{1}{2}\alpha).$$

84. PROP. IV.—*To determine the amount of parallax from observation.*



Let A and B be two places on the earth's surface which have the same meridian, and C the centre of the earth, supposed to be a spheroid. Produce CA, CB, to V and W, the geocentric zeniths of A and B. Let M be a planet, whose situation is observed at A and B when it comes to the meridian; and S a fixed star, whose declination is nearly equal to that of the planet. Then S is so very distant, that AS and BS are sensibly parallel. Draw MT parallel to AS or BS. Now, the angle AMT or MAS is known, being the difference of the meridian zenith distances of M and S, as observed at A. In like manner, the angle BMT is found from the meridian observations at B; and consequently the angle AMB is known. The angles SAM, SBM, are measured with great accuracy, by means of a micrometer.

Let the angle $CMA = p$, the parallax at A, the angle $CMB = p'$, the parallax at B, and $p + p' = \alpha =$ angle AMB; also, put the angle $VAM = z$, and $WBM = z'$. We have then,

$$\sin p = p = \frac{CA}{CM}\sin z, \qquad \sin p' = p' = \frac{CB}{CM}\sin z'$$

$$p + p' = \alpha = \frac{CA \sin z + CB \sin z'}{CM}.$$

Let $P =$ the horizontal parallax at the equator, and $r =$ the equatorial radius, then $P = \dfrac{r}{CM}$, and, consequently,

$$P = \frac{r\alpha}{CA \sin z + CB \sin z'}.$$

If the two places, *A* and *B*, be not exactly in the same meridian, it will be necessary to make a correction for the change of declination in the planet during its passage from one meridian to the other.

85. By a comparison of the observations made at the Cape of Good Hope, by de la Caille, with those made at different observatories in Europe, the parallaxes of Mars and the moon have been obtained with considerable precision. This method, however, cannot be applied with advantage, when the parallax is less than 10 or 12 seconds, because the probable errors of observation bear too great a proportion to the whole amount; but as the relative distances of the planets from the sun are known with tolerable accuracy, from the principles of the last chapter,—when the parallax of Mars has been determined, the parallaxes of the sun, and all the other planets, can thence be easily ascertained. The only other method, which is superior to this in point of accuracy, is that furnished by the transit of Venus over the sun's disk, an account of which may be found in Woodhouse, Delambre, and other writers on astronomy.

86. The following examples will serve to illustrate the method of correcting the altitudes for the effects of refraction and parallax. A table of refractions is given in all astronomical tables, which must be corrected for the heights of the barometer and thermometer (see Mathematical Tables, Tab. 6); and in many collections there are tables combining the effects of refraction and parallax in the case of the sun and moon. The parallax in altitude is easily calculated from the formula $p = P \times$ cos app. alt. (art. 79); or,

$$\text{Prop. log } p = \text{Prop. log } P + \text{log secant alt.} - \text{log radius.}$$

Examples.

Ex. 1.—The apparent double altitude of a star, taken with the sextant and artificial horizon, was 15° 20′ 10″, and the index correction of the sextant − 2′ 30″. Required the true altitude, the thermometer being at 66° and the barometer 29·87 inches.

	°	′	″
Observed double alt.	15	20	10
Index error	—	2	30
Apparent double alt.	15	17	40
———— single do. .	7	38	50
Refraction	0	6	35·6
Star's true altitude. .	7	31	14·4

		′	″
Refraction (Tab. 6)....		6	51·9
Correction for barometer	—		1·8
———— thermometer	—		14·5
True refraction........		6	35·6

Ex. 2.—The apparent double altitude of the sun's lower limb was 29° 27′ 20″; the index correction was + 2′ 10″, and the semidiameter of the sun, taken from the Nautical Almanac, 16′ 11″·2.

		°	′	″
Observed double altitude of sun's L.L....		29	27	20
Index correction..............	+		2	10
		29	29	30
Apparent altitude of sun's L.L...........		14	44	45
Refraction (Tab. 6)	—		3	37·6
Semidiameter (N. A.)..........	+		16	11·2
Parallax (Tab. 1) ..	+			8·3
True altitude of the sun's centre.........		14	57	26·9

Ex. 3.—The apparent double altitude of the moon's lower limb was 48° 10′ 20″; its horizontal parallax was 59′ 22″, and its semidiameter 16′ 10″·6. Required the true altitude of the moon's centre,

	°	′	″
Observed double alt.	48	10	20
——— single do.	24	5	10
Refraction		2	9·6
	24	3	0·4
Semidiameter		16	10 6
	24	19	11
Par. in alt.		54	6
True alt. of moon's centre.	25	13	17

Prop. log P (Tab. 17) ..	·48173
Log secant of 24° 19′	·04035
Prop. log.............	·52208
∴ Par. in alt. =	54′ 6″

THE PRECESSION OF THE EQUINOXES.

87. In the preceding sections we have supposed the earth's axis of revolution to remain always parallel to itself, and to be inclined to the plane of the ecliptic, at an angle of 66° 32′. From the observations of ancient astronomers, however, they found that the earth's axis, or the pole of the heavens, has a very slow motion of 50″·1 annually, round the pole of the ecliptic, by which it would make a complete revolution in a little more than 25,000 years.

88. A tolerably correct idea of this motion may be formed from observing the spinning of a top. If a top spin steadily, its axis will always be upright, and point to the zenith of the heavens; but we frequently see that, while it spins briskly round its axis, the axis itself has a slow conical motion round the vertical line, so that, if produced, it would describe a small circle in the heavens round the zenith. A plane passing through the middle of the top, perpendicular to its axis, would, in this case, represent the terrestrial equator gradually turning round on all sides, and the floor of the room, or rather a plane parallel to it, would represent the plane of the ecliptic. If we suppose the top to go round the room in a circle, we should have all the motions of the earth correctly represented in kind, although not in degree. The revolution of the top round its axis would represent the diurnal revolution of the earth in 24 hours; the motion round the room would represent the earth's motion about the sun in 365¼ days; and the conical motion of the axis of the top round a vertical line would resemble the motion of the earth's axis about a line passing through the pole of the ecliptic in 25868 years. They only differ in this respect, that the spinning motion and the wavering motion of the top are in the same direction, whereas the diurnal rotation and the motion of the equinoxes are in opposite directions.

89. If the earth were a perfect sphere, of uniform density, the axis round which it revolves would always remain parallel to itself; since the attraction of the sun on the different particles of the earth would exactly balance on, a line joining the centres of the two bodies. In this case, the celestial equator and the pole of the heavens would remain fixed.

But as the figure of the earth is very nearly that of an oblate spheroid, we may consider the earth as consisting of an internal sphere, surrounded by an outward shell, which is most protuberant at the equator. In this case, it is easily shown, from the principles of dynamics, that the external shell will not balance itself, but has a continual tendency to draw the equator into the plane of the ecliptic.

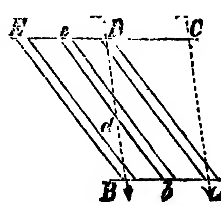
90. By combining this tendency with the earth's rotatory motion, the axis of the earth is made to move slowly round the pole of the ecliptic. The motion is not uniform, because the action of the sun and moon continually varies; but all the different effects may be thus represented. The pole of the heavens may be supposed to describe a circle, uniformly from east to west, round the pole of the ecliptic, at its mean distance. This is called the *precession*, and is about $50''\cdot34$ annually, of which the sun produces about $15''\cdot3$ and the moon about $35''$. But, from the unequal disturbing force of the sun, the pole deviates from its mean place in the heavens, and it may be supposed to describe a small circle in half a year round the mean pole, with a radius $= 0''\cdot43$. This is called the *solar inequality*. A second correction, of a similar kind, is applied on account of inequalities in the action of the moon on the protuberant parts of the earth. This may be represented by describing an ellipse round the mean pole, whose axes are $19''\cdot2$ and $15''$, and supposing the true pole to describe the circumference of this ellipse in 19 years. This correction is called *lunar nutation*. By physical astronomy we find, also, that a slight change is produced in the position of the ecliptic by the action of the planets, and that, from this cause, the equinoxes advance about $0''\cdot16$ annually, thus making the whole annual regression of the equinoxes equal to about $50''\cdot34 - 0''\cdot16$, or $50''\cdot18$.

91. In consequence of these different motions of the pole of the heavens, the right ascensions and declinations of the fixed stars undergo continual alteration. Proper formulæ have been investigated for computing these changes, but as they are inserted in the Nautical Almanac, we shall omit them in this work.

ABERRATION OF LIGHT.

92. The aberration of light, in astronomy, is an apparent change of place of the celestial bodies, occasioned by the progressive motion of light, and the earth's annual motion in its orbit.

This effect may be explained thus. Let AB represent the velocity of the earth in its orbit, and AC the velocity of light proceeding from a star or other heavenly body; also, suppose AD to be a tube or telescope, with which the star is observed. Now, if the earth be at rest, it is evident that the tube must be held in the direction AC, and the light will pass through it without interruption; but if the earth move from A to B, whilst the light passes from C to A, the light will strike the side of the tube, and will no longer pass through it. Let the tube now be moved forward in the direction of AD, the diagonal of the parallelogram $ABDC$; the light will then pass through the axis of the tube without striking its sides. Thus, if D be a particle of light just entering



the tube at D, when the tube has moved into the position be, in consequence of the progressive motion of the earth, we have

$$\begin{aligned} \text{vel. of earth} : \text{vel. of light} &:: AB \text{ or } DE : DB \\ &:: De \text{ or } Ab : Dd \text{ (by sim. triangles),} \end{aligned}$$

and therefore the light will move through Dd whilst the tube passes over the space Ab. Hence the light will be at d in the axis of the tube. As the light then passes through the tube of the telescope in the position AD without interruption, the star will be seen in the heavens in the direction AD, instead of its true direction AC.

From the triangle ABD, we have

$$\text{vel. of earth} : \text{vel. of light} :: AB : BD :: \sin ADB : \sin BAD.$$

If, therefore, the aberration $ADB = \alpha$ and $BAD = \phi$, we have

$$\sin \alpha = \frac{\text{vel. of earth}}{\text{vel. of light}} \times \sin \phi, \text{ or,}$$

$$\alpha = 20''\cdot5 \sin \phi, \text{ nearly.}$$

93. On these principles formulæ have been investigated for determining the effect of aberration in right ascension and declination, and also in longitude and latitude; but, as the right ascensions and declinations of all the principal fixed stars are given in the Nautical Almanac, corrected for aberration, precession, and nutation, and also their proper motions (if they have any), it is unnecessary to introduce them into this work.

CHAP. III.—TO FIND THE TIME AND THE LATITUDE AND LONGITUDE OF A PLACE.

SIDEREAL AND SOLAR TIME.

94. All the heavenly bodies appear to move round the axis of the heavens; and if they have no proper motion, they all return to the meridian after the same interval of time. This interval is manifestly the same as the time of the earth's revolution about its axis, and is called a *sidereal day*. It is supposed to commence when the first point of Aries is on the meridian, and the sidereal clock at that instant is placed at $0^h\ 0^m\ 0^s$. This time, however, is not adapted to the purposes of civil life. For these the period of a solar day, or the time elapsed between two successive passages of the sun over the meridian, is a more convenient measure of time. Now, a solar day differs from the time of the earth's rotation about its axis, in consequence of the increase of the sun's right ascension during this interval. But the daily increase of the sun's right ascension is variable, from two causes:—1st, From the inclination of the ecliptic to the equator; and 2ndly, From the sun's unequal motion in longitude. It is evident, therefore, that solar days must be unequal in length, and consequently clocks, which move uniformly, cannot be regulated to keep time with the sun.

95. If the sun moved uniformly in the equator, the interval between two transits of the sun over the meridian would be always the same, and would be an exact measure of time. Let us suppose, then, an *imaginary* sun moving uniformly over the equator in the same time in which the real sun appears to move over the ecliptic, and having its right ascension equal to the *mean* longitude of the sun. The time measured by this imaginary sun is called *mean solar time*, or *mean time*; and the time measured by the real sun is called *apparent solar time*, or *apparent time*; it is also sometimes called true time. The difference between mean time and apparent time is called the *equation of time*; and it is equal to the difference between the right ascension of the sun and the mean longitude of the sun converted into time, at the rate of $360°$ for 24^h, or $15°$ for 1^h. In the civil reckoning of time, the day begins at midnight, and is divided into two equal parts of 12 hours each, distinguished by A. M. (ante-meridiem) and P. M. (post meridiem): but astronomers begin the day at noon, 12 hours later than the commencement of the civil day: thus, Nov. 6, $2^h\ 0^m$ A. M., is Nov. 5, $14^h\ 0^m$, astronomical time.

96. The time elapsed from the sun's leaving an equinox until it comes to it again, is called *a solar year*. Since the equinoctial points are continually carried backward on the plane of the ecliptic, the sun returns sooner to the same equinox than to the same fixed point in the heavens. A solar year, therefore, is less than the time in which the earth makes a complete revolution round the sun; this latter interval is called a *sidereal year*. According to Delambre, a solar year is equal to $395^d\ 5^h\ 48^m\ 51^s{\cdot}6$, and the sidereal year equal to $366^d\ 6^h\ 9^m\ 11^s$. The *civil* year is made to agree as nearly as possible with the mean solar year, because the same succession of seasons, and the same variation in the lengths of the days, recur with it periodically. But, as it would be inconvenient to begin the civil year at any point of time, except the beginning of a day, a year must consist of an entire number of days; and since the length of the solar year consists of a number of days and a fraction of a day (·242264 of a day), there is some difficulty in making this agreement. According to the Julian calendar, one day was added to every fourth year. This is equivalent to making the year consist of 365·25 days, which is too great by $0{\cdot}007736^d$. Hence, according to this reckoning, the loss in the Julian calendar amounted to 1 day in 130 years, and to a little more than 3 days in 400 years; Pope Gregory XIII., therefore, directed that three of the centenary years, 17, 18, and 19 hundred (in which the number of centuries is not divisible by 4) should consist of 365 days each, and not of 366 days, as in the Julian calendar.

97. PROP. I.—*Having the time at any place, to find Greenwich mean time.*

The right ascensions, declinations, &c. of the sun, moon, and planets are inserted in the Nautical Almanac, and computed to Greenwich mean time. It is necessary, therefore, before we can make use of them, that we should know the mean time at Greenwich, at least approximately, when any observation was made. For this purpose we shall give the following rule, the reason of which will readily appear.

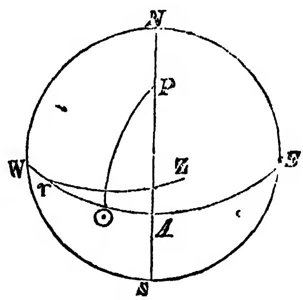
Rule.—When the time at the place is P.M., it is the same as astronomical time, the P. M. being omitted; when the time is A. M., take

the preceding day, and add 12 hours to the time for astronomical time. Convert the longitude into time, at the rate of 15° for 1 hour, then will

Greenwich mean time = mean time + west longitude in time,

Greenwich mean time = mean time — east longitude in time.

98. Prop. II.—*To convert sidereal time into mean and apparent time, and the contrary.*



Let $NESW$ represent the horizon, NZS the celestial meridian, P the pole, Z the zenith, and EAW the celestial equator. Also let ♈ represent the first point of Aries, and ☉ the intersection of a circle of declination passing through the sun's place in the heavens with the equator. Then, when ♈ was on the meridian, the sidereal time was $0^h\ 0^m\ 0^s$, and, therefore, the time of passing over A♈ will manifestly represent the time shown by the sidereal clock. In like manner A☉, converted into time, at the rate of 15° for 1 hour, will represent apparent solar time; also, ♈☉ is the right ascension of the true sun; hence, therefore,

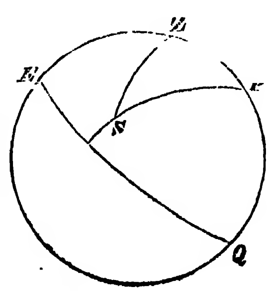
Sidereal time = R.A. of the true sun + apparent solar time;
and, for the same reason,

Sidereal time = R.A. of the mean sun + mean solar time.

If the sum be greater than 24^h, then 24^h must be rejected.

99 *Note.*—The right ascension of the *mean sun* is given every day at mean noon, in the second page of each month, in the Nautical Almanac, under the head of *Sidereal Time;* and it may be taken out for any intermediate Greenwich time, by a proportion in the usual manner. There is a table, however, given in the Nautical Almanac, for "converting intervals of mean solar time into equivalent intervals of sidereal time," which facilitates this reduction.

100. Prop. III.—*To find the time at a place by means of a single altitude of the sun.*



Observe the altitude of the sun two or three times when it is nearly *east* or *west*, and at the same instant note the time by a timekeeper. Take a mean of the altitudes, and also a mean of the times. Let PZE be the meridian, P the pole of the heavens, Z the zenith, and S the place of the sun. We have, then, in the triangle ZPS, ZS the coaltitude, ZP the colatitude, and PS the codeclination, given, to find the angle ZPS. Put $ZP = 90° - l = l'$, $PS = 90° - d = d'$, $ZS = z$, and the angle $ZPS = h$. We have then (Trig., art. 153),

$$\sin^2\frac{h}{2} = \text{rad}^2\,\frac{\sin\frac{1}{2}(z + d' - l')\,\sin\frac{1}{2}(z - d' + l')}{\sin d'\,\sin l'};\quad\text{or,}$$

$$\sin^2\frac{h}{2} = \text{rad}^2\,\frac{\sin\frac{1}{2}(z + l - d)\,\sin\frac{1}{2}(z - l + d)}{\cos l\,\cos d}.$$

In this expression the latitude and declination are supposed to have the same name; if they have different names, $-d$ must be substituted for d. Hence we readily obtain the following

Rule.—Place the latitude and declination under one another, with their proper names marked N or S. If their names are *alike*, take their *difference;* but if they are *unlike*, take their *sum*. Under the result put the true zenith distance of the sun's centre; take the sum and difference of these two quantities, and their half sum, and half difference.

Add together the log secants of the latitude and declination, and the log sines of the above half sum and half difference. Divide the sum of these four logarithms by 2, and the quotient will be the log sine of $\frac{1}{2}h$. Then

$\left.\begin{matrix} h \\ 24^{\text{h}}-h \end{matrix}\right\}$ in time, will be app. time when the sun is $\left\{\begin{matrix}\text{west}\\ \text{east}\end{matrix}\right\}$ of the meridian.

Find the equation of time from the Nautical Almanac, page I. of the month, and apply it, with its proper sign, to the apparent time; the result will be the mean time at the place.

Ex.—Sept. 11, 1841, A.M., at Croydon, in latitude 51° 22′ 30″ N., and longitude 0° 5′ 47″ W., the mean of several double altitudes of the sun's lower limb was 43° 3′ 10″, and the mean time by a chronometer $8^{\text{h}}\ 0^{\text{m}}\ 10^{\text{s}}{\cdot}2$. Required the error of the chronometer on mean time at Croydon, the index correction of the sextant being + 1′ 32″.

Greenwich Time.	h	m	s	Declination.	°	′	″	Equat. of time. m	s
10th, Ast. time	20	0	10·2	10th, at noon	4	5	11·0N.	3	8·74
Long. in time..		0	23·1W.	Correction ..		19	2·5		17·25
Gr. mean time	20	1			3	46	8·5	3	25·99

	°	′	″		°	′	″		
Doub. alt.	43	3	10	Lat.....	51	22	30N.	......sec.	·204662
Ind. error	+	1	32	Dec....	3	46	8N.	......sec.	·000911
	43	4	42		47	36	22	......sin.	9·928041
	21	32	21	z.......	68	14	2	..sin.	9·252952
Refraction	−	2	26·2		115	50	24		19·386596
	21	29	54·8		20	37	40		9·693298
Semidiam.	+	15	55·2		57	55	12..		24 34 18
Parallax	+	0	8·0		10	18	50......		2
True alt.	21	45	58					Hour angle	59 8 36
z.......	68	14	2						

	h	m	s	
Hour angle in time..........	3	56	34·4	East.
	24			
Apparent time at Croydon....	20	3	25·6	
Equation of time............	—	3	26·0	
Mean time at Croydon	19	59	59·6	
Time by chronometer........	20	0	10·2	
Error of chronometer		0	10·6	Fast.

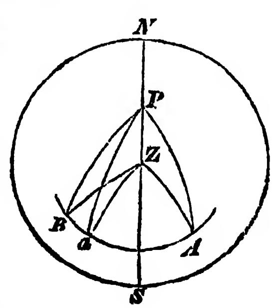
101. Prop. IV.—*To find the time at any place by means of equal altitudes of the sun taken before and after noon.*

One of the best practical methods of finding the time at any place, is to observe the sun's altitude about three or four hours before noon, when its motion in altitude is considerable, and let an assistant note the time by a chronometer. Again, in the afternoon, at nearly the same distance from noon, let the instrument be fixed at the same altitude as before, and when there is an exact contact, or the sun has descended to this altitude, let the time be noted again; then the mean between these two times would be apparent noon, or the time of the sun's passage over the meridian, if there was no change in the declination. But as this is seldom the case, a small correction will be necessary, in order to obtain the exact time of apparent noon.

In practice, it will be found best to set the zero of the index at an exact 5′ or 10′ on the instrument, and wait in the morning until the sun has risen to this altitude. At this instant let an assistant note the time, and write it down together with the sun's altitude. Again, set the index on the instrument to the next 5′ or 10′, and proceed as before, and thus take eight or ten observations. In the afternoon, as many corresponding observations must be taken as possible. Then add the times in the morning together, and divide by their number; the result will be the mean of the morning altitudes. In the same manner find the mean of the corresponding altitudes in the afternoon. The middle time between these means will be nearly the time of apparent noon. The sun should be at least two hours from noon.

102. Prop. V.—*To find the equation of equal altitudes, or the correction due to the change of declination in the sun.*

Let Z be the zenith, P the pole of the heavens, NZS the meridian, A the place of the sun at the morning observation, and AaB a small circle parallel to the horizon. If, now, Pa be taken equal to PA, the angle ZPa will be equal to ZPA (Trig., art. 125), or PZ will bisect the angle APa; but if the sun has approached towards the elevated pole P, it must pass through an additional angle aPB in its diurnal motion, before it descends to the parallel AaB.



Let $PZ = 90° - l$, $PA = 90° - D$, $ZA = z$, and the angle $ZPA = ZPa = h$.

Also, let $2t$ = the time of moving from A to B = $2t$, δ = the change of declination in the time $2t$, and α = the angle aPB. We have, then, (Trig., art. 140)

$$\cos z = \sin l \sin D + \cos l \cos D \cos h \ldots\ldots (a).$$

Now, if we suppose l and z to remain constant, and D to vary and become $D + \delta$, the angle h, which may be considered as a function of D, will also vary, and its increment, by Taylor's theorem, will be a series ascending by the powers of δ. And because δ is always very small, its second and higher powers may be neglected without sensible error. Also, since the first term of the increment is equal to the differential of any function (Diff. Calculus, art. 15), the differentials of D and h will be very nearly proportional to their increments. Hence, differentiating equation (a),

$$0 = dD \sin l \cos D - dD \cos l \sin D \cos h - dh \cos l \cos D \sin h;$$

$$\therefore \quad dh = dD (\tan l \operatorname{cosec} h - \tan D \cot h).$$

Substituting α and δ for dh and dD,

$$\alpha = \delta (\tan l \operatorname{cosec} h - \tan D \cot h) \ldots\ldots (b).$$

Now, to obtain an approximation to the meridian passage of the sun, the half interval t is added to the time of the morning observation; but $\frac{1}{15}h$ is manifestly the interval between this observation and noon. We have, also,

$$2t = \tfrac{1}{15}(2h + \alpha); \quad \text{and, therefore, } \tfrac{1}{15}h = t - \tfrac{1}{30}\alpha.$$

Hence, the correction x (seconds in time) is equal to $-\frac{1}{30}\alpha''$ (α being expressed in seconds of space). Also, if δ' be the change in the sun's declination for 1 hour of the given day, set down in the first page of each month in the Nautical Almanac,

$$\delta' : \delta :: 1^h : 2t, \quad \text{and, therefore, } \delta = 2t\delta';$$

or, if instead of δ' we take $\frac{1}{2}(\delta' + \delta'')$, δ'' being the change for 1 hour of the preceding day, we get

$$\delta = t(\delta' + \delta'') = \tfrac{1}{15}h(\delta' + \delta'') \text{ very nearly.}$$

Making these substitutions in equation (b) we obtain

$$x^s = \frac{h(\delta' + \delta'')}{450}\left\{\cot h \tan D - \operatorname{cosec} h \tan l\right\}.$$

From which expression we obtain the following

Rule.

Find apparent time at Greenwich corresponding to the apparent noon at the place of observation, and to this date find the sun's declination and the equation of time.

Find half the interval between the mean of the morning and the mean of the corresponding afternoon observations, and add this half interval to the mean of the morning observations, for approximate apparent noon. Also, convert this half interval into degrees and minutes.

Add together 7·34679 (the arithmetic complement of the logarithm of 450), the logarithm of h (in degrees and decimals of a degree), and the logarithm of $(\delta' + \delta'')$; and call the sum (A).

Add together (A), the log *cotangent of h*, and the log *tangent of the declination*, the sum will be the logarithm of the first part of the correction in seconds of time.

Add together (A), the log *cosecant of h*, and the log *tangent of the latitude*, the sum will be the logarithm of the second part of the correction in seconds of time.

Mark the 1st part $\left\{\begin{matrix}+\\-\end{matrix}\right\}$ when the declination is $\left\{\begin{matrix}\text{increasing,}\\\text{decreasing.}\end{matrix}\right.$

Mark the 2nd part $\left\{\begin{matrix}+\ \text{from midsummer to midwinter,}\\-\ \text{from midwinter to midsummer.}\end{matrix}\right.$

Apply these two corrections, with their proper signs, to the approximate time of the meridian passage, the result will be the correct time of apparent noon.

To the time of apparent noon apply the equation of time, and the time of mean noon will be obtained.

Example.

Sept. 11, 1841, at Croydon, in latitude 51° 22′ 30″N., and longitude 0° 5′ 47″ W., the following equal altitudes of the sun, and the corresponding times by a chronometer, were observed; required the error of the chronometer on mean time at Croydon.

Altitudes.		Morning.	Afternoon.
°	′	h m s	h m s
35	30	9 51 19·5	2 8 24·5
	40	52 55·6	6 48·3
	50	54 32·5	No observation
36	0	56 11·3	3 31·7
	10	57 51·3	1 51·7
	20	59 33·5	0 10·0

Here the third observation in the morning must be omitted, as there was no corresponding one in the afternoon.

	h m s
11th, Apparent noon	0 0 0
Longitude in time..	0 23·1W.
App. Gr. time	0 0 23·1
Sun's decl^{n}. (N. A.)	4° 32′ 24″
	m s
Equation of time......	3 29·4
$\delta' + \delta''$ =	112″·52

	h m s
Mean of morning obs.	9 55 34·2
——— afternoon -	14 4 9·2
Interval	4 8 35·0
Half interval	2 4 17·5
1st mean	9 55 34·2
Approximate noon...	11 59 51·7

$\frac{1}{2}$ interval in degrees 31° 4′ = 31°·07

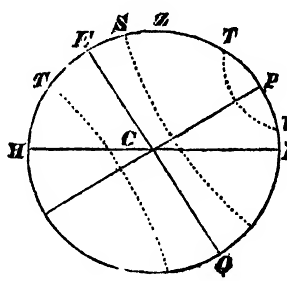
Const. log..	7·34679	(A).....	·89028	(A).......	·89028
h..	1·49234	cot h....	·22008	cosec h.....	·28732
$\delta' + \delta''$..	2·05115	tan d....	8·89920	tan l........	·09732
(A)..	0·89028	1.02.....	·00956	18·83......	1·27492

	h	m	s	
Approximate apparent noon	11	59	51·7	
1st correction	—	0	1·02	
2nd do.	+	0	18·83	
Equation of time	—	3	29·4	
Time, by chronometer, of mean noon at Croydon	11	56	40·1	
Error of chronometer	0	3	19·9	Slow.

TO FIND THE LATITUDE.

103. We have already seen (art. 29) that the latitude of any place on the surface of the earth is equal to the altitude of the pole above the horizon, or is equal to the distance of the zenith from the celestial equator. From this consideration, it is easy to find the latitude, by observing the meridian altitude of a heavenly body whose declination is known. The *meridian* altitude of any object is ascertained by observing when the altitude is the greatest.

104. Prop. VI.—*Having the meridian altitude of a heavenly body, and its declination, to find the latitude.*



Let $PZHQ$ be the celestial meridian, P the pole, Z the zenith, EQ the celestial equator, and HN the horizon. Let S be any heavenly body on the meridian, then its apparent altitude is observed, either with a sextant or a reflecting circle, and from thence its true altitude HS is found, by correcting the apparent altitude for refraction and parallax, as in art. 86. Subtracting HS from 90°, the true zenith distance ZS is obtained. Then

lat. ZE = true zen. dist. ZS + decl^n. ES.

If the heavenly body T be situated between E and H, or between Z and P, the latitude will be found by taking the difference of ZT and ET. These different cases may all be included in the following

Rule.—Call the zenith distance north or south, according as the zenith is north or south of the heavenly body. If this be of the same name with the declination, their sum will be the latitude of the same name with either; but if they have different names, their difference will be the latitude, having the name of the greater.

Note.—The declination of the sun's centre is given, in the first page of each month, in the Nautical Almanac, at *apparent* noon of each day, and in the second page it is given at *mean* noon. The declination of the moon's centre is given for every hour of the day, from page V. to page XII. in each month. The declinations of all the principal stars, corrected for aberration, precession, and nutation, are inserted, for every tenth day, at the end of the Nautical Almanac.

The equatorial horizontal parallax of the moon is given in the Nautical Almanac for every noon and midnight, in page III. of each month. When great accuracy is required, it must be reduced to the horizontal

parallax at any other place, by means of a small table calculated for that purpose (Tab. 9). The horizontal parallax of the sun is given in the 266th page; its variation, however, is so trifling, that the parallax in altitude is generally calculated from its mean value (Tab. 1).

The semidiameter of the sun is given in page I., and the semidiameter of the moon in page II., of each month. These are their values as seen from the centre of the earth. In the case of the moon, it is necessary to make a correction, on account of the observer's situation on the surface of the earth; but with the sun this correction is altogether inappreciable.

Example.

At Addiscombe, on the 9th of October, 1841, in longitude 0° 4′ 40″ W., the observed meridian altitude of the sun's lower limb was 32° 4′ 30″. Required the latitude of the place, the altitude of the thermometer being 50°, and the barometer 30 inches.

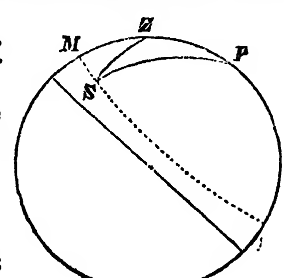
	h	m	s
Apparent time at Addiscombe	0	0	0
Longitude in time..........................	0	0	18·7W.
Apparent time at Greenwich	0	0	18·7
	°	′	″
Observed altitude of sun's L.L............	32	4	30
Refraction (Tab. 6)........................	—	1	32·8
	32	2	57·2
Semid^r. (N.A. p. II.)	+	16	2·8
Parallax (Tab. 1)..........................	+	0	7·2
True altitude of sun's centre..............	32	19	7·2
True zen. distance........................	57	40	52·8N.
Decl^n. (N.A. p. I.)	6	18	13·8 S.
Latitude....................................	51	22	39 N.

105. Prop. VII.—*To find the latitude from the altitude of a heavenly body under the pole.*

The meridian altitude of a heavenly body may sometimes be observed at its inferior passage from west to east under the pole. In this case the observed altitude diminishes, and the meridian altitude is the least. Let t be the place of the body at its inferior passage (fig. page 475): then the

Latitude PN = true altitude Nt + polar distance Pt.

106. Prop. VIII.—*To find the latitude from several observations of the sun taken near the meridian.*



Let PZM be the meridian, Z the zenith, P the pole, and S the place of the sun near the meridian. Put $PZ = 90° - l$, $PS = PM = 90° - d$, $ZS = z$, and the angle $ZPS = h$. We have, then (Trig., art. 140),

$$\cos z = \sin l \sin d + \cos h \cos l \cos d,$$

and because $\cos h = 1 - 2\sin^2\frac{1}{2}h$,

$$\cos z = \cos(l - d) - 2\sin^2\tfrac{1}{2}h \cos l \cos d.$$

Suppose $ZS = ZM + x$, or $z = l - d + x$; then, since the altitude near the meridian

varies very little, ZS will be nearly equal to ZM, or x will be very small; therefore

$$\cos z = \cos(l - d + x) = \cos(l - d)\cos x - \sin x \sin(l - d)$$
$$= \cos(l - d) - x \sin(l - d),$$

$\cos x$ being nearly $= 1$, and $\sin x = x$. Hence, comparing these two values of $\cos z$, we get

$$x \sin(l - d) = 2 \sin^2 \tfrac{1}{2}h \cos l \cos d.$$

Here x is measured in parts of the radius; if x be required in seconds, we have $x = x'' \sin 1''$, and consequently

$$x'' = \frac{2 \sin^2 \frac{1}{2}h}{\sin 1''} \times \frac{\cos l \cos d}{\sin(l - d)} \ldots\ldots\ldots (a).$$

Hence, if z, z', z'', &c. be n zenith distances, taken on both sides of the meridian, h, h', h'', &c. the corresponding hour angles, and x, x', x'', &c. the calculated corrections, then the true zenith distance, when the sun is on the meridian, will be equal to

$$\frac{(z - x) + (z' - x') + \&c.}{n} = \frac{z + z' + z'' + \&c.}{n} - \frac{x + x' + x'' + \&c.}{n}.$$

In this formulæ, z, z', &c. are known from observation; x, x' &c. are calculated from equation (a), in which d is found from the Nautical Almanac; h, h', &c. are determined by means of a timekeeper; and an approximate value of l, sufficiently near for determining the values of x, x', &c., may be found from the least value of $z + d$.

Example. (From Biot.)

Dec. 19, 1802, the following observations were made, at Dunkirk, on the star *α Polaris*. The approximate latitude was 51° 2′ 5″, the declination of the star was 88° 17′ 41″·41, and the time of its passage over the meridian, by the clock, $9^h\ 24^m\ 44^s$.

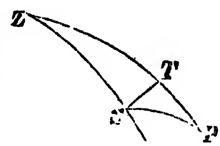
Times by the clock.				Values of h.			Values of $\frac{2 \sin^2 \frac{1}{2}h}{\sin 1''}$.
h	m	s		m	s		″
23	57	2	......	27	42	....	1504·7
	58	18	......	26	26	....	1370·4
	59	6	......	25	38	....	1288·8
......							

Sum of 26 observations	24811·8
Log 24811·8 − log 26	2·9796885
Correction for retardation of pend....	·0006986
cos l cos d sec $(l - d)$....	8·4900862
29″·55	1·4704733

Hence, therefore, we obtain

	°	′	″	
$\frac{1}{26}(z + z' + z'' + \&c.)$	37	15	20·9	
Refraction	+	0	46·4	
$\frac{1}{26}(x + x' + x'' + \&c.)$	−	0	29·55	
Meridian zenith distance..........	37	15	37·75	S.
Declination	88	17	44·41	N.
Latitude	51	2	6·66	North.

107. PROP. IX.—*To find the latitude by the altitude of the pole star, taken at any time.*



Let ZP be the celestial meridian, Z the zenith, P the pole, and S the pole star, whose distance from the pole is at present less than 100 minutes. Put $ZP = 90° - l$, $ZS = 90° - a$, the polar distance $PS = p$, and the angle $ZPS = h$, which is known, since the time is supposed to be given. From S draw ST perpendicular to ZP, then, since PS is less than 100 minutes, the arcs ZS, ZT, are nearly equal. Let $ZS - ZT = x$, $PT = y$; then

$$ZT = ZS - x = 90° - a - x; \text{ also } ZT = ZP - PT = 90° - l - y;$$

$$\therefore \quad 90° - a - x = 90° - l - y; \quad \text{or,} \quad l = a + x - y.$$

Now, in the right-angled triangle PST, we have, by Napier's rules, $\tan y = \cos h \tan p$; and, because $y = \tan y - \frac{1}{3}\tan^3 y + \&c.$ (Trig., art. 87), therefore

$$y = \cos h \tan p - \tfrac{1}{3}\cos^3 h \tan^3 p + \&c.$$

But $\tan p = p + \frac{1}{3}p^3 + \&c.$; substituting this value in the last equation, and neglecting all powers of p greater than the third, we get

$$y = p\cos h + \tfrac{1}{3}p^3\cos h - \tfrac{1}{3}p^3\cos^3 h = p\cos h + \tfrac{1}{3}p^3\sin^2 h\cos h.$$

Again we have, from Napier's rules,

$$\frac{\cos ZT}{\cos ZS} = \frac{1}{\cos ST} = \frac{\cos PT}{\cos PS}; \text{ or, } \frac{\sin(a + x)}{\sin a} = \frac{\cos y}{\cos p};$$

consequently, $\sin a \cos x + \sin x \cos a = \sin a \dfrac{\cos y}{\cos p}$, or

$$\sin x \cos a = \sin a\left(\frac{\cos y}{\cos p} - \cos x\right);$$

$$\therefore (x - \&c.)\cos a = \sin a\,(\tfrac{1}{2}p^2 - \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2 + \&c.)$$

Hence it is evident that x is of the second order with respect to p, and therefore x^2, which is of the fourth order, may be neglected; consequently

$$x = \tan a\,(\tfrac{1}{2}p^2 - \tfrac{1}{2}y^2) = \tfrac{1}{2}p^2\tan a\sin^2 h.$$

Hence, expressing p, x, and y in seconds, we obtain

$$l = a - p\cos h + \tfrac{1}{2}\sin 1''(p\sin h)^2\tan a - \tfrac{1}{3}\sin^2 1''(p\sin h)^2(p\cos h).$$

From this formula, tables are calculated, and inserted in the Nautical Almanac, by means of which the latitude can very readily be found.—(See the explanation at the end of the Nautical Almanac.)

TO FIND THE LONGITUDE.

108. When the sun is on the meridian at any place A, it is 12 o'clock at A; and as the sun appears to move from east to west in its diurnal motion, it will be on the meridian of a place B, 15° to the west of A, one hour afterwards. The time, therefore, will be noon at B, when it is 1 in the afternoon at A. The difference of *local* times, therefore, at

A and *B*, at the same instant of *absolute* time, will be proportional to the difference of longitude. All that is necessary, then, for finding the longitude of any place, is to determine the time at *B*, by any of the preceding methods, and also to find the corresponding time at some other given place *A*.

109. Nothing would appear to be more easy than this in theory. If we suppose the errors and rates of one or several good chronometers to be carefully determined at *A*, we shall have the time at *A* by these chronometers when they are transferred to *B*; and they have now attained so great a degree of perfection, that this simple method is very generally adopted. But still, as they are subject to variation, and continually liable to accidents, it becomes necessary to have some independent method of finding the time at *A*.

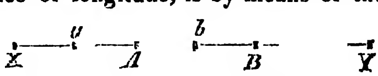
110. If we suppose that any instantaneous phenomenon, such as the explosion of a rocket, the bursting of a meteor, &c., should be observed at *A* and *B*, and the times noted at these two places, the difference of these times would give the difference of longitude. The first of these methods has been adopted by officers in the trigonometrical survey; and astronomers have proposed to make a trial of the second for places at a considerable distance from each other, as it is now conjectured that many of the meteors, or falling stars, lie far beyond the bounds of our atmosphere. It would be necessary, however, in such cases, that the two observers should communicate with each other before their difference of longitude could be determined.

111. If the phenomenon to be observed were of such a nature that the time when it happened at *A* could be previously calculated, and set down in the Nautical Almanac, the observer at *B* would be able to determine his longitude immediately, by comparing his time with that given in the Almanac. Of this kind are the *eclipses of the moon*, and the *eclipses of Jupiter's satellites*. Neither of these methods, however, is susceptible of great accuracy.

112. There are also other phenomena which do not happen at the same moment of *absolute* time at different parts of the earth, such as *occultations* and *solar eclipses*. Thus a solar eclipse may commence at some instant of *absolute* time at *A*, before it commences at *B*; and, indeed, there may be a total eclipse at *A*, whilst there is none at all at *B*. These latter phenomena generally require tedious calculations, but are capable of great exactness.

113. Prop. X.—*To find the longitude by means of signals.*

One of the easiest, and at the same time one of the most accurate, methods of determining the difference of longitude, is by means of the explosion of rockets. Let *X* and *Y* be two stations, provided with accurate means of determining their respective *local* times; and suppose that they are too far distant to be able to see the same signal. Let *A*, *B*, &c. be any intermediate stations, so selected that a signal *a* may be distinctly visible both at *X* and *A*; a signal *b* may be seen both at *A* and *B*; and so on throughout the whole line. Suppose, now, a rocket to be fired at *a*, at a time previously arranged by the observers, and let the time at *X* and *A* be



carefully noted by the observers at these places. About five minutes afterwards let another signal be made at b, and noted by the observers at A and B. Again, five minutes afterwards, let another signal be made at c, and so on. The time of the last signal is carefully noted by the observer at Y. Suppose that x is the sidereal time at X of the first signal at a, determined by a sidereal clock; that a is the difference of solar time at A between the signals at a and b, observed by a chronometer; and that α is this difference converted into sidereal time; that b is the interval of solar time observed at B between the signals b and c, and that b is equivalent to β in sidereal time, and lastly, that y is the sidereal time of Y of the last signal at c. It is evident, then, that the sidereal time of the second signal at X is $x + \alpha$, of the third signal, $x + \alpha + \beta$; and so on. In the present case, the local time at X of the signal at $c = x + \alpha + \beta$, and the local time at Y of the same signal $= y$; and therefore the

$$\text{difference of longitude of } X \text{ and } Y = (x + \alpha + \beta) - y.$$

114. In the "Philsophical Transactions" for 1826, there is an account* of a series of observations made for the purpose of connecting the observations of Greenwich and Paris. The rockets required for making the signals contained each 8 ounces of powder, and were expressly prepared at Paris for similar operations. The instantaneous explosion of these rockets at their greatest altitude constituted the signals to be observed.

The following example is taken from this paper, page 107:—

Ex.—On the 19th of July, 1825, a signal was made at Mont Javoult and noted at Paris to have happened at $18^h\ 39^m\ 52^s{\cdot}5$, true sidereal time at Paris, and at Lignieres at $10^h\ 49^m\ 41^s{\cdot}0$, by the Lignieres chronometer. About 5^m after this a signal made at La Canche was observed at Lignieres to happen at $10^h\ 54^m\ 53^s{\cdot}2$, and at Fairlight at $10^h\ 46^m\ 37^s{\cdot}5$, by the Fairlight chronometer. Finally, a third signal was made, about 5^m later still, at Wrotham, and observed at $10^h\ 51^m\ 59^s{\cdot}4$, by the Fairlight chronometer, and at $18^h\ 41^m\ 7^s{\cdot}11$, true sidereal time at Greenwich. Required the difference of longitude between Greenwich and Paris.

Paris.	Lignieres.	Fairlight.	Greenwich.
h m s 18 39 52·5..	h m s ..10 49 41·0 10 54 53·2..	 h m s ..10 46 37·5	
	$a = 5$ 12·2	10 51 59·4..	h m s ..18 41 7·11..
		$b = 5$ 21·9	

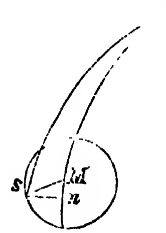
* By Sir J. Herschel. The difference of longitude between Greenwich and Paris, as determined from these observations, he conceives to be within a tenth of a second.

	h	m	s
$a =$	0	5	12·2
$b =$	0	5	21·9
$a + b =$	0	10	34·1
Reduction (Tab. 3) $=$			1·73
$\alpha + \beta =$	0	10	35·83
$x =$	18	39	52·5
$x + \alpha + \beta =$	18	50	28·33
$y =$	18	41	7·11
Difference of longitude $=$		9	21·22

115. PROP. XI.—*To find the longitude by means of an occultation of a fixed star by the moon.*

The moon, during her revolution in her orbit, frequently passes between the earth and some of the fixed stars, and thus intercepts their view from the spectator. This is called an *occultation*. As the instant of the immersion or emersion of a star can be observed with great accuracy and facility, the method of determining the longitude from an occultation may be practised with great advantage. The calculations are rather tedious, but the following rule will be attended with no difficulty, except its length; and we are disposed to think that it is as simple and direct as the nature of the case admits.

116. The principle of the solution may be briefly described as follows. At the instant of an immersion or emersion of the star, the apparent right ascension and declination of the point of the moon's limb in contact with the star are manifestly the same as the right ascension and declination of the star; and these are given in the Nautical Almanac, in a table containing the "*elements* for the *occultations*," &c. Correcting the apparent right ascension and declination of the point of the moon's limb for the effects of parallax (arts. 82, 83,) we get the right ascension and declination of the moon's limb as seen from the centre of the earth. From an approximate Greenwich date, find the right ascension and declination of the moon's centre, from the Nautical Almanac. If, then, in the annexed figure s be the true situation of the point of the moon's limb in contact with the star, and M the centre of the moon, we know the radius SM, as seen from the earth's centre, correctly from the Nautical Almanac, and sn, Mn, approximately from the assumed Greenwich date. And since the hypothenuse Ms is always less than 17 minutes, we may consider Mns as a plane triangle, without any sensible error. We have, then,



$$sn = \sqrt{Ms^2 - Mn^2} = \sqrt{(Ms + Mn)(Ms - Mn)}.$$

If this value of sn agrees with that derived from the approximate Greenwich date, then the Greenwich time has been rightly assumed; but if

not, assume a second date, one minute earlier or later than before. From these two results we may, by double position, find the Greenwich time of the occultation very nearly. Comparing this time with the time at the place, we find the longitude.

117. From the preceding considerations we readily obtain the following

Rule.

(1). Find, from the Nautical Almanac, under the head of Occultations, the star's right ascension and declination (d'); these will be the apparent right ascension and declination of the moon's limb. From the latitude of the place subtract the reduction (Math. Tables, Tab. 2), the remainder will be the *geocentric* latitude (l).

(2). Find Greenwich mean time to the nearest minute. To this date find the right ascension of the sun (page II., N. Alm.), to which add the apparent time at the place; the sum will be the right ascension of the meridian. The difference between this right ascension and that of the star will give the hour angle of the star in time. Convert this into degrees and minutes (h').

(3). With this date, also, find the moon's semidiameter in seconds (R), its horizontal parallax in seconds (P), which must be corrected for the latitude (Math. Tables, Tab. 9), and its right ascension and declination. Find, also, the right ascension and declination one minute afterwards.

(4). Find an approximate value of $\frac{1}{2}\alpha$ from the expression (art. 82)

$$\tfrac{1}{2}\alpha = P \cos l \sin h' \times \tfrac{1}{2} \sec d' = (A) \times \tfrac{1}{2} \sec d';$$

then the hour angle corrected $= h' - \frac{1}{2}\alpha = h + \frac{1}{2}\alpha$.

Find the correction in declination from the formula (art. 83)

$$\delta = -P \cos l \sin d' \cos (h + \tfrac{1}{2}\alpha) + P \sin l \cos d'.$$

The first part of this expression will have a name

$\left\{\begin{matrix}\text{different from}\\ \text{the same as}\end{matrix}\right\}$ the declination when the hour angle is $\left\{\begin{matrix}\text{less}\\ \text{greater}\end{matrix}\right\}$ than 90°.

The second part of the expression will have the same name as the latitude.

These two arcs being added to the apparent declination (d') of the moon's limb, when they have the same name, and subtracted from it when they have different names, the result will be its true declination (d).

(5). Find the correct value of α, in seconds of time, from the formula,

$$\alpha^s = \tfrac{1}{15} P \cos l \sin h' \sec d = (A) \times \tfrac{1}{15} \sec d.$$

Add this correction to the star's right ascension when it is west of the meridian, and subtract the correction when the star is east; the result will be the true right ascension of the moon's limb.

(6). Find the difference, in seconds (Δ), between the true declination of the moon's limb, and the declination of the moon's centre, for the first Greenwich date; then calculate the difference (a) between the right ascensions of the moon's limb and its centre, from the formula,

$$a = \tfrac{1}{15} \sec d \sqrt{(R + \Delta)(R - \Delta)}.$$

(7). Let b be the difference of these right ascensions, found from the 5th direction, and put $a - b = e$.

Calculate the values of a', b', and e', for the second Greenwich date, then we have, by double position,

$$e - e' : e :: 60 \text{ seconds of time} : \text{correction required.}$$

Add this correction to the first Greenwich date, and the correct Greenwich time of the occultation will be obtained very nearly.

Example.

Suppose, at Bedford, on January 7, 1836, in latitude 52° 8′ 28″ N., the immersion of ι Leonis to be observed at $10^h\ 39^m\ 22^s{\cdot}4$ P.M., apparent time, and the estimated longitude to be about $0^h\ 1^m$ W. Required the longitude.

Nautical Almanac, p. 455.

	h	m	s
R.A. of ✻	10	23	26·39
Declin. do.	14	58	38·8N.

	°	′	″	
Latitude..	52	8	28	N.
Reduction		10	57	
Geoc. lat.	51	57	31	N.

Greenwich Time.

	h	m	s
App. t...	10	39	22·4
Lon. W. +	0	1	
Gr. app. t.	10	40	22·4
Equa. t. +		6	30
Gr. mean t.	10	47	

R.A. of the Sun, p. II.

	h	m	s
At m. noon	19	10	41·7
Correction	+	1	58·0
	19	12	39·7
App. time..	10	39	22·4
R.A. M	5	52	2·1
R.A. ✻	10	23	26·4
Hour angle.	4	31	24·3
In degrees ..	67°	51′	

Moon's Semidiameter.

Noon......	15	12·4
Midnight...	15	16·6
		4·2
	h	m
Prop. parts to	10	47
		3·8
	15	12·4
	15	16·2
		916″·2

Horizontal Parallax.

	′	″
Noon....	55	48·2
Midnight..	56	3·5
		15·3
		m
Prop. parts to 10		47
		13·7
	55	48·2
	56	1·9
Reduc.		6·8
	55	55·1
		3355″·1

Moon's Right Ascension.

	h	m	s
At 10 ..	10	18	55·52
11 ..	10	20	58·47
		2	2·95
	h	m	
Prop. parts to	10	47	
		1	36·31
	10	18	55·52
$10^h\ 47^m$..	10	20	31·83
			2·05
10 48..	10	20	33·88

Moon's Declination.

h	°	′	″	
At 10...	15	58	50·1	N.
P. pts...		9	7·6	
$10^h\ 47^m$...	15	49	42·5	
			11·7	
10 48...	15	49	30·8	

log P......	3·52570	....	3·52570	....	3·52570
cos l.......	9·78975	....	9·78975	sin l.......	9·89629
sin h'......	9·96670	sin d'.....	9·41237	cos d'....	9·98499
(A).......	3·28215	cos $(h+\frac{1}{2}\alpha)$	9·58152		3·40698
sec d'......	·01501		2·30934		
const. log...	9·69897		203''·9		2552''·6N.
	2·99613				203·9 S.
991'' =	16 31				2348·7N.
	67 51				39 8·7N.
$(h+\frac{1}{2}\alpha)$..	67 34 29				14 58 38·8N.
					15 37 47·5N.

			h m s	h m
(A)......	3·28215		0 2 12·56	At 10 48
sec d.....	·01636		10 23 26·39	h m s
const.log..	8·82391	R.A. of ☽'s limb	10 21 13·83	10 21 13·83
	2·12242	Do. centre	10 20 31·83	10 20 33·88
	132''·56	Diff.........	42·00	39·95

	At 10h 47m		At 10h 48m	
	° ' ''		° ' ''	
decn. ☽'s cent	15 49 42·5	3·21240	15 49 30·9	3·20933
" " limb	15 37 47·9	2·30449	15 37 47·8	2·32858
	11 54·6	5·51689	11 43·1	5·53791
Δ......	714·6	2·75844	703·1	2·76895
R......	916·2	·01658	916·2	·01658
R + Δ	1630·8	8·82391	1619·3	8·82391
R − Δ	201·6	1·59893	213·1	1·60944
Difference of R.A.		39s·71		40s·69
Difference found above		42·00		39·95
Error		− 2·29	Error....	+ ·74

As 2·29 + ·74 : 2·29 :: 60s : 45s·3

		h m s
Hence, Greenwich mean time	=	10 47 45·3
Equation of time	=	− 6 31·0
Greenwich apparent time ..	=	10 41 14·3
Time at Bedford	=	10 39 22·4
Longitude	=	1 51·9 W.

118. Prop. XII.—*To find the longitude by means of moon culminating stars.*

If the moon had no motion in right ascension, the interval of time

between its transit and that of a fixed star would be the same at all places. The difference of the intervals arises entirely from the change of the moon's right ascension; and, if we suppose that this increases uniformly, the change in R.A. will be proportional to the absolute time elapsed between the transits of the moon over the two meridians, and therefore will be proportional to the difference of longitude. In the improved nautical almanacs, the time of transit of the moon's enlightened limb and that of certain stars differing from it but little in R.A. or declination, are *computed* for Greenwich mean time. This may be considered as equivalent to an actual observation at Greenwich; and if the difference of these transits be *observed*, by means of a portable transit instrument,* at any other place, its longitude from Greenwich may be easily and accurately determined. The following example will be sufficient to illustrate these remarks.

Ex.—Suppose that at *X*, on the 31st of March, 1841, the transit of β Geminorum was observed, by a chronometer, at $8^{h}\,30^{m}\,10^{s}{\cdot}3$, and of the moon's bright limb at $9^{h}\,16^{m}\,57^{s}{\cdot}5$; the daily rate of the chronometer being $3^{s}{\cdot}6$ gaining. Required the longitude of *X*.

Observed transits at X.

	h	m	s
Moon's limb......	9	16	57·5
Geminorum	8	30	10·3
		46	47·2
Correction for rate....		—	·12
Diff. of trans. in sol. time		46	47·08
sid. time		46	54·77

Transits at Greenwich (Naut. Alm., p. 489).

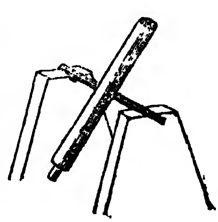
	h	m	s
	8	22	23·88
	7	35	36·91
Diff. of trans. at Gr......		46	46·97
Do. at *X*		46	54·77
Diff. of intervals			7·80

We have, then, from the Nautical Almanac, page 489,

$149^{s}{\cdot}27 \ : \ 7^{s}{\cdot}80 \ :: \ 1$ hour in long. : longitude of *X*;

and therefore longitude of $X = 3^{m}\,8^{s}{\cdot}1$ west.

* The transit instrument is a telescope, with an axis fixed at right angles to it, about which it revolves. The axis is placed in a horizontal position, due east and west; and consequently the telescope, when properly adjusted, always moves in the plane of the meridian. *Portable* transits are made for the purpose of observing the time of the moon's passage over the meridian; and as this is compared with the passage of some of the fixed stars, whose situations are well known, any slight error in the position of the instrument, or in the clock, are by this means avoided.



CHAPTER IV.—GEODESY.

119. When an extensive survey is made over large portions of the surface of the earth, either for the purpose of ascertaining the exact position of the principal places of a country, or of determining the dimensions and figure of the earth, it is usually designated by the terms *Geodesy*, or *Trigonometrical Surveying*.

120. For this purpose the country is first divided into a number of large triangles, whose sides are usually from 10 to 20 miles in length; but sometimes they extend to 50 or 60 miles, and even occasionally, as in Spain and the west of Scotland, to 100 miles in length. All the angles of the triangles are then carefully observed, and a line situated in a level tract of country, called a *base line*, is measured with extreme care and attention. These triangles may be said to form a species of polyedron, circumscribing a portion of the earth, and they are reduced to others on its surface at the level of the sea, by supposing perpendiculars to be drawn from each station to the surface. The latitudes and longitudes of the different stations are then determined; and also their heights, and the angles which the sides of the triangle make with the meridional line.

121. The great triangles, into which the country is divided in the first instance, are denominated *principal* triangles. They are afterwards, by a second series of operations, subdivided into smaller ones, called *secondary* triangles, and these again are broken up into others of still smaller dimensions, until at length a survey of the whole country is made of any degree of minuteness which may be thought necessary. The calculations are finally *verified*, by measuring other base lines, and comparing them with their lengths determined by calculation.

We shall divide this chapter into the following sections:—

1. The selection of stations and signals.
2. The measurement and reduction of a base line.
3. The measurement and reduction of the angles.
4. The calculation of the sides of the triangles.
5. The longitudes, latitudes, and azimuths.
6. The heights of the stations, and terrestrial refraction.
7. The measurement of arcs of the meridian, and arcs parallel to the equator.
8. The figure of the earth.

I.—THE SELECTION OF STATIONS AND SIGNALS.

122. In the choice of stations, in a trigonometrical survey, there are two points which ought principally to be attended to: 1st. The angles should have such a magnitude, that any small inevitable errors in the observations shall produce the least effect on the sides to be calculated. 2nd. The stations should all be distinctly visible from each other.

123. Prop. I.—*It is required to determine the most advantageous conditions of a triangle.*

Let a, b, c, be the sides of a triangle, and A, B, C, the angles respectively opposite to them. The angles are all known from observation, and the side a is either measured, or determined from previous calculation. The side b is then found from the equation

$$b \sin A = a \sin B \ldots\ldots\ldots\ldots(a).$$

Suppose, now, that the side a is accurately known, but that the angles A and B have not been correctly measured. Let α, β, be the respective errors in A and B, and let x be the corresponding error in the side b. We have, then,

$$(b + x) \sin (A + \alpha) = a \sin (B + \beta).$$

Expanding this expression (Trig. art. 71), and putting $\cos \alpha = 1$, $\sin \alpha = \alpha$, $\cos \beta = 1$, $\sin \beta = \beta$, since the errors α, β, are necessarily very small, we get

$$(b + x) (\sin A + \alpha \cos A) = a (\sin B + \beta \cos B).$$

Subtracting (a) from this equation, and omitting the term $x\, \alpha \cos A$, which is of the second order, and extremely small compared with the other terms, we get

$$x \sin A + b\, \alpha \cos A = a\, \beta \cos B = \frac{b \sin A}{\sin B}\, \beta \cos B;$$

$$x = b\, (\beta \cot B - \alpha \cot A). \qquad .(1)$$

Hence, if we suppose the errors α and β to be equal, and to have the same sign, the error x will be 0, when $A = B$; that is, there will be no error in calculating the side b, although the angles A and B are not correctly observed. If the errors α and β are equal, but have different signs, this equation becomes

$$x = b\, \alpha\, (\cot A + \cot B).$$

Now, $\quad \cot A + \cot B = \frac{A}{\ } + \frac{\cos B}{B} = \frac{(\sin A + B)}{\sin A \sin B}.$

Also, $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$
$= \cos (A - B) + \cos C$, and $\sin (A + B) = \sin C$, therefore

$$x = b\, \alpha \frac{2 \sin C}{\cos (A - B) + \cos C} \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

an expression which is evidently the least possible when $A = B$.

124. Since the same reasoning is applicable to the third side c, it follows that the most advantageous conditions of a triangle are *that its sides should be as nearly equal as possible.* But, as it is frequently impossible to fulfil these conditions, surveyors are in general satisfied with rejecting all triangles which have an angle less than 30 degrees.

125. If the angles are accurately known, but there is an error in the side a, it is evident that the errors in the sides b and c will be proportional to their lengths; for the angles being constant, the triangles will be similar. Hence, it is of the utmost importance to measure the base line correctly, for any error in this line, which is necessarily very short compared with the extent of the country to be surveyed, will be

continued through the whole chain of triangles, and magnified in proportion to the length of the sides.

126. *Description of signals.*—All the stations should be situated in the most elevated part of the country, so as to be seen from each other without difficulty. In many cases the theodolite was elevated to the top of some tower, church-steeple, or other building, and flagstaffs placed over the instruments. These can be more easily distinguished when their figures are seen in the sky, than when they are projected on the earth or on trees. For more distant stations, *Bengal* or *white lights* were at first used by General Roy. Afterwards, the reflection of the sun from a plane mirror, as recommended by Gauss, was employed by Colonel Colby and Captain Kater, in verifying that part of General Roy's triangulation which connected the meridians of Greenwich and Paris. *Drummond lights* were used as night signals at some of the stations in Ireland and the west of Scotland; but the practice of observing by night has lately been abandoned, in consequence of the unsteadiness of the light, and the quantity of vapour in the atmosphere. At present, the signals in the English survey are formed by building a temporary shed in the form of the frustum of a cone, over the point which marks the centre of the station. When the distances are not very great, a plate of metal is sometimes used, with a narrow vertical slit cut in it; in which case the line of light passing through it may be seen very distinctly.

127. In elevating a signal for the purposes of observation, it is necessary that it should be sufficiently high to be easily distinguished from the surrounding objects. From the experience of the French surveyors, they state that the angle of elevation should be at least $31''$; and as $\tan 31'' = 0{\cdot}00015$, it follows that the height of the signal must be equal to $0{\cdot}00015 \times$ distance. In practice, therefore, the French usually made the height of the signal equal to a seven thousandth part of the distance from whence it was to be observed, and the base of the signal equal to half its height. Hence, if the distance be 20 miles, a distance not unusual in the trigonometrical survey, the signal should be at least 15 feet in height.

II.—THE MEASUREMENT AND REDUCTION OF A BASE LINE.

128. Of all the operations in which the surveyor is engaged, that which appears the most simple, but which is by far the most difficult, is the measurement of a base line. Since this line is seldom more than 6 or 7 miles in length, and any error in its measurement is multiplied in the other parts of the survey in proportion to their linear dimensions, it is obvious that, in a tract of country 300 miles long, the error in this length would be 50 times the error in the base line. Every precaution, therefore, has been taken, which art or ingenuity could devise, to ensure the greatest accuracy in this most important operation.

129. The first thing to be done is to select a level piece of ground, from five to seven or eight miles in length, which shall be free from local obstructions, and commodiously situated with respect to surrounding objects. It is also desirable that it should not be far distant from an observatory, so that the whole chain of triangles may be connected with a fixed station, where astronomical observations are made with the utmost care and precision.

After the ground has been selected, a line is drawn in the same vertical plane, by means of a transit telescope, and marked out by pickets, the tops of which are brought exactly into the same level. The tract which is to be measured is then cleared of all obstructions, and made tolerably smooth; and the extremities of the base are permanently fixed by dots marked on cannon, or on massive blocks of stone.

130. *Deal rods.* In the commencement of the English survey, General Roy made use of deal rods, 20 feet three inches long, about 2 inches deep, and $1\frac{1}{4}$ inch broad, on which lengths of twenty feet were laid off by Ramsden. They were constructed in such a manner that they might be used either by butting the end of one rod against the end of another, or by bringing fine transverse lines, inlaid into the upper surface at the distance of $1\frac{1}{2}$ inch from each extremity, into exact coincidence; but the method of coincidences was attended with so much inconvenience and loss of time, that General Roy was compelled to abandon it, and to proceed solely by the method of contacts. Notwithstanding all the care, however, that was taken to select rods of the best materials, they were found liable to such irregular and sudden variations of length, from the moisture of the atmosphere, that they were entirely abandoned, after the first base on Hounslow Heath had been completed. The error in this measurement was found to be about 21 inches.

131. *Glass rods.* When it was discovered that the deal rods would not prove satisfactory, it was proposed that glass rods should be substituted in their place. Tubes were used, rather than solid rods, as it was found that a sufficient quantity of melted glass could not be taken on the irons which were used at the glass-house for drawing the rods. Three hollow tubes were, therefore, selected, and converted by Ramsden into measuring rods. They were then placed in cases, to which they were made fast in the middle, and also braced at two other points; the whole together serving as stays to keep the tubes in their true places from shaking, but not binding them too closely. The ends were ground perfectly smooth, and at right angles to the axis of the bore; one end having a fixed apparatus, or metal button, attached to it, for making the contacts, and the other end a moveable apparatus, or slider, which was pressed outwards by a slender spring. The fixed extremity of the succeeding rod was pushed against this spring until a fine line on the slider was brought into exact coincidence with another fine line on the glass rod, in which state the distance between the extremities was exactly twenty feet.

132. *Steel chains.* The third method of measuring a base line, by the English surveyors, was with a steel chain, made by Ramsden. This chain was 100 feet in length, and contained 40 links of $2\frac{1}{2}$ feet each. A transverse section of these links was a square, each of whose sides was half an inch. In using the chain, five coffers were arranged in a straight line, and supported either by trestles or courses of bricks; the chain was then placed on the coffers, and stretched with a constant weight of 56 lbs. The ends were then brought over the same point in this manner: at the extremity of the chain, but unconnected with it, and on a separate post, was placed a scale. When the chain was in any position, the scale at the preceding end was moved by means of screws, until one of its divisions coincided exactly with the mark on the handle of the chain. This scale remaining in its place, the chain was

then carried forward into its next position, and adjusted, by means of its screw apparatus, until the mark in its following end coincided exactly with that division of the scale which had been in coincidence with the mark on the preceding end.

	Feet.
The measurement of the base on Hounslow Heath, made with deal rods, reduced to the lowest extremity, was found to be	27406·26
Do. with glass tubes	27404·0843
Do. with steel chain	27404·3155

The mean of the two last results, or 27404·2 feet, was assumed as the true length of the base in the future calculations.

133. Notwithstanding the near agreement of the two last methods of measuring a base line, it has been objected to the glass rods: 1st. That some error might arise from the ends of the two consecutive rods being made to rest on the same trestle, because, when the first rod was taken off, the face of the trestle being pressed by one rod only, would have a tendency to incline a little forward, the effects of which would be to shorten the apparent length of the base. 2nd. That some error might arise from the casual deviation of the rods from a straight line in the direction of the base. 3rd. That, from the manner of supporting the rods on two trestles only, they would be liable to bend in the middle. To the steel chain it has also been objected, by Legendre and others, that, as the chain is not uniformly supported at every point, some doubt must remain whether it is perfectly straight when placed in the coffers, and also that its length is liable to vary from the rubbing and wear of the joints.

134. *Rods of platinum and brass.* In the French surveys rods of platinum were used. These were two toises, or 12 French feet, in length; their breadth was about six lines (or half a French inch), and their thickness one line. On the surface of the platinum was placed another rod of brass, firmly fixed at one end to the rod of platinum, by means of three screws, but entirely free at the other end, and throughout its whole length. It was about six inches shorter than the rod of platinum; and, from the different expansive powers of the two metals, the two rods united might be considered as a kind of metallic thermometer. Four rods were used in the measurement, three of which were always on the ground at the same time; and, in order to prevent any derangement from bringing the ends into contact, a small interval of about $\frac{1}{4}$ of an inch was left between them, which was measured by means of a slider attached to the preceding end of each rod. The slider was then pushed gently out, until it came into contact with the following end of the next rod.

135. *Colonel Colby's method.* The last method adopted, in the survey of Ireland, is an ingenious apparatus made by Troughton, which will probably supersede all other instruments. *AB* is a bar of iron, 10 feet long, $1\frac{1}{2}$ inch deep, and $\frac{5}{8}$ of an inch broad, united to a bar of brass *DE*, of the same dimensions, at the distance of inches. These bars are

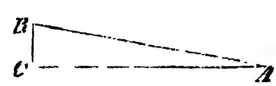
firmly riveted together at their centres, but are free to move at the extremities, according to their respective expansions. The base DE is covered with a nonconducting substance, to make the two bars equally susceptible of heat. PD, QE, are two tongues of steel, attached to the rods by double conical joints, around which they are capable of turning and forming a small angle with the lines perpendicular to the bars. P and Q are two dots of platinum, so exceedingly minute as to be almost invisible to the naked eye. At the temperature of 60° the bars are exactly of the same length, and the tongues PD, QE, are then perpendicular to the bars; but, if the temperature be increased, the bars will expand in different proportions: thus, if Pad, Qbc, represent the position of the tongues at the temperature of 70°, and the expansion of iron be to that of brass as 53 to 83, then

$$Aa \qquad Dd \qquad PA \qquad PD \qquad 53 \qquad 83.$$

Hence, the situation of the point P, about which the tongue PE revolves, is invariable, or at least is sensibly so in practice, for all moderate variations of temperature. The same thing is true with respect to the point Q, and consequently the distance PQ remains, in all moderate changes of temperature, of the exact length of ten feet. It is evident, however, that this can only be true within certain limits; for, as Pd is no longer equal to PD, the point P will have moved to p, nearer to d, making $pd = PD$; and the distance of p from PD is evidently equal to $PD \times (\tan DPd - \sin DPd)$. But as the angle DPd is, in practice, always extremely small, the difference between its tangent and its sine is altogether insensible.

136. In the Irish survey, five or six sets of bars, constructed in this manner, and placed in strong deal boxes, supported on trestles, were laid along the line to be measured, and accurately levelled. They were placed at a short distance from each other, and the distance between the dots on the adjacent steel tongues of two succeeding bars was accurately measured, by means of powerful micrometers, constructed so as to form a compensating instrument of the same nature as the measuring bars. It is stated that the greatest possible error of the base, measured on the eastern shore of Lough Foyle, cannot exceed two inches, though the length is very nearly eight miles. See Encyclopedia Britannica; Art. *Figure of the Earth.*

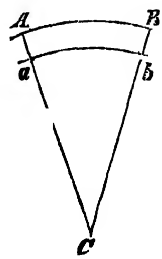
137. *The reduction of the hypothenuses.* As the ground on which the base is measured is seldom perfectly level, the whole distance is divided into a number of inclined lines in the same vertical plane. Let AB be one of those lines, whose length is l, $BC = h$, the height of this plane, and the inclination of the plane $BAC = \theta$. In the first English surveys, BC, the height of B above A was found from levelling, and therefore the base $AC = \sqrt{(l^2 - h^2)}$. But in the latter surveys, as well as in those on the continent, the angle θ was measured, and therefore the correction $AB - AC$ is equal to $l(1 - \cos\theta)$.



138. *Correction of temperature.* In the English survey, the temperature of the rods and chain was found from the mean of a number of thermometers; and the rate of expansion was previously determined by Ramsden. In the French survey, the measuring rod itself is the ther-

mometer, and the difference of the rates of expansion between the platinum and the brass is carefully ascertained before the survey commences. In either case the correction is easily found by a single proportion, or by means of tables constructed for the purpose.

139. *Reduction to the level of the sea.* Let AB be the arc which has been measured, as described above, and corrected on account of temperature and the inclinations of the hypothenuses. This arc may be supposed to be taken at a mean between the heights of the two extreme points. Let ab be a concentric arc at the level of the sea, and Ca the radius of the earth. Put $Ca = r$, $Aa = h$, $AB = L$, $ab = l$, we have, then,



$$CA : Ca :: AB : ab;$$

$$\therefore\ l = L\frac{r}{r+h} = \left(1 - \frac{h}{r} + \frac{h^2}{r^2} - \&c.\right) = L - \frac{Lh}{r} \ \ldots\ldots\ldots\ldots\ (3)$$

nearly. In order, therefore, to reduce the base to the level of the sea, we must subtract the correction $\frac{Lh}{r}$ from the length.

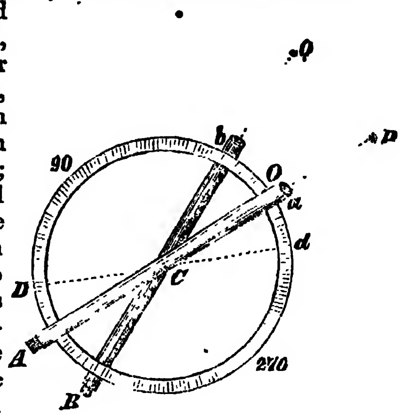
140. We will now give, as an example, the final result of the measurement of the first base, with glass rods, on Hounslow Heath. (Trig. Survey, vol. i. p. 87.)

	Feet.
Hypothenusal length of the base as measured by 1369·925521 glass rods, of 20 feet each, + 4·31 ft.	27402·8204
Reduction of the hypothenuses, to be subtracted	− 0·0714
Add the difference between the expansion of the glass above, and the contraction of it below, 62°...........	+ 0·3489
Add also for 6° difference of temperature of the standard brass scale and the glass rods	+ 0·9864
Length of the base, in temperature 62°	27404·0843
Reduction from the height of the lower end of the base above the mean level of the sea, supposed to be 54 feet..	− 0·0706
True length of the base, reduced to the mean level of the sea...	27404·0137

III. THE MEASUREMENT AND REDUCTION OF THE ANGLES.

141. In all the surveys made in the British dominions, the instrument for measuring angles has been a large theodolite, rendered as perfect as the ingenuity of English artists could make it. This instrument may be defined to be an altitude and azimuth instrument, or an instrument for measuring vertical and horizontal angles. The horizontal circle was three feet in diameter, and angles could be measured with it to the fractional part of a second.

142. The instrument used by the French and Swedish surveyors was the repeating circle of Borda. The principle of the circle of repetition is to take the angle several times successively in continuation on the circle, and then divide the whole arc by the number of observations.



Let ABD be a circle, graduated entirely round the circumference, from right to left, on the upper side only of the instrument. Aa, Bb, are two telescopes, the one on the upper, and the other on the under side of the instrument; these telescopes can either be moved independently, or they may be clamped and moved altogether with the circle. Let P and Q be two objects whose angular distance is to be measured; and let the instrument, by means of a stand, be brought into the plane PCQ. Place the upper telescope Aa at zero, and direct it to the object P; also direct the under telescope Bb to the object Q. The two telescopes are then clamped, and the entire instrument is turned in its plane, until Bb be pointed to P. Aa will now be in the position Dd, making the angle aCd equal to aCb; unclamp it and turn it back to Q, while the circle itself remains fixed; it is evident that Aa has moved through an angle dCb, equal to twice the required angle PCQ. The whole circle must now be turned again until Aa points to P, then will Bb be in the position Dd; turn Bb again through the angle dCb to Q, and clamp it. As the under side of the circle is not graduated, the angular distance of b from zero cannot be measured. Now move the whole circle until Bb points to P, and turn Aa again until it points to Q; the telescope Aa will have been turned through four times the arc ab; and, by repeating the process, the arc can be multiplied any even number of times. It will readily be seen that the circle must always be turned to the *right* through the arc ba, and the two telescopes alternately to the *left* through db, or twice the arc ab.

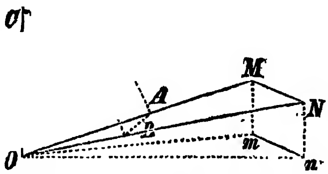
The advantages of this method are obvious. The errors of graduation may be diminished to any degree, and the errors of observation tend to destroy each other. The two circles which Delambre used were 0·21 and 0·18 metres, or about 7 inches in diameter; and, although the instruments were only graduated to minutes, yet, by successively repeating the angle ten, twelve, or even as far as twenty times, he imagined that he could determine the angle within a second.

143. Various opinions have been entertained with respect to the relative merits of the theodolite and the repeating circle. The French have imagined that they could attain any degree of accuracy with the circle, and that all errors of division, and errors of observation, might be entirely annihilated by repetition. Latter observations, however, have tended to destroy this confidence in the circle, and it is now generally admitted, that all repeating circles are liable to an error, which cannot be removed by any number of repetitions, and which, therefore, has been called the *constant error*. One of its greatest defects, perhaps, is

the smallness of the telescope attached to it, which does not exceed 24 inches in length, and which is incapable of separating the double star ζ Ursæ Majoris, although the two stars which compose it are distant from each other 14″ or 15″.—"However an instrument may be constructed, or in whatever manner it may be used, I have no faith that it can give results nearer the truth than a quantity that is visible in the telescope."* The principal advantage of the circle is its great portability, whereas the theodolite used by General Roy was 200 lbs. in weight, and required two, and sometimes four horses to convey it, with its apparatus, from one station to another.

144. When the angles are measured with a theodolite, no correction is required on account of the different altitudes of the signals, as it is the horizontal angle which is observed with the instrument; but when the sextant or repeating circle is employed, the oblique angles are observed, and these must be reduced to the plane of the horizon.

145. PROP. II.—*To reduce the oblique angles to the plane of the horizon.*



Let O be the place of the observer, MON the angle observed between two signals, M, N; Mm, Nn, two vertical lines meeting the horizontal plane mOn in the points m, n. Let OZ be a vertical line passing through O, and with the centre O and radius 1, conceive a sphere to be described, and let the planes ZOM, ZON, MON, cut this sphere in the great circles CA, CB, AB. The angle observed with the repeating circle is the oblique angle MON, which is measured by the arc AB, but the required angle is mOn, which is equal to the spherical angle C (Trig. art. 123). The angles MOm, NOn, are known from observation, and therefore the complements of these angles, or the arcs CA, CB, are known. We have, then, in the triangle CAB, the three sides CA, CB, AB, given to find the angle C.

Let h, h', be the altitudes of the two signals M and N, θ the angle between them; also, let the horizontal angle $= \theta + x$; then $CA = 90° - h$, $CB = 90° - h'$, $AB = \theta$, angle $C = \theta + x$. By Trig. (art. 140),

$$\cos C = \frac{\cos AB - \cos CA \cos CB}{\sin CA \sin CB}, \text{ or, } \cos(\theta + x) = \frac{\cos\theta - \sin h \sin h'}{\cos h \cos h'}$$

Now, in practice, h, h', are always very small, and $\theta + x$ is nearly equal to θ, therefore x also is very small. Hence

$$\cos(\theta + x) = \cos\theta\cos x - \sin\theta\sin x = \cos\theta - x\sin\theta, \text{ nearly.}$$

Also, $\cos h \cos h' = (1 - \frac{1}{2}h^2 + \&c.)(1 - \frac{1}{2}h'^2 + \&c.) = 1 - \frac{1}{2}(h^2 + h'^2) + \&c.$

$$\therefore \frac{1}{\cos h \cos h'} = \frac{1}{1 - \frac{1}{2}(h^2 + h'^2) + \&c.} = 1 + \tfrac{1}{2}(h^2 + h'^2), \text{ nearly,}$$

and $\sin h \sin h' = hh'$, nearly. Substituting these values above, we have

$$\cos\theta - x\sin\theta = (\cos\theta - hh')\left\{1 + \tfrac{1}{2}(h^2 + h'^2)\right\}.$$

* Troughton, 1st volume of the Memoirs of the Astronomical Society.

Hence $x \sin\theta = hh' - \frac{1}{2}(h^2 + h'^2)\cos\theta$

$$= \frac{(h+h')^2 - (h-h')^2}{4} - \frac{(h+h')^2 + (h-h')^2}{4}\cos\theta,$$

$$\therefore\ x = \frac{(h+h')^2}{4}\,\frac{1-\cos\theta}{\sin\theta} - \frac{(h-h')^2}{4}\,\frac{1+\cos\theta}{\sin\theta}$$

$$= \tfrac{1}{4}(h+h')^2\tan\tfrac{1}{2}\theta - \tfrac{1}{4}(h-h')^2\cot\tfrac{1}{2}\theta\,.$$

Here x is measured in parts of the radius; if it be measured in seconds, we have $x = x''\sin 1''$; therefore

$$x'' = \frac{(h+h')^2}{4}\,\frac{\tan\frac{1}{2}\theta}{\sin 1''} - \frac{(h-h')^2}{4}\,\frac{\cot\frac{1}{2}\theta}{\sin 1''}\ \ldots\ldots\ldots\ldots\ldots(4)$$

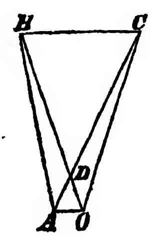
Ex. Let $\theta = 51°\ 9'\ 29''{\cdot}774$, $h = 1°\ 32'\ 45''$, $h' = 1°\ 7'\ 10''$, then $\frac{1}{2}(h+h') = 4797''{\cdot}5$, $\frac{1}{2}(h-h') = 767''{\cdot}5$.

$2\log\frac{1}{2}(h+h')$	7·362030	$2\log\frac{1}{2}(h-h')$	5·770156
$\tan\frac{1}{2}\theta$	9·680038	$\cot\frac{1}{2}\theta$	0·319962
ar. co. log sin 1″	4·685575	ar. co. log sin 1″	4·685575
53″·413	1·727643	5″·966	0·775693

	°	′	″
Observed angle................	51	9	29·744
		+	53·413
		—	5·966
Angle reduced to the horizon......	51	10	17·191

146. It sometimes happens, when the steeple of a church, or other remarkable object, is selected as a signal, that the theodolite cannot be placed immediately over the point occupied by the axis of the signal. In this case, the instrument must be removed to some convenient place near it, and a reduction is then applied to the observed angle, in order to obtain the true angle at the centre.

147. Prop. III.—*It is required to determine the "reduction to the centre."*



Let A be the situation of the axis of the signal observed from the stations B and C, O the place of the centre of the instrument. Put A, B, C, for the angles of the triangle ABC, and a, b, c, for the sides, respectively, opposite to them. Let $AO = m$, angle $AOB = \beta$, angle $AOC = \gamma$, angle $BOC = O$; also, angle $ABO = x$, $ACO = y$. Now,

angle $A = BDC - x = O - x + y$; also,

$\sin x : \sin\beta :: m : c$; $\sin y : \sin\gamma :: m : b$;

$$\therefore\quad \sin x = \frac{m\sin\beta}{c},\quad \sin y = \frac{m\sin\gamma}{b}.$$

But since O is always near the station A, the angles x and y are very small, and therefore

$\sin x = x = x'' \sin 1''$, $\sin y = y = y'' \sin 1''$, very nearly. Hence

$$A = O - \frac{m \sin \beta}{c \sin 1''} + \frac{m \sin \gamma}{b \sin 1''} \text{.................} \quad (5)$$

148. When all the angles have been observed and reduced to the plane of the horizon, if the triangle were a plane one, their sum ought to be equal to 180°, and thus the correctness of the observations might be verified. But in a spherical triangle, the sum of the three angles exceeds 180°, by a certain quantity, called the *spherical excess;* and as this can easily be calculated from the 15th theorem (Trig. art. 136), we have the same means of verifying the operation in Spherical as in Plane Trigonometry.

149. PROP. IV.—*It is required to determine the spherical excess in a small triangle measured on the surface of the earth.*

Let A, B, C, be the three angles of a spherical triangle, r the radius of the sphere expressed in feet, x the area of the triangle in square feet, and ε the spherical excess given in seconds; we have then, by theorem 15,

$$x : \pi r^2 : : A + B + C - 180° \; (= \varepsilon'') : 180 \times 60 \times 60 \text{ seconds};$$

$$\therefore \quad \varepsilon = \frac{x \times 648000''}{\pi r^2};$$

and, if we suppose the mean value of r to be 20,888761 feet, the logarithm of $\dfrac{\pi r^2}{648000}$ is equal to 9·32540. The value of x may be calculated as if the triangle were a plane one, without any sensible error. Hence we have the following

Rule.—From the logarithm of the area of the triangle, taken as a plane one in feet, subtract the constant logarithm 9·32540, the remainder will be the logarithm of the spherical excess in seconds, nearly.

150. When the triangles are very large, a more correct value of r will be obtained by computing for the mean latitude of the three stations, the radius of curvature of the meridian (art. 185), and of the arc perpendicular to the meridian (art. 195), and taking the mean of the two for the value of r.

151. The following example is taken from the Encyclopædia Britannica. The triangle connects the west of Scotland with Ireland, and is one of the largest which occurs in the Trigonometrical Survey.

The three stations are Benlomond, in Stirlingshire (A), Cairnsmuir-on-Deugh, in Kirkcudbright (B), and Knocklayd, in the county of Antrim (C); the arc c is 352037·62 feet, and the angles are as follows:

	A	B	C
	° ′ ″	° ′ ″	° ′ ″
	56 43 29·97	79 42 28·69	43 34 38·36
	27·04		35·43
	28·72		
Mean...	56 43 28·58		43 34 36·89

We shall first compute approximate values of the two sides, a, b, (which will be afterwards required), from the formulæ $a = \frac{c \sin A}{\sin C}$, $b = \frac{c \sin B}{\sin C}$; and then compute the area from the formula, area $= \frac{1}{2}bc \sin A$.

$\log c = 5{\cdot}54659$	$\log c = 5{\cdot}54659$	$\log c = 5{\cdot}54659$
$\log \sin A = 9{\cdot}92223$	$\log \sin B = 9{\cdot}99295$	$\log b = 5{\cdot}70112$
$\log \text{cosec.} C = 0{\cdot}16158$	$\log \text{cosec.} C = 0{\cdot}16158$	$\log \sin A = 9{\cdot}92223$
		ar.co.log $2 = 9{\cdot}69897$
$\log a = 5{\cdot}63040$	$\log b = 5{\cdot}70112$	
$a = 426970$	$b = 502480$	$\log \text{area} = 10{\cdot}86891$

The latitude of Benlomond (the most northern station) is 56° 11′; and that of Knocklayd (the most southern) is 55° 10′; the mean of the two is 55° 40′. The values of the radii of curvature are therefore (art. 185)

$r = 20{,}924824$ feet, $r' = 20{,}968900$ feet, mean $= 20{,}946862$ feet.

$$\log \frac{180 \times 60 \times 60}{\pi} = 5{\cdot}31443$$

$$\log r^2 = 14{\cdot}64224$$

$$9{\cdot}32781$$

$$\log \text{area} = 10{\cdot}86891$$

$$\epsilon = 34''{\cdot}76 \ldots\ldots \quad 1{\cdot}54110$$

152. The sum of the three angles of the triangle being found from observation $= 180° \; 0' \; 34''{\cdot}16$, and the true spherical excess being $34''{\cdot}76$, it appears that the errors of observation in the three angles are $= -0''{\cdot}60$. If there were no reason to suppose that one angle has been determined more accurately than another, the error should be equally divided among the three angles; but, as it generally happens that some of the angles have been determined from a greater number of observations, or from observations made under more favourable circumstances than the others, this error should be distributed among the three angles in such a manner that the respective corrections may be inversely proportional to the relative goodness of the observations. For this purpose we have the following rule, given by Gauss, but which our limits will not permit us to demonstrate in this work.

153. PROP. V.—*To apportion the error among the different angles.*

Rule. Let l, l', l'', &c. be the seconds of reading in any angle A, n the number of observations, and let m be the mean or average of the whole; then $m - l$, $m - l'$, $m - l''$, &c., are the errors of the individual observations, and the *weight* of the determination, or of the average m, will be given from this equation,

$$p = \frac{\frac{1}{2}n^2}{(m - l)^2 + (m - l')^2 + (m - l'')^2 + \&c.} \quad (6)$$

In like manner; the weights y and z are found for the angles B and C. This error in the sum of the three angles is then divided into three parts, proportional to $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, which are to be added respectively to the three angles A, B, and C.

To apply this to the last example, we have for the angle A, $l = 29''\cdot97$, $l' = 27''\cdot04$, $l'' = 28''\cdot72$; therefore $n = 3$, $m = \frac{1}{3}(l + l' + l'') = 28''\cdot58$. Hence

$$\frac{1}{x} = \frac{(1\cdot39)^2 + (1\cdot54)^2 + (0\cdot14)^2}{\frac{1}{2} \times 9} = \cdot961.$$

The angle B was given from one observation only. We may, therefore, assume the weight $y = \cdot1$, and $\frac{1}{y} = 10$.

At C the reciprocal of the weight $\frac{1}{z} = \frac{(1\cdot47)^2 + (1\cdot46)^2}{\frac{1}{2} \times 4} = 2\cdot146$.

Hence the error $- 0''\cdot60$ is to be divided into three parts proportional to the numbers $\cdot961$, 10, $2\cdot416$; and, consequently, the corrections of the angles are, respectively, $+ 0''\cdot04$, $+ 0''\cdot46$, and $+ 0''\cdot10$. The true spherical angles, therefore, are

$$A = 56^\circ\ 53'\ 28''\cdot62;\ B = 79^\circ\ 42'\ 29''\cdot15;\ C = 43^\circ\ 44'\ 36''\cdot99.$$

IV.—THE CALCULATION OF THE SIDES OF THE TRIANGLES.

154. The three spherical angles of the triangle being thus determined from observation, and corrected, and one of the sides being always known, either from actual measurement or calculation, it is necessary to show how the two other sides may be determined. The triangle may be considered as a spherical triangle, whose sides are very small, compared with the radius of the sphere; in which case three different methods have been employed for its solution: 1st. From the three given spherical angles, the angles formed by the chords are deduced, and from the given side of the triangle, the corresponding chord is calculated. With these data the other chords are found by Plane Trigonometry, and from thence the arcs themselves. 2nd. A second method is by the theorem of Legendre, by which the spherical triangle is reduced to a plane triangle, whose sides are respectively equal in length to the sides of the triangle of the sphere. 3rd. The third method is to compute the sides by Spherical Trigonometry.

FIRST METHOD.

155. Prop. VI.—*To reduce the angle of a spherical triangle to the angle formed by the chords of the containing sides.*

Let a, b, c, be the sides of the spherical triangle, and r the radius of the sphere, all measured in feet; also, let $\frac{a}{r} = \alpha$, $\frac{b}{r} = \beta$, $\frac{c}{r} = \gamma$, then will α, β, γ, be the sides of a similar triangle, on a sphere whose radius is 1. Let A be the spherical angle opposite to the side a, and let $A - x$

be the corresponding angle formed by the chords. We have then (Trig. art. 140),

$$\cos A = \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma}$$
$$= \frac{(1 - 2\sin^2 \frac{1}{2}\alpha) - (1 - 2\sin^2 \frac{1}{2}\beta)(1 - 2\sin^2 \frac{1}{2}\gamma)}{2 \sin \frac{1}{2}\beta \cos \frac{1}{2}\beta \times 2 \sin \frac{1}{2}\gamma \cos \frac{1}{2}\gamma}$$
$$= \frac{\sin^2 \frac{1}{2}\beta + \sin^2 \frac{1}{2}\gamma - \sin^2 \frac{1}{2}\alpha}{2 \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma \times \cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma} - \frac{\sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma}{\cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma}.$$

Also, because chord $\alpha = 2 \sin \frac{1}{2}\alpha$, chord $\beta =$ &c., we have, in the triangle formed by the chords,

$$\cos(A - x) = \frac{\text{chord}^2 \beta + \text{chord}^2 \gamma - \text{chord}^2 \alpha}{2 \text{ chord } \beta \text{ chord } \gamma} = \frac{\sin^2 \frac{1}{2}\beta + \sin^2 \frac{1}{2}\gamma - \sin^2 \frac{1}{2}\alpha}{2 \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma};$$

substituting this in the preceding equation, we get

$$\cos A = \frac{\cos(A - x)}{\cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma} - \frac{\sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma}{\cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma};$$

$$\therefore \cos(A - x) = \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma + \cos \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma \cos A \;\ldots\ldots\; (7)$$

This expression is exact. But, because the three arcs, α, β, γ, are very small, $A - x$ is nearly equal to A, and therefore x is also very small. Hence

$$\cos(A - x) = \cos A \cos x + \sin x \sin A = \cos A + x \sin A, \text{ nearly}.$$

Also, $\sin \frac{1}{2}\beta = \frac{1}{2}\beta$, $\cos \frac{1}{2}\beta = 1 - \frac{1}{8}\beta^2$, $\sin \frac{1}{2}\gamma =$ &c., very nearly. Hence, substituting these values in equation (7), and reducing, we obtain

$$x \sin A = \tfrac{1}{4}\beta\gamma - \tfrac{1}{8}(\beta^2 + \gamma^2) \cos A$$
$$= \frac{(\beta + \gamma)^2 - (\beta - \gamma)^2}{16} - \frac{(\beta + \gamma)^2 + (\beta - \gamma)^2}{16} \cos A;$$
$$\therefore\; x = \frac{(\beta + \gamma)^2}{16} \frac{1 - \cos A}{\sin A} - \frac{(\beta - \gamma)^2}{16} \frac{1 + \cos A}{\sin A}$$
$$= \left(\frac{b + c}{4r}\right)^2 \tan \tfrac{1}{2}A - \left(\frac{b - c}{4r}\right)^2 \cot \tfrac{1}{2}A;$$

or, if x be estimated in seconds,

$$x'' = \left(\frac{b + c}{4r}\right)^2 \frac{\tan \frac{1}{2}A}{\sin 1''} - \left(\frac{b - c}{4r}\right)^2 \frac{\cot \frac{1}{2}A}{\sin 1''} \;\ldots\; (8)$$

156. Having obtained the three reduced angles, we find the chords of the spherical arcs intercepted between the stations, from Plane Trigonometry, and from them we deduce the arcs themselves, by means of the following formula, p. 331, Ex. 5.

$$\frac{\alpha}{2} = \sin \tfrac{1}{2}\alpha + \frac{1}{2} \frac{(\sin \frac{1}{2}\alpha)^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{(\sin \frac{1}{2}\alpha)^5}{5} + \text{\&c.},$$

and, because chord $\alpha = 2 \sin \frac{1}{2}\alpha$, and α is very small, if we neglect the terms after the second, and multiply by 2, we get

$$\alpha = \text{chord } \alpha + \frac{(\text{chord } \alpha)^3}{24}; \text{ hence } \frac{a}{r} = \frac{\text{chord } a}{r} + \frac{(\text{chord } a)^3}{24r^3};$$

$$\therefore\; a = \text{chord } a + \frac{(\text{chord } a)^3}{24r^2} \;\ldots\ldots\ldots\ldots\; (9)$$

Example.

157. As an example of this method of solution, we will take that given in art. 151.

$\log r^2$	14·64224	(1)	9·46807
$16 \sin 1''$	5·88969	$(b-c)^2$	10·35468
	0·53193	$\cot \frac{1}{2}A$	0·26765
(1)	9·46807	$1''·231$	0·09040
$(b+c)^2$	11·86342		+ 11·583
$\tan A$	9·73235		− 1.231
$11''·583$	1·06384		$x =$ 10·352

In the same manner, the corrections for the angles B and C will be found to be $14''·684$ and $9''·724$, respectively. Hence the three angles formed by the chords are

$$A' = 56^\circ\ 43'\ 18''·27,\quad B' = 79^\circ\ 42'\ 14''·47,\quad C' = 43^\circ\ 34'\ 27''·26,$$

and the sum of these $= 180^\circ$, as it should be.

The chord c having been previously found equal to 352033·48 feet, we are enabled to find the lengths of the chords opposite A' and B' from the proportions,

$\sin C'' : \sin A' :: \text{chord } c : \text{chord } a$; $\sin C' : \text{in } B' :: \text{chord } c : \text{chord } b$.

cosec C'... ..	0·1615956	....	0·1615956
sin A'	9·9222144	sin B'	9·9929499
chord c......	5·5465840	....	5·5465840
chord a......	5·6303940	chord b.........	5·7011295

hence chord a = 426966·69 feet, chord b = 352033·48 feet.

We have now to determine the lengths of the arcs a and b from the corresponding chords, from formula (9). Making use of the logarithms already given in the preceding solution, we readily find

$$\frac{(\text{chord } a)^3}{24r^2} = 7·39,\ \frac{(\text{chord } b)^3}{24r^2} = 12·05,\ \frac{(\text{chord } c)^3}{24r^2} = 4·14,$$

and therefore the lengths of the arcs are

$$a = 426974·08,\quad b = 502504·51,\quad c = 35203762.$$

SECOND METHOD.—LEGENDRE'S THEOREM.

158. PROP. VII.—*If the three sides of a plane triangle be equal to the three sides of a small spherical triangle, respectively, the difference between each of the angles of the plane triangle, and the corresponding angle of the spherical triangle, will be equal to one-third of the spherical excess.*

As before, let a, b, c, be the three sides of the small spherical triangle, measured in feet, r the radius of the sphere, and $\frac{a}{r} = \alpha$, $\frac{b}{r} = \beta$,

$\frac{c}{r} = \gamma$. Also, let A be the spherical angle opposite to the side a, and A' the corresponding angle in a plane triangle, whose sides are a, b, c. We have, then, as before,

$$\cos A = \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma}.$$

If we now expand each of the quantities $\cos \alpha$, $\cos \beta$, $\sin \beta$, &c., in a series, and arrange the terms according to the powers of α, β, γ, we shall find that the terms of the first order will be the same as if the triangle were rectilineal, and those of the second order will contain the fourth powers of the arc in the numerator, and the second powers in the denominator. Neglecting, therefore, all powers higher than the fourth, we have

$$\cos \alpha = 1 - \tfrac{1}{2}\alpha^2 + \tfrac{1}{24}\alpha^4, \ \sin \beta = \beta - \tfrac{1}{6}\beta^3, \ \cos \beta = \&c.$$

Substituting these values in the preceding equation, it becomes

$$\cos A = \frac{\frac{1}{2}(\beta^2 + \gamma^2 - \alpha^2) + \frac{1}{24}(\alpha^4 - \beta^4 - \gamma^4) - \frac{1}{4}\beta^2\gamma^2}{\beta\gamma\,(1 - \frac{1}{6}\beta^2 - \frac{1}{6}\gamma^2)}$$

And because $\dfrac{1}{1 - \frac{1}{6}(\beta^2 + \gamma^2)} = 1 + \frac{1}{6}(\beta^2 + \gamma^2) + \frac{1}{36}(\beta^2 + \gamma^2)^2 + \&c.$,

if we substitute this above, and neglect all terms containing powers higher than the fourth, we get,

$$\cos A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\alpha^2\beta^2 - 2\alpha^2\gamma^2 - 2\beta^2\gamma^2}{24\beta\gamma}$$

$$= \frac{b^2 + c^2 - a^2}{2bc} + \frac{a^4 + b^4 + c^4 - 2a^2c^2 - 2a^2c^2 - 2b^2c^2}{24bc \times r^2}$$

But $\dfrac{b^2 + c^2 - a^2}{2bc} = \cos A'$ (Trig. art. 109); also (vol. i. p. 419),

$$2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 = (4 \text{ area})^2 = 16S^2;$$

$$\therefore \cos A = \cos A' - \frac{2S^2}{3bc \times r^2} \ldots\ldots\ldots\ldots(10)$$

Let $A = A' + x$, then x is evidently a very small angle, consequently

$$\cos A = \cos A' \cos x - \sin x \sin A' = \cos A' - x \sin A',$$

nearly. Comparing this value of $\cos A$ with equation (10), we have

$$x = \frac{2S^2}{3r^2 \times bc \sin A'} = \frac{S}{3r^2}. \quad \text{(Mens. prob. 2.)}$$

Hence $$A' = A - \frac{S}{3r^2} \ldots\ldots\ldots\ldots (11)$$

In like manner, $$B' = B - \frac{S}{3r^2}, \quad C' = C - \frac{S}{3r^2};$$

$$\therefore \quad A' + B' + C' = 180^\circ = A + B + C - \frac{S}{r^2}.$$

Hence $\frac{S}{r^2}$ is the excess of the three angles of the spherical triangle above two right angles, and each of the angles A, B, C, exceeds the corresponding angle of the plane triangle by one-third of this spherical excess.

Example.

159. Taking the same example as before, we find the spherical excess $= 34''\cdot76$, and one-third of this excess $= 11''\cdot59$. Hence

$$A' = 56^\circ\ 43'\ 17''\cdot04,\ B' = 79^\circ\ 42'\ 17''\cdot56,\ C' = 43^\circ\ 34'\ 25''\cdot40.$$

With these angles, and the given side $c = 352037\cdot62$ feet, we then compute the other sides, a, b, by Plane Trigonometry,

$$\sin C' : \sin A' :: c : a, \text{ and } \sin C' : \sin B' :: c : b.$$

cosec C'	0·1615997	....	0·1615997
sin A'	9·9222127	sin B'	9·9929511
c	5·5465891	....	5·5465891
a	5·6304015	b	5·7011399

Hence $a = 426974\cdot06$ feet, $b = 502504\cdot42$ feet.

THIRD METHOD.

PROP. VIII.—*To compute the sides by Spherical Trigonometry.*

160. By Trig. (art. 141), $\sin C : \sin A :: \sin c : \sin a$......($a$)
And since c and a are very small, compared with the radius of the sphere,

$$\frac{\sin c}{r} = \frac{c}{r} - \frac{c^3}{6r^3}, \text{ nearly}; \quad \therefore \sin c = c\left(1 - \frac{c^2}{6r^2}\right)$$

$$\log \sin c = \log c + \log\left(1 - \frac{c^2}{6r^2}\right) = \log c - \frac{M}{6r^2}c^2, \ldots\ldots(12)$$

nearly (Alg. art. 400), these logarithms being taken from the common tables, and M being the modulus of the system. Having found log sin c from this expression, we get log sin a from proportion (a). We then obtain a from the equation

$$\log a = \log \sin a + \frac{M}{6r^2}a^2 = \log \sin a + \frac{M}{6r^2}\sin^2 a \ldots\ldots (13)$$

Example.

161. To apply this to the last example,

log r^2	14.64224
$\frac{1}{6}M$	8.85963
(1)	4.21739
c^2	11.09318
.0000204	$\bar{5}$.31057
sin a	5.63037
	5.63037
(1)	4.21739
.0000301	$\bar{5}$.47813

log c	5.5465891
	.0000204
sin c	5.0465687
sin A	9.9222287
cosec C	0.1615740
sin a	5.6303714
	.0000301
a	5.6304015

As the logarithm of a is exactly the same as that which we obtained by Legendre's method, the arc itself will also be the same as before.

The logarithm of the side b is found in the same manner $= 5.7011398$, which only differs from the former logarithm by a unit in the last place of decimals.

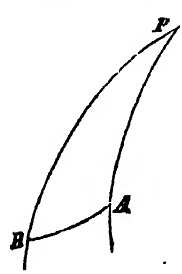
In comparing these three methods together, Legendre's certainly appears to be the most simple, and the first method perhaps the most difficult. They are all, however, rendered considerably more easy in practice, by means of auxiliary tables, previously calculated. The third method also has an advantage over the two others, in this respect, that if any of the angles (A for example) be one of the angles in another triangle, as in calculating the latitudes and azimuths, no further correction will be necessary; whereas, in the first and second methods, a new reduction must be made in order to obtain the angles for calculation. The whole may be brought under one view in the following table.

Stations.	Observed Angles.	Apportionment of Error.	Spherical Angles.	Chord Angles.	Mean Angles.	Opposite Chords.	Opposite Arcs.
A	56° 43′ 28″.58	+0″.04	28″.62	18″.27	17″.04	426966.69	426974.08
B	79 42 28.69	+0.46	29.15	14.47	17.56	502492.46	502504.51
C	43 34 36.89	+0.10	36.99	27.26	25.40	352033.48	352037.62
	180 0 34.16	0.60	34.76	0.00	0.00		

V. CALCULATION OF THE LATITUDES, LONGITUDES, AND AZIMUTHS.

162. When all the sides of the principal triangles have been found, by one of the methods described in the preceding articles, we proceed to determine the latitudes and longitudes of the different stations, and the inclinations which the sides of the triangles make with the meridian. For this purpose, it is necessary that the latitude of one of the stations and the azimuth of one of the sides should be found independently, by astronomical means; and from them we may determine the longitudes and latitudes of all the other stations, and the azimuths of the sides of the triangles. We shall first suppose the earth to be a sphere, and afterwards correct the error arising from this hypothesis.

163. PROP. IX.—*Given the latitude of a station A, the distance of A from another station B, and also the azimuth of B as seen from A, to determine the latitude of B, the earth being considered as a sphere.*



Let P be the pole of the earth, PA, PB, the meridians of the stations A and B. Let the angle $PAB = A$, $PBA = B$, arc $PA = 90° - l$, $PB = 90° - l'$, and $l - l' = \lambda$; also, let the arc AB measured in feet $= D$, and in parts of the radius $= \delta$; and let the radius of the earth measured in feet $= r$. We have then, from Spherical Trigonometry (art. 140),

$$\cos PB = \cos PA \cos AB + \sin PA \sin AB \cos A,$$

$$\text{or } \sin l' = \sin l \cos \delta + \cos l \sin \delta \cos A \ldots (a)$$

But $\qquad \sin l' = \sin (l - \lambda) = \sin l \cos \lambda - \cos l \sin \lambda$
$= \sin l \,(1 - \frac{1}{2}\lambda^2) - \lambda \cos l.$

Also, $\qquad \cos \delta = \quad - \frac{1}{2}\delta^2, \quad \sin \delta = \delta,$

neglecting all the powers of δ and λ higher than the second.

Making these substitutions in equation (a) we have,

$$\sin l \,(1 - \tfrac{1}{2}\lambda^2) - \lambda \cos l = \sin l \,(1 - \tfrac{1}{2}\delta^2) - \delta \cos l \cos A\,;$$

$$\therefore \qquad \lambda = \delta \cos A + \tfrac{1}{2}\,(\delta^2 - \lambda^2) \tan l.$$

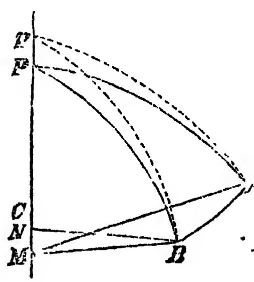
For a first approximation, we may neglect the second powers of δ and λ, and assume $\lambda = \delta \cos A$, which is the same thing as if we supposed the meridians at A and B to be parallel. Substituting this first value of λ in the second member of the last equation, we obtain

$$\lambda = \delta \cos A + \tfrac{1}{2}\delta^2 \sin^2 A \tan l.$$

Here λ and δ are measured in parts of the radius. If λ'' be the number of seconds in λ, then $\lambda = \lambda'' \sin 1''$; also $\delta = \dfrac{D}{r}$. Making these substitutions, the last equation ecomes

$$\lambda'' = \frac{D \cos A}{r \sin 1''} + \frac{D^2 \sin^2 A \tan l}{2r^2 \sin 1''} \ldots\ldots\ldots (14)$$

164. Prop. X.—*To determine the same when the spheroidal figure of the earth is taken into consideration.*



Let PA, PB, be the meridians of A and B, the earth being considered as a spheroid; let AM, BN, be the normals to the surface meeting the polar axis in M and N; join BM. Suppose ApB to be the surface of a sphere whose centre is M, and radius MA. Then, because the arc AB is very small, and AM is a normal to the spheroid, it is nearly equal to the radius of curvature at A (art. 195), therefore the surface of the sphere will very nearly pass through B, and the difference between the arc AB on the sphere and on the spheroid will be altogether insensible.* The spherical triangle pAB may be considered as that whose solution we have given in the last article, and on this supposition $BMp = 90° - l'$ is the colatitude of B. But the true colatitude of B is the angle $BNP = 90° - L$, which is greater than BMP by the angle MBN. Let $l - l' = \lambda$, $l' - L = MBN = \phi$; we have, then, in the triangle BMN

$$\sin \phi = \frac{MN}{BM} \sin BNM = \frac{CM - CN}{BM} \cos L\,;$$

but (art. 185) $CM = AM \cdot e^2 \sin l$, $\quad CN = BN \cdot e^2 \sin L$, therefore

$$\sin \phi = e^2 \cos L \left(\frac{AM}{BM} \sin l \quad \frac{BN}{BM} \sin L \right).$$

* The difference between the arc and the chord is a very small quantity, even in the largest triangles on the survey, and, therefore, the difference between two arcs of nearly equal curvature, which have the same chord, will be inappreciable.

And since $\frac{AM}{BM}$ and $\frac{BN}{BM}$ differ from unity by a quantity of a very minute order, we have

$$\sin \phi = e^2 \cos L (\sin l - \sin L), \text{ very nearly.}$$

Now, $\sin L = \sin \{l - (\lambda + \phi)\} = \sin l - (\lambda + \phi) \cos l$, nearly.

Also, $\sin \phi = \phi$, very nearly, therefore

$$\phi = e^2 (\lambda + \phi) \cos L \cos l.$$

Hence, transposing and dividing,

$$\phi = \frac{e^2 \lambda \cos L \cos l}{1 - e^2 \cos L \cos l} = e^2 \lambda \cos L \cos l, \text{ nearly};$$

$$\therefore \phi = e^2 \lambda \cos^2 l, \text{ nearly, and } \lambda + \phi = \lambda (1 + e^2 \cos^2 l).$$

Hence, on the spheroid, the difference of latitude

$$l - L = \left\{\frac{D \cos A}{r \sin 1''} + \frac{D^2 \sin^2 A \tan l}{2r^2 \sin 1''}\right\}(1 + e^2 \cos^2 l) \ldots (15)$$

where $r = AM$, the normal to the surface at the station A.

165. Prop. XI.—*The same things being given, to find the difference of longitude.*

The difference of longitude on the sphere is the angle ApB, which is equal to APB, the difference of longitude on the spheroid. We have, then, by Spherical Trigonometry,

$$\sin Bp : \sin A :: \sin \delta : \sin p :: \delta : p.$$

But $\sin Bp = \cos l' = \cos L$, very nearly, $\delta = D \div r$, $p = P = P'' \sin 1''$, therefore

$$P'' = \frac{D \sin A}{r \cos L \sin 1''} \qquad .(16)$$

166. Prop. XII.—*To find the azimuth of* A *as seen from* B.

In the spherical triangle ApB we have, from Napier's analogies (Trig. art. 155),

$$\cos \tfrac{1}{2}(pB + pA) : \cos \tfrac{1}{2}(pB - pA) :: \cot \tfrac{1}{2}p : \tan \tfrac{1}{2}(A + B).$$

Now, $\frac{1}{2}(pB + pA) = \frac{1}{2}(90^\circ - l') + \frac{1}{2}(90^\circ - l) = 90^\circ - \frac{1}{2}(l + l')$,

$\frac{1}{2}(pB - pA) = \frac{1}{2}(l - l')$, $\frac{1}{2}(A + B) = 90^\circ - \frac{1}{2}(180 - A - B)$.

Making these substitutions, this proportion becomes

$$\sin \tfrac{1}{2}(l + l') : \cos \tfrac{1}{2}(l - l') :: \tan \tfrac{1}{2}p : \cot \tfrac{1}{2}(180^\circ - A - B)$$
$$:: \tan \tfrac{1}{2}(180^\circ - A - B) : \tan \tfrac{1}{2}p.$$

And because the distance AB is always very small, compared with the radius of the earth, $A + B$ is nearly equal to 180°, and, therefore, $180^\circ - A - B$ is a very small angle. Also, $\frac{1}{2}p$ or $\frac{1}{2}P$ is very small. We may therefore substitute the arcs for the tangents, and also L for l', without sensible error. Hence, forming an equation, we obtain,

$$B = 180^\circ - A - P \frac{\sin \frac{1}{2}(l + L)}{\cos \frac{1}{2}(l - L)} \ldots\ldots \qquad .(17)$$

The angle B, which we have calculated, is the spherical angle pBA, or the angle contained between the planes MBp, MBA; but the true

azimuth is the spheroidal angle contained between the planes NBP, NBA; the difference, however, between these angles has been proved, by Delambre, to be so small as not to be sensible in practice.

In the trigonometrical survey, the angles are measured either from the north or south to the east or west; but in the "base du systeme metrique," the angles are measured from the south towards the west, entirely round the circle.

167. Prop. XIII.—*To determine the azimuth of one of the signals independently from astronomical observations.*

The general principle of the method is this. The error of a clock or chronometer is found, either by means of a transit instrument, or by observations of equal altitudes, or by single altitudes, if the latitude of the place be well known. The observer then takes the angle (θ) between the signal and the sun, or a star, when near the horizon, and notes the time when the observation was made. The azimuth of the heavenly body is also calculated for this time; the latitude and declination being known. Then the sum or difference of the angle θ and the azimuth of the heavenly body will give the azimuth of the signal required. The refraction will scarcely affect the result, but a small error in the time would produce a considerable error in the azimuth.

The method adopted in the trigonometrical survey was to take the mean of the two angles observed with the theodolite, between a flagstaff and the pole star at its greatest elongation east and west. But, from the great altitude of the pole star in our latitudes, any error in the adjustment of the cross axis of the theodolite to horizontality, would materially affect the resulting azimuth:

Example.

168. From the Trigonometrical Survey, vol. ii. p. 88, the distance of Black Down from Dunnose = 314397·5 feet, the latitude of Dunnose = 50° 37′ 7″.3 N., and azimuth of Black Down, as seen from Dunnose, = 84° 54′ 52″·5 N.W. Required the latitude and longitude of Black Down, and the azimuth of Dunnose, as seen from Black Down.

To find the Latitude.

The normal AM, which is equal to r the radius of the curvature at A, perpendicular to the meridian, is found (from art. 185) = 20,963000, nearly.

log r	7·32145	$2r^2 \sin 1''$	9.62950
sin 1″	4·68557		————
	————	ar. co.	0·37050
	2·00702	D^2	10·99470
	————	$\sin^2 A$	9·99658
ar. co.	7·99298	$\tan l$	0·08573
D	5·49735	$(1 + e^2 \cos^2 l)$	0·00116
$\cos A$	8.94763		————
$(1 + e^2 \cos^2 l)$ ..	0·00116	28″·10	1·44867
	————		
274″·87........	2·43912		

Hence $l - L = -274''\cdot87 + 28''\cdot10 = -4'\ 6''\cdot77$, and
$L = l + 4'\ 6''\cdot77 = 50°\ 41'\ 14''\cdot07$.

To find the Difference of Longitude.

ar. co. log r sin 1″	7·99298
D	5·49735
sin A	9·99829
sec L	0·19822
4862″·3	3·68684

Hence $P = 1° \ 21' \ 2''·3$; and since the longitude of Dunnose was previously found = 1° 11′ 36″, therefore, the

long. of Black Down = 2° 32′ 38″·3

To find the Azimuth.

sin $\frac{1}{2}(l + L)$ ..	9·88836
cos $\frac{1}{2}(L - l)$ ..	0·00000
P	3·68684
3760″·12	3·57520

Hence

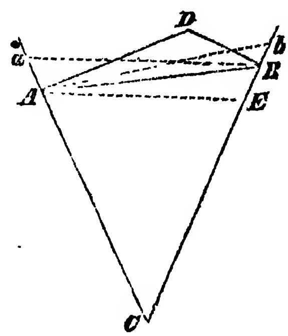
$B \quad 180° - A - 1° \ 2' \ 40''·12$
$= 94° \ 2' \ 27''·38.$

The observed angle PBA was 94° 2′ 22″·75.

In the survey, the value of P is found to be 1° 20′ 46″·4. The difference (15″.9) arises from an erroneous assumption in the length of the perpendicular degree, which gives all the longitudes on the southern coast of England too small.

VI.—HEIGHTS OF THE STATIONS, AND TERRESTRIAL REFRACTION.

169. Prop. XIV.—*To find the altitude of the station B above the station A.*



Let C be the centre of the earth, supposed to be a sphere, and A and B two stations on its surface. Draw AD, BD, perpendicular to the radii CA, CB, respectively in the plane CAB; and suppose a and b to be the apparent places of A and B as seen from each other, and elevated by refraction. If the rays of light proceeded in straight lines, the angle DAB would be the depression of B below the horizon of A, and DBA the depression of A below the horizon of B. And because DAC, DBC, are right angles,

$$C + D = 180° = DAB + DBA + D, \text{ and } \therefore C = DAB + DBA.$$

Also, since the distance AB is known, and the radius of the earth (sufficiently near for this purpose) the angle C can easily be found.

Let α, β, be the observed depressions at A and B respectively, and ρ, ρ', the two refractions, then,

$$DAB = \alpha + \rho, \ DBA = \beta + \rho', \text{ and } (\alpha + \rho) + (\beta + \rho') = C;$$

$$\therefore \text{ mean refraction } \tfrac{1}{2}(\rho + \rho') = \tfrac{1}{2}(C - \alpha - \beta) \ldots\ldots\ldots (18)$$

Let E be the point in CB which is on the same level with A, then $CE = CA$, and EB is the altitude of B above A, which is to be

determined. Join AE, then the angle $DAE = 90° - CAE = \frac{1}{2}C$, therefore the

$$\text{angle } BAE = \phi = DAE - DAB = \tfrac{1}{2}C - (\alpha + \rho) \ldots\ldots (19)$$

and since the angle BAE is always very small, and BEA very nearly a right angle,

$$BE = AE \times \phi = D \times \phi'' \sin 1'' \ldots\ldots\ldots\ldots (20)$$

If one of the stations (B for example) is elevated above the horizon of A, β must be considered negative. Also, each observation must be reduced, previously to the calculation, to the place of the axis of the instrument.

Example.

170. At Allington Knoll, the top of the staff on Tenterden steeple was depressed $3'\ 5''$; and the axis of the instrument was $5\frac{1}{2}$ feet above the ground: on Tenterden steeple the ground at Allington Knoll was depressed $3'\ 35''$, and the axis of the instrument was 3·1 feet below the top of the staff. The distance between the stations being 61777 feet, it is required to calculate the mean refraction, and also the height of Tenterden steeple above Allington Knoll. (Trig. Surv. vol. i. p. 176.)

The angle which a perpendicular height of 5.5 feet subtends at the distance = 61777 feet is

$$\frac{5.5}{61777 \times \sin 1''} = 18''{\cdot}4; \text{ and}$$

in like manner the angle which 3·1 feet subtends is $10''{\cdot}4$. Hence

Depression of the top of the staff..............	$3'$	$51''$
Correction due to 3·1 ft.	+	10·4
Depression of instrument	4	1·4
Depression of the ground	3	35
Correction due to $5\frac{1}{2}$ ft.	−	18·4
Depression of instrument	3	16·6

Length of perpendicular degree at Tenterden, vol. i. p. 168 61185 fathoms.

Fathoms. Feet.		
61185 : 61777	1 :	10 6
β	4	1·4
	3	16·6
$\alpha + \beta$	[illegible]	18·0
C	10	6
$\rho + \rho'$	2	48
Mean refraction	1	24

Hence $\phi'' = \frac{1}{2}C - (\alpha + \text{mean refr.}) = 22''{\cdot}4$,

and $h = D \times \phi'' \sin 1'' = 6{\cdot}7$ feet.

The vertical height of the axis at Allington Knoll had been previously found to be 329 feet, so that the height of the axis on Tenterden steeple was 322·3 feet.

171. To find the absolute altitudes, it is necessary that the heights of one or more of the stations be ascertained, by actually levelling down to the surface of the sea. The heights of all the intermediate stations are then determined by the reciprocal angles of elevation or depression, carried on from station to station, and it is obvious that a verification

will be obtained for every three stations; for the difference of altitude between A and B, when found from direct observation, ought to be the same as when deduced from the difference of the heights of each of those stations and a third station C.

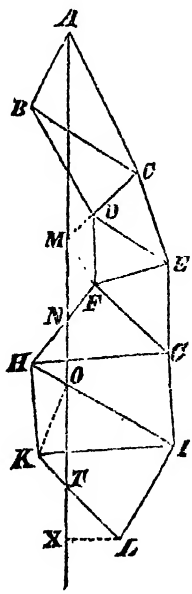
172. In the preceding example, the effect of refraction is $\frac{1}{4}$th of the intercepted arc. In other cases, the refraction varied from $\frac{1}{8}$th to $\frac{1}{20}$th of the contained arc. When reciprocal observations could not be obtained, $\frac{1}{12}$th of C was generally assumed as a mean value of ρ, in order to obtain the angle ϕ in equation (19).

VII.—MEASUREMENT OF THE ARCS OF THE MERIDIAN, AND THE ARCS PARALLEL TO THE EQUATOR.

173. When a chain of triangles has been formed nearly in the direction of the arc of a meridian, and all the sides have been computed, according to the preceding rules, we are enabled to determine the length of the arc of the meridian intercepted between the parallels of the extreme stations. For this purpose, two different methods have been adopted, which we shall briefly explain.

THE METHOD OF OBLIQUE-ANGLED TRIANGLES.

174. PROP. XV.—*To measure the arc of the meridian intercepted between the parallels of A and L.*



Let $A\,B\,C\,D\ldots.$ be a chain of triangles lying nearly in the direction of the meridian AX. All the sides of the triangles are supposed to have been previously computed, and the angle CAX is given from observation. Produce CD to M, join FM; and, from the last station L, draw LX perpendicular to the meridian AX. The following spherical triangles will then be most easily solved, according to Legendre's method, by first computing the spherical excess in each case, and then deducting one-third of this excess from each of the spherical angles.

In the triangle ACM, there are given AC, $\angle ACM$, $\angle CAM$, to find AM, CM, and $\angle AMC$.

Then $DM = CM - CD$, and $\angle MDF = 180^\circ - CDF$.

In the triangle DMF are given DF, DM, $\angle D$, to find MF, $\angle DMF$, $\angle DFM$.

$$\angle FMN = 180^\circ - (AMC + DMF);$$
$$\angle\ MFN = DFN - DFM.$$

In the triangle MFN are given MF, $\angle FMN$, $\angle MFN$, to find MN, FN, and $\angle MNF$.

$HN = FH - FN$, and $\angle FNM = HNO$.

In the triangle HNO are given HN, $\angle HNO$, $\angle NHO$, to find NO, HO, and $\angle HON$.

Lastly, in resolving the triangles HOK, OKT, LXT, we find OT and TX. Hence we have, by addition,

$$AX = AM + MN + NO + OT + TX.$$

175. It must be observed, however, that the point X is not in the same parallel of latitude with L. Suppose the latitude of A to be greater than that of X, and let $L =$ the latitude of L, and $L + x$ = the latitude of X, also put $LX = p$; if, then, we suppose XA produced to meet the meridian of L in the pole P, we shall have $\cos PL = \cos PX \cos LX$, or

$$\sin (L + x) = \sin x \cos L + \sin L \cos x = \frac{\sin L}{\cos p}.$$

But $\sin x = x - \frac{1}{6}x^3 + \&c.$, $\qquad 1 - \frac{1}{2}x^2 + \&c.$, $\cos p = 1 - \frac{1}{2}p^2 + \&c.$, consequently, we have

$$\cos L\,(x - \&c.) + \sin L\,(1 - \tfrac{1}{2}x^2 + \&c.) = \sin L\,(1 + \tfrac{1}{2}p^2 + \&c.);$$

$$\therefore\ x = \tan L\,(\tfrac{1}{2}p^2 + \tfrac{1}{2}x^2) + \&c.$$

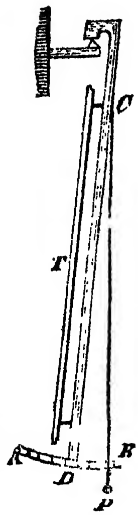
Hence x is of the second order, with respect to p, and therefore the term involving x^2, being of the fourth order, may be neglected. Hence $x = \frac{1}{2}p^2 \tan L$. In this expression p and x are measured in parts of the radius; if we suppose them to be measured in feet, we must substitute $\frac{p}{r}$ and $\frac{x}{r}$ for p and x, therefore the correction to be added to AX (the latitude of A being greater than that of X) is

$$x = \frac{p^2}{2r} \tan L \qquad (21)$$

THE METHOD OF PARALLELS.

176. The method employed by Delambre, to determine the length of the arc of the meridian, was to project on the principal meridian all the stations to the east, by means of circles parallel to the equator. He then computed the distance between every two succeeding parallels from formula (15); the sum of all these distances will give the entire length of the meridian between the extreme stations. Having done the same for the stations to the west of the meridian, these two sums ought to give the same value for the length of the total arc of the meridian, and thus the two computations serve to verify each other. If the two sums do not agree, a mean should be taken between the two for the total arc. This method, however, can only be applied when the dimensions of the earth are previously known with tolerable accuracy.

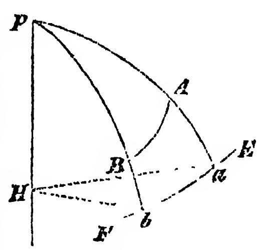
177. When the distance between the parallels of the extreme stations has been determined in this manner, it only remains to determine the latitudes of these stations, and the amplitude of the corresponding celestial arc. This is the most difficult part of the whole operation. The error of a single second in the difference of latitude is equivalent to about 100 feet on the terrestrial meridian, and therefore it is obvious that an error in the latitude is of far more importance than any which can affect the measurement of the base, the angles of the triangles, or the direction of the meridian. In the English survey, and in India, the latitudes were observed with a zenith sector, made expressly for this purpose, by Ramsden.



The principle of the zenith sector is this:—AB is an arc of a circle, having a long radius CD, to which is firmly fixed a telescope T, of the same length. The instrument is suspended vertically, and the telescope (with the arc fixed to it) can be moved in the plane of the meridian a few degrees on each side of the vertical line, so as to observe stars within a few degrees of the zenith. A plumb line CP, suspended from the centre of the instrument, and passing over the arc AB, shows the angle between CD and the vertical line CP. This instrument can be turned half round in azimuth, so that, if observations be made on the same stars in the two positions, any error in the place of the zero of graduation will be entirely removed; for the zenith distance will be as much too great in the one case as it was too little in the other. The telescope of the sector used in the British survey was 8 feet in length.

In France, the small repeating circle, described in art. 139, was used to determine the latitudes; and it has justly been doubted whether this instrument can be safely relied on for determining so important an element as the latitude.

178. PROP. XVI.—*To determine the length of an arc parallel to the equator.*



Let AB be one of the sides of a chain of triangles, which lie in a direction nearly perpendicular to the meridian, and let EF be the parallel on which the sides of all the triangles are to be projected. Draw the meridians PAa, PBb, then it is required to determine the length of the arc ab in feet. Let L = the latitude of the parallel EF, l = the latitude of B, z = the azimuth PAB, N = the normal at b, and N' = do. at B; also, let AB measured in feet $= D$, and in parts of the radius $= \delta$. We have, then, in the spherical triangle APB,

$$\sin P \ : \ \sin z \ :: \ \sin \delta \ : \ \cos l,$$

and, because P and δ are small arcs, $\sin P = P - \frac{1}{6}P^3$, $\sin \delta = \delta - \frac{1}{6}\delta^3$, very nearly; therefore, making an equation and transposing,

$$P = (\delta - \tfrac{1}{6}\delta^3)\frac{\sin z}{\cos l} + \tfrac{1}{6}P^3.$$

As a first approximation we have $P = \delta\dfrac{\sin z}{\cos l}$; substituting this value of P in the second member of the last equation, we get

$$P = \delta\frac{\sin z}{\cos l} - \tfrac{1}{6}\delta^3\frac{\sin z}{\cos l}\left(1 - \frac{\sin^2 z}{\cos^2 l}\right) \ldots\ldots\ldots (a)$$

Let H be the centre of the circle EF, and let ab measured in feet $= p$, then

$$p \ : \ bH \ :: \ \text{measure of the angle } aHb \text{ or } P \ : \ 1;$$

$$\therefore\ p = P \times bH = P \times N \cos L, \text{ equation (26); also } \delta = \frac{D}{N'}$$

Making these substitutions in equation (*a*), we get

$$p = \frac{N \cos L}{N' \cos l}\left\{D \sin z - \frac{D^3 \sin z}{6\,N'^2}\quad\left(1 - \frac{\sin^2 z}{\cos^2 l}\right)\right\}\text{.........}(22)$$

By applying this formulâ to all the sides of the triangles, the sum of those projections will give the required length of the total arc.

179. We have now to determine the astronomical difference of longitude from observation. In the *Philosophical Transactions* for 1824 an account is given of some experiments performed by Dr. Tiarks, for determining the differences of longitude of Dover and Falmouth. Twenty-four chronometers were transported by sea three several times from the one place to the other, by which means the difference of longitude was determined to be 6° 22′ 6″; and as the length of the parallel found from the survey was 1,474672 feet, we have the length of a degree of parallel in latitude 50° 44′ 24′, equal to 231563 feet. The difference of longitude of Marennes and Padua was determined, by five signals, at five intermediate stations (see art. 118). The length of the parallel in feet was found, from triangulation, to be 1,010996 metres, or 3,316976 English feet, and the difference of longitude was 12° 59′ 3″·75. This gives, for the mean length of a degree in latitude, 45° 43′ 12″, found from the whole arc between Marennes and Padua, 255470 feet; the length of the degree found from the partial arc between Marennes and Geneva was 255546 feet. Both these results are greater than a degree in the same parallel of latitude on a regular spheroid, which most nearly represents the meridional arcs; but no great reliance can be placed on these numbers, as the determination of the longitudes was attended with considerable difficulty.

VIII.—THE FIGURE OF THE EARTH.

180. If the earth were perfectly fluid, and had no motion of rotation about an axis, it would assume a spherical form; for in this case, there would be no tendency in the fluid to run in any direction, and therefore it would be in a state of equilibrium. But if any portion of the surface were further removed from the centre than the rest, the pressure arising from the protuberant would be greater than that from the less elevated parts, and therefore the equilibrium would be destroyed.

181. But, since the earth revolves on its axis, every particle has a tendency to recede from that axis proportional to its distance; consequently its gravity will be diminished, and the columns of fluid at the equator, being composed of parts that are lighter, must be extended in length, in order to balance the columns in the direction of the axis. It has been proved, by Maclaurin, and succeeding writers, that a mass of homogeneous fluid will be in equilibrium if it be formed into an oblate spheroid, such that the polar diameter shall be to the equatorial diameter as the attraction at the equator, diminished by the centrifugal force there, to the attraction at the pole. And as it appears, from experiments on the vibration of pendulums, that the centrifugal force is to the force of gravity at the equator as 1 to 289, it may be demonstrated, that a homo-

geneous fluid of the same mean density of the earth would be in equilibrium if the ratio $\frac{a-b}{a} = \frac{5}{4}\frac{1}{289} = \frac{1}{231}$, nearly, a being the equatorial, and b the polar diameter; that is, if $b : a :: 230 : 231$.

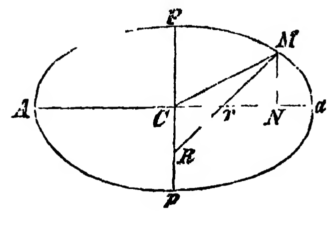
182. If the fluid mass of the earth be supposed not to be homogeneous, but to be formed of strata that increase in density towards its centre, the solid of equilibrium will still be an elliptic spheroid, but less oblate than before. Now, as it appears, from experiments made on the density of the mountain Schehallien, in Scotland, and also from those of Cavendish,* that the mean density of the earth is greater than the density at the surface, it follows, that if the earth be a solid of equilibrium, the ratio $\frac{a-b}{b}$ will be less than before, or less than $\frac{1}{230}$.

183. If the earth were homogeneous, the increase of gravity from the equator to the pole would be $\frac{1}{230}G$, G being the gravity at the equator; and the gravity g, at any latitude l, would be represented by the equation $g = G\left(1 + \frac{1}{230}\sin^2 l\right)$. But if the density of the earth increase towards the centre, the ratio $\frac{a-b}{b}$, and the increase of gravity from the equator to the pole, divided by the gravity at the equator (γ), will no longer be expressed by the same fraction, but the sum of the two fractions is constant, and equal to twice the value of $\frac{a-b}{b}$, which the spheroid would have if it were homogeneous, that is,

$$\left.\begin{array}{l} \dfrac{a-b}{b} + \gamma = \dfrac{5}{2}\,\dfrac{1}{289} = \cdot 00865 \\ \text{and } g = G\left(1 + \gamma \sin^2 l\right) \end{array}\right\} \quad \ldots\ldots(23)$$

This theorem was first given by Clairaut, and is of great importance in determining the figure of the earth from experiments with the pendulum.

184. As it would far exceed the limits of this work to demonstrate these different theorems, we must refer the student to Airy's Mathematical Tracts, and the Encyclopædias Metropolitana and Britannica (art. Figure of the Earth). We shall now proceed to show how the figure of the earth is to be determined from geodetic operations. We shall, therefore, first consider the different properties of an oblate spheroid, and then compare them with the results deduced from observation.



185. Let $APap$ be an ellipse, which by its revolution about its minor axis Pp, generates an oblate spheroid. Let $AC = a$, $CP = b$, the excentricity $= ae$, the ordinate $MN = y$, $CN = x$, the normal $Mr = n$, $MR = N$, the radius of curvature at $M = \rho$, and the latitude of M, or the angle $Mra = l$. Now, the equation to the ellipse is (art. 57)

* The experiments of Cavendish have lately been repeated, with the greatest care, by Mr. Baily, at the request of the Astronomical Society, and the results which have been obtained are extremely satisfactory. The mean density of the earth, as deduced from these experiments, is 5·67 nearly.

$$a^2y^2 + b^2x^2 = a^2b^2.$$

Also $y = n \sin l$, and $x = \frac{a^2}{b^2} \times Nr = \frac{a^2}{b^2} n \cos l$ (art. 91);

$$\therefore \quad a^2n^2 \sin^2 l + \frac{a^2}{b^2} n^2 \cos^2 l = a^2b^2;$$

consequently $\quad n = \dfrac{b^2}{\sqrt{(a^2 \cos^2 l + b^2 \sin^2 l)}}$.

And, because $b^2 = a^2(1 - e^2)$, therefore $a^2 \cos^2 l + b^2 \sin^2 l = a^2 (1 - e^2 \sin^2 l)$; hence

$$n = \frac{a (1 - e^2)}{\sqrt{(1 - e^2 \sin^2 l)}} \text{ ....(24)}$$

$$x = \frac{a \cos l}{\sqrt{(1 - e^2 \sin^2 l)}}, \qquad y = \frac{a (1 - e^2) \sin l}{\sqrt{(1 - e^2 \sin^2 l)}}, \text{...(25)}$$

$$N = \frac{x}{\cos l} = \frac{a}{\sqrt{(1 - e^2 \sin^2 l)}} \text{ ......(26)}$$

$$Cr = \frac{ae^2 \cos l}{\sqrt{(1 - e^2 \sin^2 l)}} = Ne^2 \cos l \text{............(27)}$$

$$CR = \frac{ae^2 \sin l}{\sqrt{(1 - e^2 \sin^2 l)}}, = Ne^2 \sin l \text{............(28)}$$

$$CM = \sqrt{x^2 + y^2} = a \sqrt{\left(\frac{1 - (2e^2 - e^4) \sin^2 l}{1 - e^2 \sin^2 l}\right)} \text{ ...(29)}$$

$$\rho = \frac{a^2}{b^4} n^3 = \frac{a (1 - e^2)}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}} \text{ ....(30)}$$

186. PROP. XVII.—*The lengths of two degrees on the meridian in given latitudes being known from measurement, it is required to determine the polar and equatorial diameters.*

Let D, D', be the lengths of two degrees in feet; l, l', the latitudes of their middle points; ρ, ρ', the radii of curvature at those points; then, since the two arcs are very small compared with their radii, we may suppose them to be arcs of two circles whose radii are ρ, ρ', without sensible error. Hence

$$180^\circ : 1^\circ :: \pi\rho : D;$$

$$\therefore \rho = \frac{180}{\pi} D = \mu D; \quad \text{and } \rho' = \mu D',$$

μ being substituted for $\frac{180}{\pi}$. Hence, therefore, expanding the value of ρ, and neglecting higher powers of e than the second, we have, from equation (30),

$$D = \frac{\rho}{\mu} = \frac{a (1 - e^2)}{\mu} (1 + \tfrac{3}{2}e^2 \sin^2 l) \text{...... .(31)}$$

$$D' = \frac{\rho'}{\mu} = \frac{a (1 - e^2)}{\mu} (1 + \tfrac{3}{2}e^2 \sin^2 l')$$

$$\therefore \frac{D}{D'} = \frac{1 + \frac{3}{2}e^2 \sin^2 l}{1 + \frac{3}{2}e^2 \sin^2 l'} = 1 + \frac{3}{2}e^2 \sin^2 l - \frac{3}{2}e^2 \sin^2 l'$$

$$\therefore e^2 = \frac{2}{3}\frac{D - D'}{D'(\sin^2 l - \sin^2 l')} = \frac{2}{3}\frac{D - D'}{D' \sin(l + l') \sin(l - l')} \ldots (32)$$

187. If $l' = 0$, or the degree is at the equator, the length of the degree $D' = \frac{a(1 - e^2)}{\mu}$. Hence it follows, that the excess of the degrees of the meridian above a degree of the meridian at the equator, is as the square of the sine of latitude.

188. Prop. XVIII.—*The length of a degree parallel to the equator, and the length of a degree of the meridian, being known from measurement, to determine the polar and equatorial diameters.*

Let Δ be the length of a degree parallel to the equator, at a place whose latitude $= \phi$. Then the radius of this circle $x = \frac{a \cos l}{\sqrt{(1 - e^2 \sin^2 l)}}$, equation (25); therefore

$$\Delta = \frac{x}{\mu} = \frac{a \cos l}{\mu \sqrt{(1 - e^2 \sin^2 l)}}.$$

Expanding this expression, and neglecting the powers of e higher than the second,

$$\Delta = \frac{a \cos l}{\mu}(1 + \tfrac{1}{2}e^2 \sin^2 l) \ldots\ldots\ldots (33)$$

From this equation, and equation (31), we can determine the values of e^2 and a, when D and Δ are known.

189. We shall now give some examples of the geodetic measurements which have been executed in our own country and in India. They are part of those which M. Schmidt has selected as the best for the purpose of determining the magnitude and figure of the earth. With these data, he has found

$$a = 20{,}921665 \text{ feet}, \qquad b = 20{,}852394 \text{ feet}.$$

$$\text{Ellipticity} = \frac{a - b}{a} = \frac{1}{302 \cdot 03}.$$

Degree at the equator = 362732; degree in latitude 45° = 364543·5.

No.	Country.	Latitude of Middle Points.	Arc measured.	Length in Feet.	Length of a Degree.	Diff.
		° ′ ″	° ′ ″			
1	India....	12 32 21	1 34 56·4	574368	362988	+ 83
2	"	9 34 43	2 50 10·5	1,029171	362863	+ 29
3	"	13 2 54	4 6 11·3	1,489198	362873	— 46
4	"	16 34 42	2 57 21·7	1,073409	363125	+ 96
5	"	19 34 34	3 2 35·9	1,105499	363257	+118
6	"	22 36 32	3 1 19·9	1,097320	363084	—184
7	England	51 25 18	1 36 20·0	586319	364952	+256
8	"	52 50 30	1 14 3·4	450018	365036	—411
9	"	54 0 56	1 6 49·7	406516	365109	—107

The last column in this table is the difference between the length of a degree computed with the values of a and b, given above, and the length of a degree given by measurement. These differences must be supposed to arise either from errors in the observations, or from local irregularity of form or density. The most probable source of error is in determining the latitudes; for an error of a single second in the difference of latitude is equivalent to 100 feet measured on the ground. On this account, the largest arcs may be considered the best; for the probable error is the same, whether the arcs be great or small.

190. To these examples we may add the results of four arcs of parallel, measured in different countries, and also their errors, compared with the degrees computed from formula (33).

No.	Country.	Latitude.	Measured Degree.	Diff.
	Mouth of the Rhone	43° 31′ 50″	266345	+1191
	Beechy Head to Dunnose ..	50 44 24	232331	+ 789
	Dover to Falmouth........	50 44 24	231579	+ 37
	Padua to Marennes........	45 43 12	255480	+ 110

191. Prop. XIX.—*To determine the length of any arc of the meridian.*

Let the arc aM (fig. p. 513) measured from the equator $= s$, then $ds = \sqrt{dx^2 + dy^2}$, and if we differentiate the values of x and y, given in formula (25), we shall readily find

$$ds = \frac{a(1-e^2)\,dl}{(1 - e^2\sin^2 l)^{\frac{3}{2}}} = \rho dl.$$

Expanding this expression, and neglecting all powers of e higher than the fourth, we get

$$ds = adl\,(1 - e^2)\,(1 + \tfrac{3}{2}e^2\sin^2 l + \tfrac{15}{8}e^4\sin^4 l)\,;$$

and, since $\sin^2 l = \frac{1}{2}(1 - \cos 2l)$, $\sin^4 l = \frac{1}{8}(3 - 4\cos 2l + \cos 4l)$ (Trig. art. 90), this equation becomes

$$ds = adl\,(1 - e^2)\,(A - B\cos 2l + C\cos 4l),$$

where $A = 1 + \frac{3}{4}e^2 + \frac{45}{64}e^4$, $B = \frac{3}{4}e^2 + \frac{15}{16}e^4$, $C = \frac{15}{64}e^4$.

And integrating

$$s = a\,(1 - e^2)\,(Al - \tfrac{1}{2}B\sin 2l + \tfrac{1}{4}C\sin 4l)\ldots\ldots \quad (34)$$

No constant is necessary, because, at the equator, s and l vanish together.

192. Prop. XX.—*The lengths of any two arcs of the meridian being given from measurement, to determine the polar and equatorial diameters.*

If l and l' be the latitudes of the two extremities of the first arc, and s, s', their distances measured from the equator, then we have, from equation (34),

$$s = a\,(1 - e^2)\,(Al - \tfrac{1}{2}B\sin 2l + \tfrac{1}{4}C\sin 4l)$$
$$s' = a\,(1 - e^2)\,(Al' - \tfrac{1}{2}B\sin 2l' + \tfrac{1}{4}C\sin 4l')$$

Taking the difference of these equations, and putting $s - s' = S$, $l - l' = \lambda$, $l + l' = L$, we have, from Trigonometry (art. 75),

$$S = a\,(1 - e^2)\,(A\lambda - B\sin\lambda\cos L + \tfrac{1}{2}C\sin 2\lambda\cos 2L).$$

In like manner we have, for the second arc,

$$S' = a\,(1 - e^2)\,(A\lambda' - B\sin\lambda'\cos L' + \tfrac{1}{2}C\sin 2\lambda'\cos 2L');$$

and since, in these two equations, the values of S, λ, L, S' λ', L', are all known from observation, the quantities a and e can easily be found, and the polar radius b from the expression $b = a\sqrt{(1 - e^2)}$.

193. If $b = a\,(1 - \alpha)$, the small fraction α is called the *ellipticity** of the spheroid. Hence

$$a\,(1 - \alpha) = a\sqrt{(1 - e^2)} = a\,(1 - \tfrac{1}{2}e^2 - \tfrac{1}{8}e^4)$$

$$\therefore\ \alpha = \tfrac{1}{2}e^2 + \tfrac{1}{8}e^4 \ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

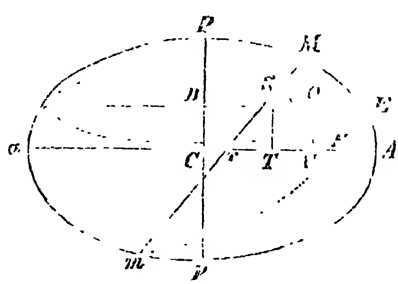
194. If Q be put for the elliptic quadrant, measured from the equator to the pole, we have $l = \frac{1}{2}\pi$ in equation (34); therefore

$$Q = \tfrac{1}{2}a\,(1 - e^2)\,A\pi = \tfrac{1}{2}\pi a\,(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4) \ldots. \quad (36)$$

195. PROP. XXI.—*If the earth be cut by a vertical plane perpendicular to the meridian, the radius of curvature of this section, at the point where it cuts the meridian, is equal to the normal* MR. (See the last figure.)

For, since the earth is supposed to be a solid of revolution, the direction of gravity always passes through the axis of the earth. If, therefore, we conceive the plumb line to be carried over an indefinitely small arc perpendicular to the meridian, its direction will intersect the axis at the same point R as before; and, therefore, R is the centre, and MR the radius of curvature of this arc. The value of MR is given in formula (26).

196. PROP. XXII.—*To find the radius of the curvature at any place, when the earth is cut by a vertical plane making an angle* θ *with the meridian.*



Let PAp be an oblate spheriod, formed by the revolution of the ellipse PAp about its minor axis Pp. Let PMA be the meridian of the given place M, MNm any section passing through the normal Mr, making an angle θ with the meridian; then it is required to find the radius of curvature of the section MNm at the point M.

From any point N in the arc MNm draw NS perpendicular to Mm, and SQ also perpendicular to Mm in the plane PAp. Let the plane NSQ cut the plane DNE drawn through N, parallel to the equator in the line QN. Because MS is perpendicular to SN and SQ, it is perpendicular to the plane NSQ, and therefore the plane MAn passing

* The ellipticity $\alpha = \dfrac{a - b}{a}$; but, by some writers, $\dfrac{a - b}{b}$ is called the ellipticity. The difference is not of much moment.

through MS is perpendicular to the plane NSQ. And because the planes NSQ, DEN, are perpendicular to the plane MAm, their common intersection QN is perpendicular to this plane; therefore NQS, NQD, are right angles. Let

$$rS = x,\ SN = y,\ \angle NSQ = AMN = \theta,\ \angle SrA = QSZ = l,$$

then will $\quad SQ = y\cos\theta,\quad SZ = SQ\cos QSZ = y\cos\theta\cos l,$

$$QZ = SQ\sin QSZ = y\cos\theta\sin l.$$

And because $DE = DN$, we have, from the ellipse (art. 67),

$$a^2b^2 = a^2\,.\,CD^2 + b^2\,.\,DE^2 = a^2\,.\,CD^2 + b^2(DQ^2 + QN^2)\ldots(\alpha)$$

But $\quad CD = ST - SZ = x\sin l - y\cos\theta\cos l.$

$$DQ = Cr + rT + TU = c + x\cos l + y\cos\theta\sin l.$$

$$QN = y\sin\theta.$$

Making these substitutions in equation (α), it will be of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\ldots\ldots\ldots(37)$$

where $\quad A = a^2\sin^2 l + b^2\cos^2 l.\qquad D = 2b^2c\cos l.$

$\quad B = -2(a^2 - b^2)\sin l\cos l\cos\theta.\qquad E = 2b^2c\cos\theta\sin l.$

$\quad C = b^2 + (a^2 - b^2)\cos^2 l\cos^2\theta.\qquad F = -(a^2 - c^2)b^2.$

This is the equation to the ellipse, and we shall find the radius of curvature from the expression $\rho = \dfrac{ds^3}{d^2x\,dy}$, dy being considered constant. Now, at the point M, $\quad x = n,\quad y = 0,\quad \dfrac{dx}{dy} = 0,$ $\dfrac{ds}{dy} = -1$; therefore $\rho = -\dfrac{dy^2}{d^2x}$. Hence, differentiating equation (37) twice, we have

$$x\frac{dx}{dy} + By\frac{dx}{dy} + Bx + 2Cy + D\frac{dx}{dy} + E = 0,$$

$$2Ax\frac{d^2x}{dy^2} + 2A\frac{dx^2}{dy^2} + By\frac{d^2x}{dy^2} + 2B\frac{dx}{dy} + 2C + D\frac{d^2x}{dy^2} = 0,$$

and because $\dfrac{dx}{dy} = 0$, $y = 0$, we get

$$-\frac{d^2x}{dy^2} = \frac{2C}{2Ax + D}\ \text{and}\ \rho = \frac{2Ax + D}{2C};$$

and, since $A = a^2\sin^2 l + b^2\cos^2 l = a^2(1 - e^2\cos^2 l)$,

$$x = Mr = \frac{a(1 - e^2)}{\sqrt{(1 - e^2\sin^2 l)}},\ b^2 = a^2(1 - e^2),\ c = \frac{ae^2\cos l}{\sqrt{(1 - e^2\sin^2 l)}},$$

$$\therefore\ 2Ax + D = \frac{2a^3(1 - e^2)}{\sqrt{(1 - e^2\sin^2 l)}},\ \text{and}$$

$$C = b^2 + (a^2 - b^2)\cos^2\theta\cos^2 l = a^2(1 - e^2 + e^2\cos^2\theta\cos^2 l);$$

$$\therefore\ \rho = \frac{a(1 - e^2)}{\sqrt{1 - e^2\sin^2 l}\,(1 - e^2 + e^2\cos^2\theta\cos^2 l)}\ \ldots,(38)$$

197. *Cor.* Because $1 - e^2 + e^2 \cos^2\theta \cos^2 l$

$$= (1 - e^2)(\sin^2\theta + \cos^2\theta) + e^2\cos^2\theta\cos^2 l$$
$$= (1 - e^2)\sin^2\theta + (1 - e^2\sin^2 l)\cos^2\theta,$$

We obtain from formula (38)

$$\frac{1}{\rho} = \frac{\sqrt{(1 - e^2\sin^2 l)}}{a(1 - e^2)}\left[(1 - e^2)\sin^2\theta + (1 - e^2\sin^2 l)\cos^2\theta\right].$$

And if r be the radius of curvature of the meridian at the point M, and r' the radius of curvature of a section perpendicular to the meridian, we have

$$r = \frac{a(1 - e^2)}{(1 - e^2\sin^2 l)^{\frac{3}{2}}}, \qquad r' = \frac{\quad}{\sqrt{(1 - e^2\sin^2 l)}}$$

Hence it follows that

$$\frac{1}{\rho} = \frac{\sin^2\theta}{r'} + \frac{\cos^2\theta}{r} = \frac{r\sin^2\theta + r'\cos^2\theta}{rr'}$$

$$\therefore \rho = \frac{rr'}{r\sin^2\theta + r'\cos^2\theta} \quad \ldots\ldots\ldots \quad (39)$$

an elegant expression, which may be proved, by the differential calculus, to be true of all surfaces, when r and r' are the radii of greatest and least curvature of all sections passing through the normal at the point M.

198. PROP. XXIII.—*To determine the figure of the earth from the vibrations of pendulums.*

This method, which is now very generally practised, on account of its great facility, may be thus briefly explained. It appears, from Mechanics (art. 279), that the time of vibration of a simple pendulum in a vacuum, when the arcs are indefinitely small, is determined by the equation $t = \pi\sqrt{\frac{L}{g}}$. If, therefore, t and L be given, the value of g may easily be found. Let G represent the force of gravity at the equator, and g the force of gravity in any latitude l; then we have, from Clairaut's theorem (art. 183),

$$\frac{a - b}{b} + \gamma = \frac{5}{2}\cdot\frac{1}{289}\ ; \text{ and } g = G(1 + \gamma\sin^2 l) \ldots (23)$$

Suppose, now, that a pendulum, of either of the forms described in Mechanics (art. 329), is made to vibrate, and its vibrations are compared with those of the pendulum of a clock, as explained in that article; then, if n be the number of vibrations which the clock pendulum makes between two successive coincidences, the experimental pendulum will make $n \pm 2$ vibrations. Let τ be the rate of the clock in seconds, or its gain in 24 hours, then the number of vibrations which the clock makes in a day is $24 \times 60 \times 60 + \tau = 86400 + \tau$. If, therefore, N be the number of vibrations made by the experimental pendulum in a day, we have, manifestly,

$$n : n \pm 2 :: 86400 + \tau : N; \text{ therefore}$$

$$N = \frac{n \pm 2}{n}(86400 + \tau) = 86400 + \tau \pm \frac{172800 + 2\tau}{n} \ldots (40)$$

Let N' be the number of vibrations which the same pendulum makes

in any other latitude l', and g' the force of gravity at this place. We have, then (Mech. art. 281),

$$\frac{N^2}{N'^2} = \frac{g}{g'} = \frac{G(1 + \gamma \sin^2 l)}{G(1 + \gamma \sin^2 l')} \quad 1 + \gamma(\sin^2 l - \sin^2 l')$$

nearly, γ being a very small quantity; therefore

$$\gamma = \frac{N^2 - N'^2}{N'^2(\sin^2 l - \sin^2 l')} \qquad (41)$$

The value of γ being determined in this manner from experiment, the ratio of a to b will be found from the first of equations (23).

In this investigation several corrections have been omitted, which must be taken into consideration when great accuracy is required.

199. (1). *Correction for the amplitude of the arc of vibration.* In the expression given for the time of vibration, in the last article, the arc is supposed to be indefinitely small. Let t be the observed time of vibration, φ the amplitude or semiarc of vibration, and t_1 the time of vibration, when the arc is indefinitely small; then we have (Mech. art. 286),

$$t = \pi\sqrt{\frac{L}{g}}\left(1 + \frac{\varphi^2}{16}\right) = t_1\left(1 + \frac{\varphi^2}{16}\right).$$

Hence, if N be the observed number of vibrations made in a day, and N_1 the number in an indefinitely small arc, φ $N_1 t_1 = 24$ hours $= Nt$, therefore

$$N_1 = N\frac{t}{t_1} = N\left(1 + \frac{\varphi^2}{16}\right) \qquad (42)$$

If, therefore, φ remains nearly constant during the time of observation, the number of vibrations N must be multiplied by the quantity $1 + \frac{1}{16}\varphi^2$. But as the amplitude is continually diminishing, on account of friction and the resistance of the air, it is necessary to make an allowance for this change. Now it is proved, both by theory and experiment, that the arcs decrease very nearly in geometrical progression. Let, therefore, φ be the first arc, φ' the last, and m the number of terms, which is always a very large number. Also, let q be the ratio of the square of each arc to the square of the preceding arc; then the whole time of vibration will be represented by the equation

$$T = \quad \sqrt{\frac{L}{g}}\left\{m + \frac{\varphi^2}{16}(1 + q^2 + q^4 \qquad q^{m-1})\right\}$$

$$\sqrt{\frac{L}{g}}\left(m + \qquad \frac{1 - q^m}{1 - q}\right)$$

Let $q = 1 - x$, then x is a very small quantity, and

$$\log(1 - x) = M(-x - \tfrac{1}{2}x^2 - \&c.) = -Mx, \text{ nearly,}$$

M being the modulus in the common system of logarithms (Alg. art. 400); hence

$$x = -\frac{\log(1 - x)}{M}, \quad \text{or,} \quad q = -\frac{\log q}{M};$$

and since $q^{m-1}\varphi^2 = \varphi'^2$, we have

$$\log q = \frac{2 \log \phi - 2 \log \phi'}{m - 1} = \frac{2(\log \phi - \log \phi')}{m}, \text{ nearly;}$$

$$\therefore \quad 1 - q = \frac{2(\log \phi - \log \phi')}{Mm}.$$

Also, $\phi^2(1 - q^m) = \phi^2 - q\phi'^2 = \phi^2 - \phi'^2$, very nearly.

Making these substitutions in the expression for T given above, we have

$$T = \pi \sqrt{\frac{L}{g}} \left\{ m + \frac{Mm}{32} \frac{\phi^2 - \phi'^2}{\log \phi - \log \phi'} \right\},$$

and therefore the mean time of one vibration is

$$t = \pi \sqrt{\frac{L}{g}} \left(1 + \frac{M}{32} \frac{\phi^2 - \phi'^2}{\log \phi - \log \phi'}\right), \text{ or}$$

$$t = t_1 \left(1 + \frac{M}{32} \frac{\phi^2 - \phi'^2}{\log \phi - \log \phi'}\right).$$

Hence, if ν_1 be the correction to be added to the observed number of vibrations N in a day, we manifestly have

$$= N \frac{M \sin^2 1^\circ}{32} \frac{\phi^2 - \phi'^2}{\log \phi - \log \phi'} \text{.........} (43)$$

the arcs ϕ and ϕ' being estimated in degrees.

200. (2.) *Correction for temperature.* When a pendulum is made to vibrate at different times, its length will vary with the temperature, and therefore the time of vibration will also vary; hence it is necessary to reduce the number of vibrations to a given standard (62°). Let T be the mean height of all the thermometers employed during the experiments, and e the rate of expansion of the metal for 1° of Fahrenheit, then, if L, L' be the lengths of the pendulum at the temperature of T°, and 62°, and N, N_2, be the corresponding numbers of vibrations in a day, we shall have $L = L'\left[1 + e(T^\circ - 62^\circ)\right]$, and consequently (Mech. art. 281),

$$\frac{N_2}{N} = \sqrt{\frac{L}{L'}} = \sqrt{1 + e(T - 62)} = 1 + \tfrac{1}{2}e(T^\circ - 62^\circ), \text{ nearly.}$$

Hence, if ν_2 be the correction to be added on account of the increase of temperature,

$$\nu_2 = \tfrac{1}{2}Ne(T^\circ - 62^\circ) \text{ ....} (44)$$

201. (3.) *Correction for the buoyancy of the atmosphere.* When a body moves in a fluid, its weight is diminished by the weight of an equal bulk of fluid, and therefore the accelerating force is diminished in the same proportion (Hydr. art. 387). Let N be the number of vibrations made in a day in air, N_3 do. in a vacuum; g the force of gravity in air, g' do. in a vacuum; σ the specific gravity of air, S that of the pendulum, during the experiments; then

$$\frac{g'}{g} = \frac{S}{S - \sigma} = \left(1 + \frac{\sigma}{S - \sigma}\right), \text{ also (Mech. art. 281).}$$

$$\frac{N_3}{N} = \sqrt{\frac{g'}{g}} = \sqrt{\left(1 + \frac{\sigma}{S - \sigma}\right)} = 1 + \tfrac{1}{2}\frac{\sigma}{S - \sigma}, \text{ nearly.}$$

Let h be the height of the barometer, and T the temperature of the air during the experiments; also, let σ' be the specific gravity of the air at the temperature of 32°, when the barometer stands at a given altitude H, and h' the height of the same weight of mercury reduced to the temperature T. It appears, then, from hydrostatics (art. 397), that

the specific gravity of the air $= \dfrac{p}{k\left[1 + \alpha(T^\circ - 32^\circ)\right]}$, when p is the pressure on a unit of surface, α is the expansion of air for 1° of temperature, and k is a constant quantity; hence

$$\sigma : \sigma' :: \frac{h}{1 + \alpha\,(T^\circ - 32^\circ)} : h'.$$

Also, if μ be the expansion of mercury for 1° of temperature, $h' = H\left[1 + \mu\,(T^\circ - 32^\circ)\right]$. Substituting this value of h' in the proportion above, and forming an equation, we get

$$\sigma = \sigma' \frac{h}{H}\,\frac{1}{1 + (\alpha + \mu)\,(T^\circ - 32^\circ)}, \text{ very nearly.}$$

According to MM. Arago and Biot, when $H = 29{\cdot}9218$, and the temperature is 32°, σ' is equal to $\frac{1}{770}$, therefore $\frac{1}{2}\,\dfrac{\sigma'}{H} = {\cdot}0000217$. Also, $\alpha = \frac{1}{450} = {\cdot}00222$, $\mu = {\cdot}0001$ (Hydr. art. 399), and therefore $\alpha + \mu = {\cdot}0023$. Hence

$$\tfrac{1}{2}\sigma = \frac{{\cdot}0000217}{1 + {\cdot}0023(T - 32)}, \text{ and } \frac{N_3}{N} = 1 + \frac{{\cdot}0000217\,h}{(S - \sigma)\left[1 + {\cdot}0023(T^\circ - 32^\circ)\right]},$$

or, if we put $\frac{1}{770}$ $(= {\cdot}0013)$ for σ in the denominator, we shall have, for the correction to be added to N,

$$= N\,\frac{{\cdot}0000217}{S - {\cdot}0013}\,\frac{h}{1 + {\cdot}0023\,(T^\circ - 32^\circ)} \;\ldots\ldots\; (45)$$

202. From several experiments, first made by Bessel, and afterwards repeated in this country, by Lieut.-Colonel Sabine and Mr. Francis Baily, this correction is found to be far too small. It appears that a quantity of air adheres to the pendulum and moves along with it, and thus the mass moved and the moment of inertia are both changed. The effect of this varies in different pendulums, and, therefore, it must be ascertained by actual experiments in each case. In the Philosophical Transactions for 1832, Mr. Baily shows that the correction on account of the air may be expressed by the formula

$$\nu_3 = C\,\frac{h}{1 + {\cdot}0023\,(T^\circ - 32^\circ)} \;\ldots\ldots\ldots\; (46)$$

in which the value of C is constant for the same pendulum, but varies for different pendulums. We cannot pursue the subject further, but must refer the student to this paper, and also to an excellent report of Captain Foster's experiments, drawn up by Mr. Baily, at the request of the Royal Astronomical Society.

203. (4.) *Reduction to the level of the sea.*—Let g' be the force of gravity at the level of the sea, g the force of gravity at a plane h feet above this level, and r the radius of the earth; also let N be the number of vibrations in a day at the place of observation, and N_4 do. at the level of the sea. We have, then (Mech. art. 281),

$$N_4 \ : \ N \ :: \ \sqrt{g'} \ : \ \sqrt{g} \ :: \ r + h \ : \ r.$$

Hence $N_4 = N\left(1 + \frac{h}{r}\right)$; and $\nu_4 = N \ \frac{h}{r}$ ....(47)

204. In this correction the mass of the globe is supposed to be entirely below the level of the sea; but as a certain portion of the earth is always interposed between the observer and this level, this correction is too great. In the Philosophical Transactions for 1819, Dr. Young proposes that it shall be multiplied by a fraction which varies from 0·50 to 0·75, according to the form and nature of the ground. In the report of Capt. Foster's experiments, Mr. Baily assumes the fraction to be 0·666.

205. We shall apply these formulæ to an example taken from Mr. Baily's report, p. 14.

Experiment, No. 10, *made at South Shetland, on January* 20, 1829, *with the No.* 11 *pendulum. The clock making* 86478·80 *vibrations in a mean solar day. The height of the pendulum above the level of the sea* = 19 *ft.* 2 *in.*

It was Captain Foster's practice to record the first three and the last three coincidences, in *printed* blank forms, in order to guard against any error in noting down the precise times; a method recommended to be pursued in all observations of this kind. The following is a specimen of the mode of registering adopted in these experiments.

No.	Time of Disappearance.	Reappearance.	Coincidence.	Arc.	Thermometers. Upper.	Thermometers. Middle.	Thermometers. Lower.	Barometer.
	h m s	s	s					
1	11 7 42	44	43·0	0·91	38·6	38·4	38·4	29·442
2	16 52	54	53·0	·87				
3	26 2	5	3·5	·82	39·0	38·8	38·8	
....	....	......	......	.....	....	......	......	......
....	....	......	......	......	.....	......	......	......
17	1 35 22	30	26·0	·36	38·5	38·4	38·4	
18	44 38	49	43·5	·33				
19	53 57	67	62·0	·30	39·0	38·6	38·6	29·429

The mean value of h is 29·436. The mean value of T is 38·625. Mr. Baily states it to be 38·65; we will, therefore, take this number for T in the following corrections. In the second correction, Mr. Baily assumes the value of e to be ·00000982. In the third correction the value of C was found, from experiments, to be ·3541.

	h.	m.	s.
First coincidence	11	7	43·0
Last do.	1	54	2·0
18 intervals =	2	46	19·0

18)9979

n = 554·389

$$\therefore\ N = \frac{n-2}{n} 86478.8 = 86166{\cdot}822.$$

$$\nu_1 = N \frac{M \sin^2 1^\circ}{32} \quad \frac{\varphi^2 - \varphi'^2}{\log \varphi - \log \varphi'} = {\cdot}545.$$

$$\nu_2 = N \times \tfrac{1}{2}e\,(T - 62) \qquad = -\ 9{\cdot}879.$$

$$\nu_3 = \frac{{\cdot}3541\ h}{1 + {\cdot}0023\ (T - 32)} \qquad = \quad 10{\cdot}266.$$

$$\nu_4 = N \times \frac{h}{r} \times 0{\cdot}666 \qquad = \quad {\cdot}052.$$

Hence, the number of vibrations at South Shetland, in a mean solar day, reduced to the level of the sea, is

$$N + \nu_1 + \nu_2 + \nu_3 + \nu_4 = 86167{\cdot}806.$$

206. By a comparison of various experiments, which have been made at different places of the earth, by Kater, Goldingham, Hall, Brisbane, Sabine, Fallows, Freycinet, Duperrey, and Leuthe, Mr. Baily finds that the value of $\frac{a-b}{b}$ varies from $\frac{1}{289{\cdot}48}$ to $\frac{1}{266{\cdot}40}$, but the observations on the whole may be tolerably well represented by the fraction $\frac{1}{285{\cdot}26}$. The value of γ therefore, given above, will be ·00514491. All the pendulum experiments agree, however, in giving a greater ellipticity to the earth than that which is deduced from the comparison of arcs of the meridian. To whatever cause this discrepancy may be assigned, we cannot hesitate in giving the preference to the results of the geodetic measures.

FINIS.

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